2. DYNAMIC EQUILIBRIUM MODELS I: TWO-PERIOD ECONOMIES

Interest always carrieth with it an ensurance praemium.
– Sir William Petty (1623–1687)

To be able to study intertemporal prices (interest rates) or plans for behaviour over several periods, as well as growth, we need a dynamic, equilibrium model. The simplest dynamic models have two periods and that is where we begin.

(a) Exchange Economy

Macroeconomics is about dynamics, and about how the different decisions made by consumers and firms – saving, investment, employment, and so on – interact. We shall next look at some of these decisions, and the resulting equilibrium, in the simplest possible dynamic economy: a two-period, single consumer, exchange economy. The idea, which dates back at least to Irving Fisher, is to look at the allocation of goods over time the same way we treat the allocation of resources to different goods at a point in time. We simply treat goods at different dates as different commodities.

The essential elements of our dynamic economy are:

(a) The list of commodities: consumption now (c_1) and later (c_2).

(b) Endowments of the commodities, y_1 and y_2, which are nonstorable.

(c) The preferences of a single 'representative' consumer, characterized by the concave utility function $u(c_1, c_2)$.

(a) and (b) summarize the feasible allocations of this economy, namely $c_t \leq y_t$, for $t = 1, 2$. There is no uncertainty in this environment; we'll allow for that later.

This summarizes the environment. The question is how the consumer behaves and what determines prices and quantities, the objects of interest. One theoretical assumption which is simple and the implications of which are testable is that the agent operates in a system of competitive markets. The goods, consumption now and later, sell at prices p_1 and p_2, respectively. The consumer is endowed with quantities y_1 and y_2, respectively, of the two goods and maximizes utility subject to the budget constraint

$$p_1 c_1 + p_2 c_2 \leq p_1 y_1 + p_2 y_2$$

and prices adjust so that, in equilibrium, supply and demand for each good are equal.

In other words, we define a competitive equilibrium as a price system $\{p_1, p_2\}$ and allocation $\{c_1, c_2\}$ such that:

(a) the consumer maximizes utility subject to the budget constraint for given prices; and

(b) (feasibility) supply equals demand for each good: $c_t = y_t$ for $t = 1, 2$.

Let us take an example that we can solve explicitly. Suppose utility is

$$u(c_1, c_2) = \ln c_1 + \beta \ln c_2$$

for $\beta > 0$. We shall use log preferences frequently. Typically we'll choose $0 < \beta < 1$, implying that consumers discount future consumption relative to current consumption. It takes more than one unit of future consumption to compensate the agent – in the sense of maintaining the level of utility – for the loss of one unit of current consumption at any point along the ray $c_1 = c_2$. [Graph the isoquant, show β as the slope along the 45° line.] Sometimes we shall write U for the lifetime utility function and u for the period utility function. With this notation we have $U = u(c_1) + \beta u(c_2)$, $u(c) = \ln c$.

We find the equilibrium as follows. First, the consumer maximizes utility subject to the budget constraint. This yields the "demand functions"

$$c_1 = \frac{1}{1+\beta}\left[y_1 + \frac{p_2}{p_1} y_2\right]$$
$$c_2 = \frac{\beta}{1+\beta}\left[\frac{p_1}{p_2} y_1 + y_2\right]$$

Now we impose the second part of the definition of equilibrium, that supply equals demand. Note that both conditions, $c_1 = y_1$ and $c_2 = y_2$, produce the same result: $p_2/p_1 = \beta(y_1/y_2)$. At these prices the allocation is, obviously, $c_t = y_t$.

This brings us to an important point about exchange economies that sometimes is confusing. We work with these models because they are simple; in particular the consumption allocation can be read off from the endowments, because the good is perishable and cannot be stored or invested. But if people cannot save why do we write down a multiperiod budget constraint and how is an interest rate or intertemporal relative price determined? A practical note first. To solve these models first work out the household planning problem in general, then use the technology and equilibrium conditions to determine the allocation and prices. Second, one can think of this two-period model without storage as an auction. At the beginning of time the auctioneer can deduce the relative values people attach to c_1 and c_2 by offering them trades between the goods at different dates. If people value c_1 more than c_2 then p_1 will exceed p_2, for example. Thus the relative prices tell us something about people's impatience or willingness to substitute between the two periods, even if they are not able to do so on their own if there is no storage or investment. And this result will continue to apply in more realistic models. There, as we shall see, the interest rate or intertemporal relative price tells us about investment possibilities, but it also tells us about people's preferences, a fact illustrated most simply in the endowment economy.

As a rule, the definition of equilibrium only determines relative prices, here p_2/p_1, not prices themselves. If we multiply p_1 and p_2 by a positive constant the new prices are also an equilibrium for the same allocation. Thus we are free to choose one price arbitrarily, say $p_1 = 1$. In this case all prices are measured in units of the first good. The price p_2 then means that one unit of consumption tomorrow costs p_2 units of consumption today.

Thus, the allocation $\{c_1 = y_1,\ c_2 = y_2\}$ and prices $\{p_1 = 1,\ p_2 = \beta(y_1/y_2)\}$ constitute a competitive equilibrium for this economy.

Now one of the things we know about competitive equilibria in such economies (in fact, for a large class of economies) is the fundamental theorems of welfare economics: competitive allocations are Pareto optimal, and optimal allocations can be supported as competitive equilibria. Given the resource constraints on the economy, no consumer's utility can be increased without lowering someone else's. (This is the sense in which competitive markets are said to produce an efficient allocation of resources.) In this economy there is a single agent, and thus no difficulty of comparing utilities across them and no sense in which distribution matters for macroeconomic properties. A Pareto optimum is an allocation $\{c_1, c_2\}$ that maximizes utility subject to the economy's resource constraints, $c_t \leq y_t$. The solution to this problem is, obviously, for the representative agent to consume the entire endowment, $c_1 = y_1$ and $c_2 = y_2$, so we have verified the first welfare theorem for this economy (the competitive allocation derived above is the same). Moreover, the relative price implicit in this allocation can be computed from the marginal rate of substitution. The *mrs* between the two goods is the ratio of the marginal utility of consumption tomorrow to the marginal utility of consumption today, with both derivatives evaluated at equilibrium or optimum quantities:

$$mrs = (\beta/c_2)/(1/c_1) = \beta(y_1/y_2).$$

Thus $p_2/p_1 = mrs = \beta(y_1/y_2)$ as before. [Practice: derive $p_2/p_1 = (du/dc_2)/(du/dc_1) =$ mrs from the consumer's first-order conditions.]

It is sometimes convenient, in model economies to which the welfare theorems apply, to compute a competitive equilibrium by solving a welfare maximization, rather than going through the consumers' problem and imposing the market clearing conditions. Exercise 3 illustrates this point in a related two-person economy.

We have, to this point, done little to distinguish this dynamic economy from a static one with two goods, haggis and oatcakes, say. There has been no mention of the rate of interest, for example. Consider, then, a slight change in notation in which the intertemporal character of the economy is made more explicit. We can think of our consumer as deciding, in the first period, how much to save: choosing, that is,

$$s = y_1 - c_1.$$

In the second period the agent takes the proceeds from saving, after interest, plus second-period income, and spends it:

$$c_2 = (1 + r)s + y_2,$$

where r is the rate of interest paid on saving. The consumption-saving decision clearly has an intertemporal character to it. In equilibrium the rate of interest equates supply and demand: $c_t = y_t$. Note that r is a real, or commodity, rate of interest, since it is measured in units of goods: goods tomorrow for goods today.

Thus the rate of interest is defined by

$$1/(1+r) = p_2/p_1 = \beta(y_1/y_2).$$

If $y_1 = y_2$ this takes a particularly simple form. The discount factor β is often expressed in terms of a discount rate θ, with $\beta = 1/(1+\theta)$. When $y_1 = y_2$, the equilibrium interest rate is $r = \theta$. In general the interest rate depends on both the discount factor and the endowments.

We usually think of r as being positive (reflecting, in this last, steady-state example, impatience). Since $1 + r = p_1/p_2$ that means that $p_1 > p_2$. The endowment at time 1 commands a higher price since no waiting is involved. It may seem odd that the interest rate is determined, since the economy as a whole does no saving in equilibrium. But consumers can still value marginal units of the endowments and hence determine a relative price. (Try drawing a picture of this).

If you have studied interest rates before, then you might be thinking: This r is simply a commodity own-rate or real rate of interest. So shouldn't rises in the price ($p_2 > p_1$) correspond to positive interest rates? In fact, this way of thinking is correct. Let us clarify the units in which these prices are quoted:

$$\begin{aligned}\frac{p_1}{p_2} &= \frac{\text{utility per goods today}}{\text{utility per goods tomorrow}} \\ &= \frac{\text{goods tomorrow}}{\text{goods today}} \\ &= 1 + r\end{aligned}$$

For future reference, suppose that prices are quoted in money terms. Let P denote the good's price in dollars. Then

$$\begin{aligned}\frac{P_2}{P_1} &= \frac{\text{\$ per goods tomorrow}}{\text{\$ per goods today}} \\ &= 1 + \pi\end{aligned}$$

where π is the inflation rate. Combining these two results,

$$\begin{aligned}\frac{p_1}{p_2} \cdot \frac{P_2}{P_1} &= (1+r)(1+\pi) \\ &= \frac{\text{goods tomorrow}}{\text{goods today}} \cdot \frac{\text{\$ per goods tomorrow}}{\text{\$ per goods today}} \\ &= \frac{\text{\$ tomorrow}}{\text{\$ today}} \\ &= 1 + R\end{aligned}$$

where R is the nominal interest rate.

(b) Production Economy

Our first theoretical economy was an exchange economy, by which we mean that there was no opportunity for producing goods – consumers simply exchange what is available in competitive markets. In that economy the rate of interest is determined by the representative consumer's preferences for present and future consumption, including the discount factor β and the endowments y_1 and y_2. More generally the rate of interest will depend, as well, on the technological opportunities for transforming current consumption into future consumption. For example, if the endowment y_1 is storable and a unit stored at time 1 yields $(1+g)$ units at time 2 then the interest rate will be g (show this).

More generally, consider the possibility of investment in physical capital. We shall say that k units of investment leads to $f(k)$ extra units of the good next period; if we set aside k units in period 1 we can have $k(1-\delta)+f(k)$ units in period 2. Here δ is the rate of depreciation of the capital stock. We assume that the following conditions apply to the function f: (i) No investment, no extra output: $f(0)=0$. (ii) f is increasing and concave: $f' > 0$, $f'' < 0$. (iii) The Inada conditions, $f'(0)=\infty$ and $f'(\infty)=0$. [Graph $f(k)$ vs k and show how the conditions apply. Show that the marginal rate of transformation is the slope of the production possibilities frontier.]

The possibility of production introduces another actor into the economy, a firm which buys (or rents) capital from consumers at price q per unit and uses it to produce extra output in the second period. The firm chooses the amount of capital to maximize profit:

$$\mathrm{PR} = p_2[k(1-\delta)+f(k)] - qk.$$

[Graph this vs k, show that there is a finite interior maximum (due to the Inada conditions).] q is the price of capital.

The representative consumer owns both the capital stock and the firm, and her budget constraint changes as follows. Let x be the amount of capital rented to firms and PR be profits from owning the firm. Then the consumer chooses consumption quantities, c_1 and c_2, and capital rentals x to maximize utility subject to the budget constraint

$$p_1c_1 + p_2c_2 \leq p_1(y_1 - x) + p_2y_2 + qx + \mathrm{PR}.$$

Clearly we must have $p_1 = q$ in equilibrium (or the consumer would choose x equal to plus or minus infinity).

In this expanded economy the definition of equilibrium changes as follows. The list of commodities is $\{c_1,\ c_2,\ x,\ k\}$. A competitive equilibrium is a price system $\{p_1,\ p_2,\ q\}$ and allocation $\{c_1,\ c_2,\ x,\ k\}$ such that:

(a) the consumer maximizes utility given prices and subject to the budget constraint;

(b) the firm maximizes profits given prices and technology; and

(c) supply equals demand for each good:

$$c_1 + k = y_1, \quad c_2 = y_2 + k(1-\delta) + f(k), \quad x = k.$$

The definition of a Pareto optimum also changes. An optimum in this economy is an allocation that maximizes the utility of our representative agent, $U(c_1, c_2)$, given the endowments and technological possibilities of the economy: subject, that is, to the feasibility constraints

$$\begin{aligned} c_1 + k &\leq y_1, \\ c_2 &\leq y_2 + k(1-\delta) + f(k). \end{aligned}$$

[Graph the production possibilities set.] Again, the optimum problem often provides a shortcut to finding a competitive equilibrium. Notice that the planning problem involves no prices.

Let us again turn to a concrete example. Let $u = \ln c_1 + \beta \ln c_2$, as before; $y_1 = y$ and $y_2 = 0$; and $f(k) = k^\alpha$, with $0 < \alpha < 1$. Also assume that $\delta = 1$ so that there is 100% depreciation. It is convenient to substitute the feasibility constraints into the objective function, giving us the optimum problem,

$$\max \ \ln(y-k) + \beta \ln(k^\alpha).$$

The first-order condition is:

$$\frac{-1}{y-k} + \beta \frac{\alpha k^{\alpha-1}}{k^\alpha} = 0$$

which implies

$$k = (\frac{\alpha\beta}{1+\alpha\beta})y.$$

In short, a constant fraction of the first-period endowment y is saved (this is a result of using log utility), and used to produce output next period. The consumption decisions are readily derived from the resource constraints:

$$\begin{aligned} c_1 &= y - k = (\frac{1}{1+\alpha\beta})y, \\ c_2 &= f(k) = \big[(\frac{\alpha\beta}{1+\alpha\beta})y\big]^\alpha. \end{aligned}$$

Thus the optimal allocation, $\{c_1 = (1-\omega)y,\ c_2 = (\omega y)^\alpha,\ k = \omega y\}$, for $\omega = \alpha\beta/(1+\alpha\beta)$, depends on preferences (through β), technology (α), and the endowment (y).

Prices. It is clear, first, that in any equilibrium we will have $q = p_1$ (in a sense, both prices pertain to the same good). We can find the relative price $p_2/p_1 = p_2/q$ from either the marginal rate of substitution or the marginal rate of transformation. The first gives us

$$\frac{p_2}{p_1} = \text{mrs} = \frac{\beta/c_2}{1/c_1} = \frac{\beta(1-\omega)y}{\omega y^\alpha},$$

the second

$$\frac{p_2}{p_1} = \text{mrt} = \frac{1}{f'(k)} = \frac{\omega y}{\alpha(\omega y)^\alpha}.$$

These reflect, respectively, the first-order conditions of the consumer and the firm. [Practice: Verify these relations for the general case.] Notice that a first-order condition for profit maximization is

$$(1-\delta) + f'(k) = \frac{p_1}{p_2} = 1 + r,$$
$$f'(k) = r + \delta.$$

One can think of the interest rate $r = p_1/p_2 - 1$ as reflecting either impatience or productivity. From the marginal rate of transformation, if there is a storage such that investing 1 at time 1 yields $1+g$ at time 2 then $r = g$. In the example above investing k at time 1 yields $k(1-\delta) + f(k)$ at time 2 so that investing an additional unit at time 1 yields $f'(k) + (1-\delta)$ at time 2. Thus in the example the interest rate is $r = f'(k) - 1$ because $\delta = 1$. In this notation we assume that one gets back the original investment (*i.e.* that f is a net rather than gross production function, so that its marginal product is the net rather than gross interest rate).

Incidentally, you should be able to verify that solving this simple example would be much more difficult had we not assumed 100% depreciation. For more realistic δ's, numerical methods might be used.

A lot of macroeconomics can be done with the two-period models we have seen so far in this section. Aside from adding more periods (which we won't do explicitly) there are at least two obvious extensions: allowing for leisure and elastic labour supply, and adding a government sector. We'll make these additions in applications below. To make these extensions interesting we also need to introduce some uncertainty in these two-period economies. So far we have assumed that when decisions are made in period 1 the values of period-2 exogenous variables are known.

(c) Uncertainty and Expectations

To make progress in introducing some realistic uncertainty we shall assume that it can be described by standard tools of probability. The randomness or uncertainty arises from exogenous variables which can take on one of several different values. In the two-period economy a good example of such a variable is the exogenous endowment $\{y_1, y_2\}$. Other examples might include tax rates, the money supply, or some property of the production function (*i.e.* there might be lean years and fat years).

As a simple stationary example suppose in the endowment version that $\{y_1, y_2\}$ have the following joint density:

Outcome	Probability
(1,1)	0.3
(2,1)	0.2
(1,2)	0.2
(2,2)	0.3

which is a valid density because the probabilities sum to one. This is the simplest example because there are only two possible outcomes. More generally we might think of modelling with continuous random variables, but this way we can calculate means and variances by simple addition.

Next we shall assume that agents know that the realizations of income are draws from this urn. Assuming that they know the true urn is sometimes called the hypothesis of rational expectations – more on this in section 3. Thus when we speak of their expectations of y_2, for example, we identify those with the actual mathematical expectation from this probability density function.

What is $\mathrm{E}(y_2)$? We know that y_2 can take on two values: 1 and 2. To find the probabilities attached to these we need to find the marginal density of y_2. It is easy to see that $\mathrm{Prob}(y_2 = 1) = .5$ and $\mathrm{Prob}(y_2 = 2) = .5$ simply by summing the cases from the joint density in which y_2 takes on each value. Thus $\mathrm{E}(y_2) = 1.5$. This is sometimes called the unconditional expectation.

However, in this example y_1 occurs before y_2 does. One obvious property of the joint density is that states tend to be persistent. If y_1 is low then y_2 is more likely to be low than high, and conversely. Thus is a realistic feature of business cycles; periods of low income are likely to follow periods of low income and there is some persistence or autocorrelation to deviations from trend.

But that fact built into this example suggests that agents can improve their forecast of y_2 if they observe y_1, simply because the two random variables are not independent. Once period 1 events occur agents can condition on them. The conditional forecast or expectation is denoted $\mathrm{E}_1 y_2$ to show that it is the expectation based on information known about period 1. An alternative notation which is often clearer is $\mathrm{E}(y_2|y_1)$ which shows precisely what variable in period 1 can be used to help forecast y_2. To calculate this expectation we again use the two possible outcomes for y_2 but now weight them by the conditional probabilities.

Conditional probabilities are given by dividing joint by marginal. So if we are in state 1 in period 1 then $\mathrm{Prob}(y_2 = 1|y_1 = 1) = .3/.5 = .6$ and $\mathrm{Prob}(y_2 = 2|y_1 = 1) = .2/.5 = .4$. Thus $\mathrm{E}(y_2|y_1 = 1) = 2 \cdot .4 + 1 \cdot .6 = 1.4$. This is lower than the unconditional expectation, as one would expect. Knowing that y_1 has taken a low value causes one to revise downward the expectation or forecast of y_2. This reflects the fact that the two random variables are positively correlated. One could work out the covariance or correlation from the probabilities given; if it were zero then the conditional and marginal expectations would coincide.

Now we are ready to introduce uncertainty in two-period models. We simply have to be careful to specify joint distributions and then say what agents know or condition on at time 1, or else to write down a conditional density directly. Think of this as an important part of the model. Take the endowment economy as an example.

Now agents maximize expected utility. Suppose that

$$\mathrm{E}U = \mathrm{E}(\ln c_1 + \beta \ln c_2|y_1) = \ln c_1 + \beta \mathrm{E}(\ln c_2|y_1),$$

and notice that the expected value of the log of a function does *not* equal the log of the expected value. We know in this economy that the allocation is $c_1 = y_1$ and $c_2 = y_2$ so let us study the interest rate. A first-order condition is

$$\frac{1}{c_1} = \mathrm{E}[\frac{\beta(1+r)}{c_2}|y_1].$$

It may seem that we have differentiated through the expectations operator, which is not correct for nonlinear functions. But we now can show that this Euler equation is correct. To derive the Euler equation, suppose that the household knows the current state of the world (y_1) but that there are two possible states for the next period, $\underline{y}_2$ and $\overline{y}_2$. Then there are also two possible values for c_2 and, in general, for the interest rate. The budgetting problem now involves choosing c_1 and a plan for c_2 in each state. The plan consists of $\underline{c}_2$ and $\overline{c}_2$. The idea is that we treat second-period consumption as two different goods, depending on the state which occurs. Let us assume in the example that the probability of the low state is 0.6 and of the high state 0.4.

Maximizing expected utility in the log example means:

$$EU = log(c_1) + \beta[0.6\ln(\underline{c}_2) + 0.4\ln(\overline{c}_2)]$$

subject to

$$\underline{c}_2 = \underline{y}_2 + (1+\underline{r})(y_1 - c_1)$$
$$\overline{c}_2 = \overline{y}_2 + (1+\overline{r})(y_1 - c_1)$$

In problems like this it is best to use the sequence form of the budget constraint.

Call the Lagrange multipliers on these two constraints (only one of which can end up applying) $\underline{\lambda}$ and $\overline{\lambda}$. The first three first-order conditions are:

$$\frac{1}{c_1} + \underline{\lambda}(1+\underline{r}) + \overline{\lambda}(1+\overline{r}) = 0$$

$$\frac{\beta\, 0.6}{\underline{c}_2} + \underline{\lambda} = 0$$

$$\frac{\beta\, 0.4}{\overline{c}_2} + \overline{\lambda} = 0.$$

Then putting these together gives

$$\frac{1}{c_1} = \beta[\frac{0.6(1+\underline{r})}{\underline{c}_2} + \frac{0.4(1+\overline{r})}{\overline{c}_2}] = \beta E_1 \frac{(1+r)}{c_2},$$

which confirms our result.

The Euler equation links two endogenous variables: interest rates and consumption. The general version is

$$u'(c_1) = E_1[\beta(1+r)u'(c_2)].$$

This makes intuitive sense. Suppose you are deciding how much to save. The cost of saving a bit more is the marginal utility of current consumption, because that is what you forego by saving. The benefit is that you can take a saved unit and invest it, earning $(1+r)$ units next period. That benefit in goods is converted into utility terms by multiplying it by marginal utility then $(u'(c_2))$. Then the future utility benefit is discounted, so as to be comparable to today's cost.

To show that this uncertainty can make a difference let us look at a very specific type of interest contract, in which one invests 1 at time 1 and receives $1+r$ at time 2 no matter which state occurs then. In that case there is no uncertainty about the return so we can take it outside the E operator:

$$\frac{1}{1+r} = \beta c_1 \mathrm{E}[(\frac{1}{c_2})|y_1].$$

Jensen's inequality holds that for a function which opens up, as the function $y = 1/x$ does, the expected value of the function is greater than the function of the expected value. Thus the right-hand side is greater than $\beta c_1/E(c_2)$. Therefore, in this economy the interest rate is lower than in an otherwise identical economy with the same average second-period consumption but lower variance in it. The intuition is that with uncertainty in future income (and with this utility function) people try to save more (this is sometimes called precautionary saving) and so they bid up p_2 relative to p_1. Notice that the risk in this example is in income and not in the asset's payoff. In a few pages we shall see the effect of payoff risk too.

In dynamic models agents may be uncertain about productivity or about aspects of government policy. Later sections will provide some illustrations of the effects of such uncertainty.

We'll next outline two interesting applications of these two-period model economies. Further applications are found in later sections of the course.

(d) Application: Labour Supply and Lucas's Critique

So far our household has consumed but not worked. It is simple to add labour supply and demand to our two-period framework, though.

In growth theory (section 4) we shall often assume that labour is supplied inelastically: households have an endowment of time and they supply this in a labour market so they labour supply curve is vertical. The idea is that cycles in unemployment and participation rates can be ignored for long-run analysis. In that case, real wages depend on the size of the labour force and on the demand for labour.

Exercise Suppose that in the two-period production economy the consumer has endowments $y_1 = y$ and $y_2 = 0$ of the good, and endowment $n_2 = 1$ of labour in the second period. The consumer has log utility, as a function only of the two consumptions. There is 100 percent depreciation, and the production function is

$$f(k, n) = k^{\alpha} n^{1-\alpha}.$$

Solve for the real wage.

As you can see from this exercise, changes in the real wage must come from shifts in the labour demand curve (or else changes in population or participation) in this environment. What explains a trend in real wages in developed economies?

In business-cycle theory (section 5), in contrast, we usually focus on the elasticity of labour supply. That is modelled by assuming that leisure enters the utility function. We won't look at a general equilibrium example here, but simply study the agent's labour supply decision.

Consider a two-period model in which a representative agent has preferences

$$U = \ln(c_1) + \ln(1 - n_1) + \beta \ln(c_2) + \beta \ln(1 - n_2)$$

where c is consumption, n is labour supply (hours worked), and hence $1 - n$ is leisure. The budget constraint is

$$c_1 + \frac{c_2}{1+r} = n_1 w_1 + \frac{n_2 w_2}{1+r},$$

where w is the wage rate. We need not specify the production side or solve the general equilibrium. Instead we shall focus only on the household's plans and incentives, so that we take wages as given. There is no uncertainty.

To find the optimal plan, simply form the Lagrangean and solve for c_1, c_2, n_1, and n_2. The first-order conditions are the budget constraint and

$$\begin{aligned}
\frac{1}{c_1} + \lambda &= 0 \\
\frac{\beta}{c_2} + \frac{\lambda}{1+r} &= 0 \\
\frac{-1}{1-n_1} - \lambda w_1 &= 0 \\
\frac{-\beta}{1-n_2} - \frac{\lambda w_2}{1+r} &= 0
\end{aligned}$$

Exercise: Eliminate λ three different ways, giving Euler equations which describe the choice between c_1 and c_2, between c_1 and n_1, and between n_1 and n_2.

Eliminating λ and substituting in the budget constraint gives:

$$c_1(2 + 2\beta) = w_1 + \frac{w_2}{1+r},$$

so that

$$\begin{aligned}
c_1 &= \frac{1}{2+2\beta}[w_1 + \frac{w_2}{1+r}] \\
c_2 &= \frac{\beta}{2+2\beta}[w_1(1+r) + w_2] \\
n_1 &= 1 - \frac{1}{2+2\beta}[1 + \frac{w_2}{(1+r)w_1}].
\end{aligned}$$

where wages, and the interest rate are taken as given by the agent. Combining Euler equations and the budget constraint gives us the *consumption function* and the *labour supply function*. However, we often cannot solve for these explicitly when there is uncertainty. In the exercises, you'll see that with uncertainty we usually need to restrict ourselves to some special cases of the utlity function (such as quadratic utility) in order to solve for the consumption function, for example.

Two very general macroeconomic ideas show up in these decision rules. One is the *permanent income hypothesis*: consumption depends on the present value of labour income. We'll study this idea in detail in section 6.

The other idea is *intertemporal substitution*. As usual, a quantity decision depends on a relative price, in this case relative wages in the two periods. You can see that if w_2 rises relative to w_1 then n_1 falls; labour supply in period 2 is substituted for labour supply in period 1. Section 5 discusses some of the evidence for this effect.

You might think that wages do not vary much from period, unless the comparison is between straight-time and overtime. One potential source of variation in the intertemporal prices which affect labour supply, consumption, and investment is government policy. For example, suppose that there is a tax τ on wage income. Then the budget constraint becomes

$$c_1 + \frac{c_2}{1+r} = n_1 w_1 (1-\tau_1) + \frac{n_2 w_2 (1-\tau_2)}{1+r},$$

and first-period labour supply is:

$$n_1 = 1 - \frac{1}{2+2\beta}[1 + \frac{w_2(1-\tau_2)}{(1+r)w_1(1-\tau_1)}].$$

To isolate the effect of tax changes let us assume that $w_1 = w_2 = w$. Next, we can use the two-period model to find the effect on labour supply of a policy change. Suppose that we want to see the effect of a tax cut ($d\tau_1 < 0$). It is natural to start by finding $dn_1/d\tau_1$ (it is negative).

Why would we want to know this? The answer is that we need to predict the effects of the tax cut on tax revenue and on output. Tax revenue is here given by $T_1 = \tau_1 w_1 n_1$. Thus the effect on revenue is:

$$\frac{dT_1}{d\tau_1} = w_1 n_1 + \tau_1 n_1 \frac{dw_1}{d\tau_1} + \tau_1 w_1 \frac{dn_1}{d\tau_1}.$$

The first term is just the tax base, the third term seems to be given from our labour-supply decision rule, and the second term requires a general equilibrium model of the labour market (and not just the agent's problem as outlined here). Usually one imagines that the total effect is positive. The Laffer curve graphs the case in which it may be negative at large values of τ (as for cigarettes), which requires a very large labour-supply elasticity. That raises the question of how elastic labour supply is in reality. Perhaps supply-side economics is more relevant at low incomes (the ‘poverty trap’) than at high incomes.

In fact, we’ve made two fundamental mistakes in evaluating this policy. The first mistake is that we have tried to assess the effect using a decision rule, whereas we need a general equilibrium model. For the term $dn_1/d\tau_1$ in our expression for the change in tax revenue we need the change in equilibrium labour supply, not the horizontal shift in the labour supply curve. With a downward-sloping labour demand curve, wages may fall when the supply curve shifts, so we need the labour demand curve to find the intersection. We also would need that intersection to find the effect of the tax cut on output.

The second mistake we’ve made is to hold τ_2 fixed when changing τ_1. It is easy to see that the predicted effect on labour supply depends on what people think will happen to taxes next period. Will the tax cut be reversed? Will it be continued?

Exercise: Compare the effects of the policy $d\tau_1 < 0$, $d\tau_2 = 0$ with those of the policies $d\tau_1 < 0$, $d\tau_2 > 0$, and $d\tau_1 < 0$, $d\tau_2 < 0$. Which case yields the largest response of n_1?

To predict the effect of the tax cut requires an assumption about future tax rates. It seems that we can make predictions only for policies that include credible announcements of future policy, sometimes called policy rules. We have to think that these rules or policy plans are well-understood by households. This does not mean that all policy changes are of this form, but it does mean it may be hard to predict the effects of changes which aren’t.

Our second mistake illustrates two of the lessons for policy evaluation drawn by Lucas (1976):

• In predicting the effects of changes in government policy we should allow for resulting changes in private behaviour, and

• to predict those changes in private behaviour (and hence the total effect of the policy change) may require well-understood policies.

These lessons sometimes are referred to as the Lucas critique, because they go along with criticism of some more mechanical ways to predict the effects of policy changes. Perhaps the simplest way to convey the critique is to give an example of what not to do. Again suppose we want to measure the effect of the tax cut, and we collect data on labour supply and tax rates. We then run a linear regression of n on τ and use the fitted coefficients with the proposed new tax rate to predict the new n. Our mistake is that the relationship between current n and current τ may depend on expected future τ. If it does (and this is an empirical question) then this econometric relationship may be unstable; it may shift when the policy rule describing current and future taxes changes. In other words, $dn_1/d\tau_1$ depends on expectations of τ_2. We need a behavioural model *and* a policy rule – not a static linear regression – to study that effect.

(e) Application: Asset Pricing

As a second application, we can use our two-period model to predict the prices of various assets. Financial markets seem like a place where competitive general equilibrium theory should apply: there are many participants with good information and transactions costs are low. These markets also are a good place to test theories because the data from them are available readily, for long time spans, and at high frequency. Even if you are not interested in finance *per se* studying these markets may tell us something about attitudes to risk, which would be useful knowledge in answering other economic questions.

To introduce this subject, consider the fact that the average rate of return on stocks (over decades, say) is higher than the average rate of return on bonds. Why do investors still buy bonds? Why do firms issue equity? In each case the answer must be that asset prices reflect varying amounts of risk.

Suppose that the underlying attitudes to risk are based on the expected utility model we have already introduced. Consider a two-period model of asset-pricing. With uncertainty we can now think of rates of interest on several different assets, rather than there being only one interest rate. One of our tasks as macroeconomists is to account for interest rates that differ according to risk, maturity, and country, as well as variation over time in the general level of rates. We shall attempt to do that by finding attitudes to risk from an underlying problem of maximizing expected utility.

We shall use an important, general relationship between prices and rates of return:

$$1 + r = \frac{[\text{payoff} + \text{resale price}]}{\text{price}},$$

where r is a one-period return. In most of our examples in this section, time ends at period 2 and so there is no resale value to the asset.

A key insight is that an asset is defined by its payoffs. If we are told when and what amounts an asset (contract) pays, then we can calculate its price. How can we do this? Our Euler equation tells us something about $(1 + r)$. If we know the payoffs, then we can calculate the price.

This leads to a 'cookbook' method for figuring out predictions for interest rates. There are three steps in the recipe:

- describe completely the payoffs on an asset, for each time and state
- calculate the asset's price using the payoffs and information on consumption, using the Euler equation
- calculate expected returns

Again for simplicity let us assume that there are only two possible states in period 2. To see how the recipe works, consider an asset which costs q_1 in period 1 and pays 1 unit in period 2 no matter which state occurs. This is sometimes called a riskless asset, for obvious reasons. It also is called a zero-coupon bond or a discount bond (because q_1

usually is less than 1). Its rate of return is known in period 1 (because it pays 1 in period 2 in each state) and is

$$r = \frac{1}{q_1} - 1.$$

For this bond the payoff is 1 and the resale price is zero.

Suppose that we are in state 1 in period 1 (so $y_1 = c_1 = 1$), and use the probabilities from part (b). For the riskless asset

$$q_1 u'(c_1) = \mathrm{E}_1 u'(c_2)\beta \cdot 1.$$

Also suppose that $u(c) = \ln(c)$ and $\beta = 0.95$. Then

$$\frac{q_1}{1} = .95 \cdot [\frac{0.6}{1} + \frac{0.4}{2}] = 0.76,$$

$$r = \frac{1}{q_1} - 1 = 0.315$$

using the probabilities from our two-state example in part (b).

This model bond return is not random, yet we observe changes over time in the interest rate on bonds. So, as you might guess, we attribute that to variation in the initial state. Bond prices are just forecasts, so for the bond price to vary, the forecast must vary depending on the initial state. That means that *iid* consumption growth, for example, won't give an interesting pattern in bond returns.

If we begin in the high state, then the bond price is:

$$\frac{q_1}{2} = 0.95[\frac{0.6}{2} + \frac{0.4}{1}] = 1.33,$$

so that the interest rate is -0.248. We get these unrealistic values because we have used integers for consumption, for simplicity. In reality consumption growth rates are not this enormous and so interest rates aren't either.

To calculate the unconditional mean bond price, we use the unconditional probabilities:

$$0.5 \cdot \mathrm{E}[\beta c_1/c_2|y_1 = 1] + 0.5 \cdot \mathrm{E}[\beta c_1/c_2|y_1 = 2].$$

Then please note carefully that the average bond return is *not* the inverse of the average bond price, minus one.

We can use exactly the same method for assets which have more complicated patterns of payoffs. We also may consider assets whose payoffs depend on the state. For example consider an asset with price $\underline{q}_1$ which pays 1 in state 1 and zero in state 2, and an asset with price $\overline{q}_1$ with the opposite payoffs. These are sometimes called contingent claims, because their returns depend on the state which occurs in period 2. They also are referred to as Arrow-Debreu basis securities because their returns span the space of possible outcomes in period 2. It turns out that knowing their prices is sufficient to tell us the prices of more complicated assets, because we can write the latter as combinations of the basis securities. For example, holding the riskless asset is equivalent to holding both the AD assets, so its price is simply the sum of the prices of the AD assets.

The key is to note that our Euler equation links $1+r$ (for any asset) to consumption. The asset returns r_i must all satisfy the Euler equation:

$$u'(c_1) = \mathrm{E}_1 u'(c_2)\beta(1+r_i).$$

But this does not mean that their rates of interest are equal! In fact in the two-period model with only two states we can easily solve for their rates of return and explain the pattern of interest rates across assets, given the current state.

So what are the prices of the AD securities? By the same recipe:

$$\begin{aligned} \underline{q}_1 u'(c_1) &= \mathrm{E}_1 \beta u'(c_2)[\text{payoff}], \\ \underline{q}_1 &= .95 \cdot [1 \cdot 0.6/1 + 0 \cdot 0.4/2] = 0.57. \end{aligned}$$

Likewise,

$$\overline{q}_1 = .95 \cdot [0 \cdot 0.6/1 + 1 \cdot 0.4/2] = 0.19.$$

Notice that $\underline{q}_1 + \overline{q}_1 = q_1$. This is an example of *arbitrage*: if two portfolios give the same payoffs in all states then they have the same price.

We found the bond return, starting in the low state, to be 0.315. What about the returns on the two equities? First consider the stock which pays off if $y_2 = 1$. We know that its price is $\underline{q}_1 = 0.57$. If in fact $y_2 = 1$ then the payoff is 1 and the return is 0.754. But if $y_2 = 2$ then the payoff is zero, and the return is -1.00. Hence the return is random, and we usually study the *expected return*, which here is $0.6(0.754) + 0.4(-1.0) = 0.0524$. What is the expected return on the other asset?

In our calculation of the AD security prices you might suspect that $\underline{q}_1 > \overline{q}_1$ simply because the former asset pays off in the low state, we have assumed we are in state 1 in period 1, and there is some persistence to states. You can think of one asset as a stock in a countercyclical industry (say a firm of bailiffs and auctioneers) and the other as a stock in a procyclical industry, which pays off in booms but not in recessions.

Exercise: Find the asset prices $\underline{q}_1 > \overline{q}_1$ if the economy is in the high state in period 1. Thus find the unconditional means of the prices and rates of return. Does the premium on defensive stocks vary over the business cycle (*i.e.* depending on the initial state)?

In fact, the unconditional mean price of the defensive stock is higher than that of the procyclical stock, even though the payoffs are completely symmetric. Why? The rationale is that holding the former asset is more desirable because it pays off (can be redeemed) in the state in which income is low and hence the marginal utility of an additional unit of consumption is high. Thus it provides some insurance and commands a higher price than the equity which pays off in the good state. This gives a well-defined meaning to risk; it is not simply variance.

The differences between asset prices that result from differences in risk are called risk premiums (or premia). A very useful tool for describing them is the *covariance decomposition*, which is simply a definition. Recall from section 1 (rule 8) that for two random variables, say x and y:

$$E(xy) = E(x)E(y) + cov(x, y).$$

Now imagine that we have a two-period asset, with price q and payoff or dividend in period 2 of d_2, which in general will be random. We know already that with log utility the price satisfies:

$$q = \beta E_1 \frac{c_1}{c_2} d_2.$$

The right-hand side is the expectation of a product, so using the covariance decomposition gives:

$$q = \beta E_1 \frac{c_1}{c_2} E_1(d_2) + \beta cov(\frac{c_1}{c_2}, d_2).$$

It is obvious that assets with higher average payoffs should have higher prices, so let us assume that this asset has an average payoff of one: $E_1(d_2) = 1$. Then the first term is simply the bond price, and

$$q = q_1 + \beta cov(\frac{c_1}{c_2}, d_2).$$

Here the second term is the risk premium. If the asset's payoff is high when c_2 is high then the covariance is negative and the asset has a lower price than the riskless bond. Conversely, if the asset tends to pay off when c_2 is low then it will have a positive risk premium.

This result is called the consumption capital asset pricing model (CCAPM). It suggests that idiosyncratic risk is not priced; in other words, it is covariance between payoffs and marginal utility that matters, not the variance of payoffs. For example, if d_2 were random but unrelated to c_2, the theory predicts that the asset should have the same price as the riskless bond.

The risk premiums also can be expressed in returns, as opposed to prices, and we'll see some examples below and in the exercises. One of the most well-known is the *equity premium*, the difference between average returns on stocks and on bonds. Historically, there has been an equity premium of several percentage points. According to the theory, higher returns (lower prices) on equity are explained by the positive covariance between dividends and consumption. However, when economists directly calculate this covariance using historical time series, they predict a premium much smaller than the observed one. This is the so-called equity premium puzzle.

One way to raise the risk premium predicted by the theory is to increase risk aversion (the examples here use log utility). We'll see examples of this effect in the exercises. Intuitively, if people are more risk averse then equilibrium features higher expected returns on equity to compensate them for bearing risk. But it seems that explaining the scale of historical equity premia requires unrealistically high risk aversion.

As an application of the same CCAPM reasoning, let us next look at the interest rate on a junk bond. The idea here is that this asset pays 1 unit at time 2 in all states, except that the issuer may default with some probability λ, as viewed by the market (of course this could be higher or lower than the actual probability). The event of defaulting is assumed to be uncorrelated with the state of the economy. We know, then, that the risk premium will be zero, because there is no covariance. The only effect of the junkiness is on the expected payoff. The resulting difference in price, relative to the riskless bond, is called a default premium.

To see this, simply calculate the junk bond price in state 1 as

$$q_{J1} = .95 \cdot [0.6/1 + 0.4/2] \cdot [1(1-\lambda) + 0\lambda] = 0.76 \cdot (1-\lambda) = q_1(1-\lambda).$$

Thus the prediction is that the junk bond will have a lower price than the safe bond. The same thinking could be applied to sovereign debt (*i.e.* debt issued by governments) and the price differences could be used to estimate default probabilities, though it may be difficult to identify separately the scale and probability of default. What is the expected return on the junk bond? Is your answer consistent with the covariance model of risk?

Most assets are priced in dollars, not units of consumption. Let us next briefly see how the theory deals with this realistic feature. Consider a one-period nominal bond with price Q_1 in dollars today, and payoff one dollar next year. Let P denote the general price level. The bond's price satisfies:

$$\frac{Q_1}{P_1} = E_1 \beta \frac{c_1}{c_2} \frac{\$1}{P_2}.$$

Thus

$$Q_1 = q_1 E_1 \frac{P_1}{P_2} + \text{cov},$$

or

$$\frac{1}{1+R} = \frac{1}{1+r} E_1 \frac{1}{1+\pi} + \text{cov},$$

which is simply the Fisher relation between a nominal interest rate, a real interest rate, and expected inflation.

Exercise: What happens to the Fisher relationship when interest income is taxed? If real interest rates are to be unaffected by inflation then how much will R rise when π rises?

Our theory also can be used to see what information about the economy might be contained in the prices of assets which differ in maturity, not in payoff risk or default probability. Imagine now that there are three periods, and two assets with the following payoffs:

Period	1	2	3
Cashflow	$-q_1^S$	1	

$$-q_1^L \qquad\qquad\qquad\qquad\qquad\qquad\qquad 1$$

The first asset is our original riskless bond. It costs q_1^S and pays 1 a period later. The second asset costs q_1^L and pays 1 two periods later. We might call these short and long bonds.

These two prices must satisfy:

$$q_1^S u'(c_1) = \mathrm{E}_1 \beta u'(c_2),$$
$$q_1^L u'(c_1) = \mathrm{E}_1 \beta^2 u'(c_3).$$

For simplicity I shall use log utility so that

$$q_1^S = \mathrm{E}_1 \beta \frac{c_1}{c_2},$$
$$q_1^L = \mathrm{E}_1 \beta^2 \frac{c_1}{c_3}.$$

which give prices in terms of risk-adjusted discount factors.

The pattern of interest rates on assets that differ only by maturity, graphed against maturity, is called the term structure (sometimes a curve fitted to them is called the yield curve). Understanding the term structure is crucial to monetary policy and to investment decisions. The gross interest rate on the short bond is simply

$$1 + r_1^S = \frac{1}{q_1^S} = \frac{1}{\mathrm{E}_1 \beta c_1/c_2}.$$

The gross return on the long bond satisfies

$$(1 + r_1^L)^2 = \frac{1}{q_1^L} = \frac{1}{\mathrm{E}_1 \beta^2 c_1/c_3}.$$

where r_1^L is the yield to maturity (or internal rate of return) on the long bond. The term structure $\{r_1^S \ r_1^L\}$ thus embodies forecasts of future consumption growth. In this model it can be used to forecast recessions. To see that result in a simple way, suppose that c_3 is expected to be very low. Then marginal utility at time 3 will be high, the price of a bond which pays off at time 3 will be high and so its interest rate will be low and the term structure will slope down. In practice the yield curve generally slopes up, reflecting growth (as well as inflation, in the case of nominal bonds). An inverted yield curve often forecasts a recession.

Exercise: Show that an upward sloping yield curve means that investors and borrowers expect short-term interest rates to rise.

To describe the connection between long and short bond prices, we again may use the covariance decomposition. The price of the long bond is

$$q_1^L = E_1 \beta^2 \frac{c_1}{c_3}$$
$$= E_1 \beta \frac{c_1}{c_2} \beta \frac{c_2}{c_3}$$
$$= q_1^S E_1 q_2^S + cov(\frac{\beta c_1}{c_2}, \frac{\beta c_2}{c_3})$$

where *cov* refers to the covariance between consumption growth in successive periods.

If we ignore the risk premium (it usually is small in practice) for a moment, then this expression relates the long bond price to current and expected future short bond prices. Suppose that $q_1^S = 0.95$ and you have the expectation $E_1 q_2^s = 0.94$. Then the product is 0.893, which should be approximately the long bond price. The intuition comes from arbitrage: you can assure yourself of a unit payoff in period 3 by buying a long bond, or by rolling over short bonds. The two methods should have similar prices. They do not have identical prices because of the risk premium. For example, buying the long bond exposes you to risk that the price of the short bond in period 2 will be lower than you expected.

We also can express the predictions in terms of interest rates, instead of bond prices. If we ignore the risk premium then we get

$$\left(\frac{1}{1+r_1^L}\right)^2 = \left(\frac{1}{1+r_1^S}\right) E_1 \left(\frac{1}{1+r_2^S}\right).$$

Then if we take logarithms and completely ignore Jensen's inequality we get

$$r_1^l \approx \frac{r_1^S + E_1 r_2^S}{2}.$$

So as an approximation, long bond yields are the arithmetic average of current and expected short bond yields. This is sometimes called the *expectations hypothesis* of the term structure.

Exercise: Our predictions for the prices and interest rates of different assets depend on how risk averse we assume investors are. A utility function which is often used (along with the hypothesis of expected utility maximization) in asset-pricing is:

$$u(c) = \frac{c^{1-\alpha}}{1-\alpha},$$

with $\alpha > 0$ and $u(c) = \ln(c)$ if $\alpha = 1$. Several exercises at the end of this section apply this utility function. How risk averse are you? This amounts to asking: What is your α? Suppose that I offer you a choice: You can have \$10 with certainty, or else take a gamble. In the gamble you win \$100 with probability p and \$1 with probability $1 - p$. Obviously if $p = 0$ you will take the \$10. But suppose p increases. At what value of p will you decide to take the gamble instead? Let us call this switch point p^*. You can show that if $p^* = 0.24$ then $\alpha = 0.5$, if $p^* = 0.5$ then $\alpha = 1$, and if $p^* = 0.90$ then $\alpha = 2$. This thought experiment seems to rule out very large values for α.

Exercise: Show that nominal bond prices may forecast inflation.

We also can use the theory to price *forward* contracts. Again suppose there are three periods. In period 1 you sign a contract, which requires you to pay f_1 in period 2 in exchange for a payoff of 1 in period 3. Notice that we write f_1 because the forward price is agreed to in period 1.

How do you value this contract? The expected utility value of the price is $\mathrm{E}_1 \beta u'(c_2) f_1$. The expected utility value of the payoff is $\mathrm{E}_1 \beta^2 u'(c_3)$. In equilibrium these are equal, so

$$\begin{aligned} f_1 &= \frac{\mathrm{E}_1 \beta^2 u'(c_3)}{\mathrm{E}_1 \beta u'(c_2)} \\ &= \frac{q_1^L}{q_1^S} \end{aligned}$$

This is an arbitrage result. You can guarantee yourself a payoff of 1 in period 3 two different ways. First, you can purchase a long bond, at price q_1^L. Second, you can buy an amount f_1 of short bonds, and sign a forward contract. When the short bonds mature, they are used to pay the forward price in period 2. These two investment strategies have the same price.

Exercise: Show that the expectations hypothesis implies that forward rates are equal to expected future short rates.

Exercise: Find the relation between forward rates and short and long rates. Also, use a covariance decomposition to define the risk premium in forward prices.

In theory, forward prices should be useful predictors of future short (spot) prices. So I should mention that one of the most mysterious findings in international finance is that currencies which are at a forward premium (*i.e.* for which the forward price exceeds the spot price) tend to depreciate.

The same logic we've found in examples with two or three periods applies to assets which payoff (in coupons or dividends) in several periods. For example, suppose an asset has a stream of payoffs $\{d_t\}$. Its price at time t is:

$$q_t = E_t \sum_{i=1}^{\infty} \beta^i \frac{u'(c_{t+i})}{u'(c_t)} d_{t+i},$$

where the theory leads to risk-adjusted discount factors. This now involves a sum of forecasts of products of random variables. Traditionally there has been some focus on the special case when agents are risk-neutral, which gives

$$q_t = E_t \sum_{i=1}^{\infty} \beta^i d_{t+i},$$

a present-discounted value model. To study models like this we need some tools for multi-step forecasts like $E_t d_{t+2}$, $E_t d_{t+3}$, and so on. That is the subject of section 3.

(f) Application: Ricardian Equivalence

So far we've ignored the government, so you may be wondering what effect taxes have on the competitive equilibrium. We can examine this issue (and others) by a simple extension of our two-period economy. Our goals are: (i) to describe how the economy, including the definition of equilibrium, changes when we introduce government spending and taxation; and (ii) to explore the impact of government policy on equilibrium prices and quantities. The latter will include a discussion of the Ricardian equivalence theorem and of the interaction between government policies and private decisions. To keep things simple, we will stick with the exchange version of the two-period economy.

The government does two things: it consumes quantities $\{g_1, g_2\}$ of the good in each period, and it collects lump-sum taxes $\{t_1, t_2\}$ from consumers to finance it. The tax in period t is measured in units of the date t good. Tax and spending decisions are related through the government's budget constraint,

$$p_1 g_1 + p_2 g_2 = p_1 t_1 + p_2 t_2;$$

the present value of taxes equals the present value of government spending (recall that r can be thought of as a commodity own rate of return in this one-good economy). We shall assume, as well, that government spending has no effect either on utility or production (neither u nor f depends on g). This isn't necessary, but it makes the analysis simpler. Likewise, the representative agent pays taxes to the government, and faces the intertemporal budget constraint,

$$p_1 c_1 + p_2 c_2 = p_1(y_1 - t_1) + p_2(y_2 - t_2).$$

The definition of equilibrium changes as follows. The list of commodities is, as it was in section 2, the good today and tomorrow. A competitive equilibrium consists of prices $\{p_1, p_2\}$, private decisions $\{c_1, c_2\}$, and government policies $\{g_1, g_2, t_1, t_2\}$ such that:

∘ the consumer maximises utility, given prices and policies, subject to the budget constraint;

∘ the government's policies satisfy its budget constraint;

∘ supply equals demand for each good:

$$c_1 + g_1 = y_1, c_2 + g_2 = y_2.$$

Consider a simple example: endowments $\{y_1, y_2\}$, utility $\ln c_1 + \beta \ln c_2$, government spending $\{g_1, g_2\}$ is given. Clearly equilibrium quantities are $c_t = y_t - g_t$. The relative price can be computed from the marginal rate of substitution evaluated at these quantities:

$$(1+r)^{-1} = p_2/p_1 = \text{mrs} = [\beta/c_2]/[1/c_1] = \beta(y_1 - g_1)/(y_2 - g_2).$$

Note that neither prices nor quantities depends on the timing of taxes. You may, however, verify that the taxes satisfy the government's budget constraint.

This illustrates a more general result that we can summarize as a theorem:

Theorem (Ricardian Equivalence). If prices $\{p_1, p_2\}$, private decisions $\{c_1, c_2\}$, and government policies $\{g_1, g_2, t_1, t_2\}$ constitute a competitive equilibrium, then so do the same prices, private decisions, and government spending with any new tax policy $\{t_1^*, t_2^*\}$ satisfying

$$p_1 g_1 + p_2 g_2 = p_1 t_1^* + p_2 t_2^*.$$

Proof. We simply show that the new equilibrium satisfies all the requirements of a competitive equilibrium. Note that $p_1 g_1 + p_2 g_2 = \{\bar{g}\}$, a constant that does not vary across the two equilibria since prices and government spending are the same in each. (a) The consumer's budget constraint only depends on the present value of taxes, $p_1 t_1^* + p_2 t_2^*$, not the division between t_1 and t_2. By assumption this sum is the same in both equilibria. The consumer's budget constraint, hence the decisions $\{c_1, c_2\}$, are the same. (b) Clearly the government's budget constraint is not affected by the tax change. (c) Obvious because consumption and government spending are the same.‖

This theorem defines a class of 'equivalent' policies $\{t_1^*, t_2^*\}$: for any policies in this class, allocations and prices are the same. In this sense we can say that government deficits are irrelevant. Implicitly we are saying that debt financing and tax financing are equivalent in this economy. The government's debt after the first period is $b = g_1 - t_1$. The theorem says that any choice of b produces the same equilibrium prices and allocations. If we issue more debt in the first period, it is matched by higher taxes in the second:

$$t_2 = g_2 + (p_1/p_2)b = g_2 + (1+r)b.$$

When the taxes are collected makes no difference to consumption or interest rates (though it does affect saving, as we shall see).

The intuition comes simply from budget constraints (see exercise 1 below). Imagine two employees who are identical except that one has income tax deducted at source while the other pays tax in a lump-sum at the end of the fiscal year. The present values of their tax liabilities are equal. Would you expect their spending patterns to differ?

The general point is not that the timing of taxes does not matter, although that is true in this economy. In other environments the timing of taxes can influence equilibrium prices and quantities by distorting the prices faced by consumers (we ruled this out by positing lump-sum taxation), by changing the allocation of resources across agents (this is irrelevant with a single consumer), or a number of other mechanisms. The points of the theorem are more fundamental. First, in almost all models we use, government policies must obey budget constraints. Second, for policies to affect agent's decisions they must affect the problem the agent solves. We typically presume that preferences are invariant to policies, so to affect the agent's decisions they must alter the constraints the agent faces. In this economy the only constraint is the present value budget constraint, and the theorem states that these are not affected by the timing of government tax receipts. More generally our task is to describe how government policies affect the choice sets faced by private agents, and from this to predict how policies influence the economy's equilibrium.

The Ricardian equivalence theorem has the sound of an 'irrelevance proposition,' that government policy doesn't matter. In fact government spending has a strong affect on the equilibrium of this economy, since it changes the amount of resources available for private consumption. (See the price relation above.) There is thus a sharp contrast between the model's strong predictions for the effects of government spending and it absence of predictions regarding taxes and deficits. The general point here is that 'equivalence,' 'irrelevance,' and 'neutrality' theorems say not that particular policies are uncorrelated with equilibrium prices and quantities, but that theory permits literally any correlation. When you see these theorems you should think: 'theory has little to say about this aspect of the data.' This makes testing such theorems, if that's what you want to do, fairly difficult. A test cannot really be based on a partial correlation between deficits and interest rates, for example.

Notice that under the theorem a deficit-financed tax cut has no effect and that government bonds are not net wealth – they signal deferred taxes and do not affect lifetime budget constraints in the private sector. This contrasts with the view that deficits crowd out investment (through increasing the interest rate) and net exports (through appreciating the exchange rate). This second view of deficits (confusingly) is sometimes called neoclassical. The Ricardian view also contrasts with a Keynesian view, in which resources are not fully employed and deficit finance expands output and employment.

Exercise: Consider the two-period economy with preferences

$$U_t = log c_1 + \beta log c_2,$$

and budget constraint

$$c_1 + c_2/(1+r) = y_1 - t_1 + (y_2 - t_2)/(1+r).$$

First solve for saving. Use the government budget constraint to relate dt_1 to dt_2 and then show that $ds/dt_1 = -1$. One can think of interest rates being determined by total saving (public and private).

Exercise: What is the effect of a change in tax timing in a small open economy? ∘

Let us consider environments in which Ricardian equivalence might not hold (although we shall not model them formally). See Barro's survey for more:

• uncertainty: The example above involves perfect foresight. What if future taxes and incomes are uncertain? Some people have argued that since future taxes on an individual are uncertain, they will be discounted at a rate above r so that net wealth will rise with current deficits. It turns out that this effect can go either way. Suppose that future taxes implied by a current deficit have an uncertain incidence – we know they will be paid but we do not know by whom and cannot insure against this risk. Then a deficit adds to uncertainty about everyone's future disposable income and this may actually reduce current consumption. Conversely, if we know we'll be paying a poll tax in the future then the uncertainty about future disposable income is reduced by a deficit so that current saving will fall.

• finite lives: People don't live forever and don't care about tax liabilities due after death. In the two-period example above there is debt neutrality. More generally, when will a bequest motive give rise to decision rules that look like those from an economy with infinitely-lived agents? Suppose that agents live for one period. We know that if there were two-period lived agents with preferences:

$$U_1 = u_1 + \beta u_2 \quad \text{(subscripts denote time, not derivatives)}$$

then there would be debt neutrality as in the simple example above. The decision rules in an economy with one-period lived agents will be the same as those in the two-period lived agent economy if U_1 is the same, that is if at time 1 agents are concerned about *utility* at time 2, even if this utility is someone else's. This is the idea in Barro (1974).

Other formal models with finite lives feature deficit financing which is non-neutral (*e.g.* Blanchard 1985). Even the OLG model used by Barro with intergenerational transfers may feature too weak a bequest motive to give rise to neutrality. But there is some empirical evidence that intergenerational transfers are large, and it seems likely that they are altruistic. Also people do live a long time.

• One criticism of the equivalence/neutrality results is based on the opposite view of interpersonal economic links. Rather than arguing that these are less pervasive than neutrality results require, Bernheim and Bagwell (1988) argue that they are more pervasive than Barro assumes. For example, humans are linked not only to their descendants but, since they have two parents, to other families as well. Taking this argument to its logical conclusion leads to an apparent refutation of neutrality based on a reductio ad absurdam. If all links between humans were as well-developed as assumed of intergenerational links then not just deficit-financed tax cuts but virtually all changes in policy would have no effects, since they could be offset by a private sector acting virtually as one. For example, if my salary is frozen to prevent an increase in tuition fees, then you could undo the effect of this policy simply be writing me a cheque. Since we do not usually think of intratemporal income distribution in this way, the neutrality theorem may be empirically misleading.

• imperfect capital markets: Suppose a group in the economy has no collateral for loans. Its members can borrow only at rate $r' > r$, say the Mastercard rate as opposed to the Tbill rate. You can show then that consumption will increase as a result of the deficit financing decision. Here the government basically provides intermediation services. Its actions allow consumption loans at rate r to those who would not otherwise have access to them. However, there also are examples of capital market imperfections under which equivalence still holds or under which deficits lower interest rates.

• taxes are not lump-sum: In this case the deficit certainly can be non-neutral, since it affects the timing of taxes and thus affect incentives to work and save in different periods. Again, this affect can go either way. For example, consider our two-period example. Suppose that tax revenue t is raised with an income tax at rate τ_1 in the first period and τ_2 in the second period. Suppose that the government cuts τ_1, issues bonds, and increases τ_2. Then work effort and saving will tend to increase in period one (expenditures are not taxed and income earned in period two is now less valuable) so that after-tax income can be carried over to period two. Here the deficit is non-neutral; but it reduces the interest rate (since bond prices are bid up).

• empirical evidence: There are many tests of the implications of Ricardian equivalence. They generally use statistical models to see whether consumption, interest rates, and current account deficit are related to deficits. Cardia (1997) provides some interesting evidence on tests of Ricardian equivalence using 'artificial economy' methods, by examining the tests within simulation models where we know whether or not equivalence holds.

Ricardian equivalence implies that the time path of the government budget deficit does not matter. This conclusion depends on there being lump-sum taxes, though. In section 6 we'll discuss some theories of tax-setting when the taxes are distortionary.

Further Reading

For a complete treatment of the theory used in general equilibrium economics, begin with Hal Varian's *Microeconomic Analysis* (1992). First study chapters 1, 4, 9, and 11 for the building blocks. The general equilibrium tools are found in chapters 17-19.

Ph.D. students should be familiar with Andreu Mas-Colell, Michael Whinston, and Jerry Green's *Microeconomic Theory* (1995) chapters 6, 15, 16, 19, and 20.

On the Lucas critique, begin with excerpts from two textbooks on rational expectations: Steve Sheffrin's *Rational Expectations* (1983), 100-111 and David Begg's *The Rational Expectations Revolution in Macroeconomics* (1982), 80-86. Next, you should read Lucas's classic (1976) article, "Econometric policy evaluation: a critique," reprinted in his *Studies in Business Cycle Theory.*

On the equity premium puzzle, M.A. students should read Jeremy Siegel and Richard Thaler's "Anomalies: the equity premium puzzle," *Journal of Economic Perspectives*, (Winter 1997), 191-200. Ph.D students should read the original work by Edward Prescott and R. Mehra, "The equity premium: a puzzle," *Journal of Monetary Economics* (1985) 145-161, and also a survey by Narayana Kocherlakota, "The equity premium: it's still a puzzle," *Journal of Economic Literature* (1996) 42-71.

A fine introduction to asset prices and arbitrage is given by Donald Smith in "The apartment story: to introduce the study of spot, forward, and futures markets, the term structure, and arbitrage," *Journal of Economic Education* (1988), 275-286. On using the term structure of interest rates for forecasting, see Campbell Harvey's "The relation between the term structure of interest rates and Canadian economic growth," *Canadian Journal of Economics* (1997), 169-193. On the forward premium anomaly, see Kenneth Froot and Richard Thaler's "Anomalies: foreign exchange," *Journal of Economic Perspectives*, (Summer 1990), 179-192.

A review of Ricardian equivalence is offerred by Robert Barro's "The Ricardian approach to budget deficits," *Journal of Economic Perspectives* (1989).

More advanced readers could see Barro's original reearch in "Are government bonds net wealth?" *Journal of Political Economy* (1974), 1095-1117. Other important contributions have been made by Olivier Blanchard, "Debt, deficits, and finite horizons," *Journal* *of Political Economy* (1985), 223-247, and in Douglas Bernheim and Kyle Bagwell's "Is everything neutral?" *Journal of Political Economy* (1988), 308-338. Also see the method and findings in Emanuela Cardia's "Replicating Ricardian equivalence tests with simulated series," *American Economic Review* (1997), 65-79.

Exercises.

1. Show that the 'sequence budget constraint', plus the definition of s, are equivalent to the 'date-0' budget constraint with relative price $p_2/p_1 = 1/(1+r)$:

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}.$$

This version of the budget constraint holds that the present value of consumption equals the present value of income.

2. Consider a two-person extension of the two-period economy. Agents 1 and 2 consume quantities $\{c_1, c_2\}$ and $\{d_1, d_2\}$, respectively, today and tomorrow. Agent 1 is endowed with λ units of the good in the first period, λg in the second period. Agent 2 gets $1-\lambda$ and $(1-\lambda)g$, resp. Each consumer has utility function $u(x,z) = \ln x + \beta \ln z$.

We define a competitive equilibrium for this economy as a price system $\{p_1, p_2\}$ and an allocation $\{c_1, c_2, d_1, d_2\}$ such that (a) both agents maximize utility subject to their budget constraints and (b) supply equals demand for each good: $c_1 + d_1 = 1$, $c_2 + d_2 = g$.

(a) Solve the optimum problem: choose an allocation $\{c_1, c_2, d_1, d_2\}$ to maximize $\lambda u(c_1, c_2) + (1-\lambda)u(d_1, d_2)$ subject to the feasibility (resource) constraints: $c_1 + d_1 \leq 1$, $c_2 + d_2 \leq g$. What is the implied allocation? How do (one plus) the growth rates of consumption, c_2/c_1 and d_2/d_1, compare across the two consumers?

(b) Compute the implicit price of the second period good relative to the first-period good from the marginal rate of substitution for consumer 1, evaluated at the optimal allocation. Show that the mrs for consumer 2 is the same. What is the rate of interest in this economy?

(c) Show, by verifying each element of the definition, that the allocation (a) and price system (b) constitute a competitive equilibrium.

3. (Difficult) Consider the economy of the previous exercise, but where the endowments of the two consumers are $\{y_1, 0\}$ and $\{0, y_2\}$.

(a) Find the set of Pareto optima by solving the planning problem: maximize $\lambda u(c_1, c_2) + (1-\lambda)u(d_1, d_2)$ subject to the feasibility (resource) constraints: $c_1 + d_1 \leq y_1$, $c_2 + d_2 \leq y_2$, for all $\lambda \epsilon (0,1)$. What are the implied prices for each allocation?

(b) For a given arbitrary λ, substitute the implied prices and allocations into the first consumer's budget constraint. What value of λ balances the value of the consumer's endowment and expenditures?

(c) What method of computing equilibria in multi-agent economies does this suggest?

Remarks. Thus we have two methods of finding an equilibrium. One, we can find agents' demand functions and impose market-clearing conditions to determine prices. Two, we can solve a planning problem, and use budget constraints to find the appropriate welfare weights for a given distribution of resources.

4. Does the technology $f(k) = k^\alpha$, with $0 < \alpha < 1$, satisfy the Inada conditions?

5. Consider the one-person economy, with endowments $(y_1, y_2) = (1, g)$ and linear investment, $f(k) = sk$ for $k > 0$. (This guarantees that the investment activity cannot be run backwards, with tomorrow's good made into today's good.) The parameter s is positive. Suppose that $\delta = 1$.

(a) Does this technology satisfy the Inada conditions?

(b) Compute the competitive equilibrium. What is the equilibrium rate of interest when g is small? Large?

(c) Give a graphical representation based on the production possibilities set.

Answer

This question can be solved as

$$\max \ln(1-k) + \beta \ln(g + sk).$$

When g is small, $r = s - 1$. When g is large, $r = g/\beta - 1$.

6. It is sometimes argued that cross-country differences in savings rates or interest rates reflect different uncertainties about future income. To study this question, consider a two-period exchange economy with many identical, price-taking agents which can be treated as one. The aggregate endowment is y_1 in the first period and y_2 in the second period. There is a single good, and it is nonstorable. Preferences are given by

$$U = u_1 + \beta u_2$$

where for $i = 1, 2$ $u_i = (c_i^{1-\alpha})/1 - \alpha$

(a) For what values of α is the period utility function increasing and concave?

(b) Define a competitive equilibrium allocation and price system.

(c) Solve for a competitive equilibrium.

(d) Now suppose that at period one there is uncertainty about the endowment in period two. For example, suppose that given y_1, $y_2 \sim (\mu_2, \sigma_2^2)$. Will increasing uncertainty about future income (*i.e.* larger σ_2) lead to an increased interest rate?

Answer

(a) $u_i = (c_i^{1-\alpha})/1 - a$

$u' = c^{-\alpha} > 0$ for any α if $c > 0$

$u'' = -\alpha c^{-\alpha-1} < 0$ for $\alpha > 0$

(b) $\{c_1, c_2, r\}$: markets clear and agents optimize subject to their budget constraint.

(c) $c_1 = y_1$; $c_2 = y_2$; $1 + r = p_1/p_2 = c_1^{-\alpha}/\beta c_2^{-\alpha}$

(d) With uncertainty: $1 + r = p_1/p_2 = c_1^{-\alpha}/\beta \mathrm{E}_1 c_2^{-\alpha}$. Call marginal utility $f(c) = u'(c) = c^{-\alpha}$. Then $f' < 0$ and $f'' > 0$, so this function opens up. So more variability or uncertainty raises the expectation which raises the denominator. So r falls with the income uncertainty or risk.

7. This question derives the consumption function from a simple optimization problem, in a two-period model. The two periods are denoted 1 and 2. Suppose that a typical consumer solves the following problem:

$$\max_{c_1,c_2} U(c_1, c_2) = Kc_1 - (c_1 - \bar{c})^2 + \mathrm{E}_1\beta[Kc_2 - (c_2 - \bar{c})^2]$$

subject to

$$c_1(1 + r) + \mathrm{E}_1 c_2 = y_1(1 + r) + \mathrm{E}_1 y_2$$

where r is the real interest rate (taken as given by the consumer), K is a large constant, $\bar{c}$ is a constant, and y is exogenous to the consumer. $\mathrm{E}_1 y_2$ is the expected value as of time 1 of income at time 2. Suppose that $\beta = 1/(1 + r)$.

(a) Find the Euler equation linking consumptions intertemporally.

(b) Use your answer to part (a) in the budget constraint, and hence solve for the consumption function, expressing c_1 in terms of current and expected future income.

(c) Will an increase in the variance of second period income affect first period consumption?

Answer

Note that we are not told anything about the technology, so here we simply solve the consumer's problem.

(a) $c_1 = \mathrm{E}_1 c_2$ if $\beta = 1/(1 + r)$. or $K - 2c_1 + 2\bar{c} = \mathrm{E}_1\beta(1 + r)(K - 2c_2 + 2\bar{c})$

(b) $c_1(1+r)+c_1 = y_1(1+r)+\mathrm{E}_1 y_2$ so that can solve for c_1. One gets a longer answer without the approximation. Do not use any properties of a complete competitive equilibrium; take y as given here. The point of the question is to try to account for observed consumption smoothness over business cycles.

(c) No. This is the certainty equivalence result, and we shall see it in more detail in section 6. Notice that utility is affected, though.

8. With quadratic utility, consumption and savings do not depend on income risk. With other utility functions that is not so, although it takes some work to show the effects. In this question we shall try to do that, for one example, lest these effects be important empirically.

Consider a two-period model of representative household consumption spending:

$$\max \mathrm{E}U = \ln(c_1) + \beta \mathrm{E}_1 \ln(c_2)$$

subject to

$$c_1 + \mathrm{E}_1 c_2/(1+r) = y_1 + \mathrm{E}_1 y_2/(1+r),$$

in standard notation.

(a) Find the Euler equation linking c_1, c_2, and r.

(b) Suppose that $y_1 = 3$ and that $y_2 = 2$ with probability 0.5 and $y_2 = 4$ with probability 0.5 (these are the conditional probabilities). Also suppose that a storage technology determines that $r = g$ and that, coincidentally, $\beta = 1$ and $g = 0$. Solve for c_1, perhaps by experimenting using a hand calculator, to two decimal places.

(c) Consider another economy which is identical except that y_2 has a larger conditional variance. In which economy will the savings rate be higher?

(d) Do you see any problems with explaining cross-country differences in savings rates by income variabilities?

Answer

(a) $1/c_1 = \mathrm{E}_1 \beta (1+r)/c_2$

(b) $1/c_1 = 0.5 \cdot [1/(4 + (3 - c_1))] + 0.5 \cdot [1/(2 + (3 - c_1))]$ and $2/c_1 = 1/(4 + (3 - c_1)) + 1/(2 + (3 - c_1))$ This gives 2.85, to two decimal places. This is difficult, and few arrive at it the first time.

(c) More income risk will increase savings (I know this from CRRA but it is easy to show by increasing the variance of y_2 in the numerical example of part (b)) so c_1 will fall. Thus this second economy will have a higher savings rate. Note from the preamble to part (b) that this is not an endowment economy and that r is fixed by the technology and so does not change here.

(d) Among other problems (such as parameters which vary across countries, or wealth effects from non-homotheticity of utility), if there is a competitive equilibrium with trade then the two countries perhaps can pool the income risk. Also there may be wealth effects with alternative, non-homothetic preferences. Also parameters may differ across countries.

The four parts to this question are of increasing difficulty.

9. Over long time periods the real return on equities (stocks) seems to be higher than the real return on bonds, presumably to compensate investors for relatively greater risk. Let us see whether the simplest, consumption-based, asset-pricing model can account for that equity premium. Consider a two-period, endowment economy in which $c_1 = y_1$ and $c_2 = y_2$. The representative agent has preferences described by the utility function:

$$U = \ln c_1 + \mathrm{E}_1 \beta \ln c_2.$$

Let $\beta = 0.85$.

(a) State the Euler equation linking consumption with any rate of return.

(b) Suppose that y can take on one of two values, 0.5 (state L) and 1.5 (state H) in each period. The joint probabilities are

Outcome:	$(0.5, 0.5)$	$(0.5, 1.5)$	$(1.5, 0.5)$	$(1.5, 1.5)$
Prob:	0.4	0.1	0.1	0.4

Find the conditional probabilities of income's being in each state at time 2 given each state at time 1.

(c) Denote by q the price of a one-period bond which pays 1 unit of income in period 2 in exchange for q units in period 1. Thus its return is $r = (1/q) - 1$. Using your answers to parts (a) and (b) solve for q in each state at time 1 and hence for the average value.

(d) Now consider an equity which costs h_L in period 1 in state L and h_H in period 1 in state H. Next period it pays y_2. Find its price in each state and hence its average price.

(e) Does the model produce a positive equity premium *i.e.* a higher average return on equity than on bonds? Which asset price has higher variance?

Answer

(a) $1/c_1 = \mathrm{E}_1\beta(1+r)/c_2$ so $1/y_1 = \mathrm{E}_1.85(1+r)/y_2$

(b) In each case .80 that stay and .20 that go.

(c) $q_L = \mathrm{E}_L.85 \cdot y_1/y_2 = 0.85(0.5)[.8/.5 + .2/1.5] = 0.737$
$q_H = \mathrm{E}_H.85y_1/y_2 = 0.85(1.5)[.8/1.5 + .2/.5] = 1.19$ Thus $\overline{q} = 0.9635$. Note that $\overline{r} \neq (1/\overline{q}) - 1$.

(d) Here the payoff is simply y_2:

$$h_L = 0.85\mathrm{E}_L(y_1/y_2) \cdot y_2 = 0.85 \cdot .5 = .425$$

$$h_H = 0.85 \cdot 1.5 = 1.275$$

Thus $\overline{h} = 0.85$. What is the average return on equity?

(e) There is a premium. The stock price is much more volatile, obviously.

10. It often is argued that asset returns are related to business cycles, despite the bromide that the stock market has forecasted ten of the last three recessions. In this question we shall study how real returns on stocks and bonds might be related to persistence in the state of the economy.

Consider a two-period, endowment economy in which $c_1 = y_1$ and $c_2 = y_2$. The representative agent has preferences described by the utility function:

$$U = c_1^{1-\alpha}/(1-\alpha) + \mathrm{E}_1\beta c_2^{1-\alpha}/(1-\alpha)$$

Let $\beta = 0.90$ and suppose that $\alpha > 0$.

(a) State the Euler equation linking consumption with any rate of return.

(b) Suppose that y can take on one of two values, 0.8 (state L) and 1.2 (state H) in each period. The joint probabilities are

Outcome:	$(0.8, 0.8)$	$(0.8, 1.2)$	$(1.2, 0.8)$	$(1.2, 1.2)$
Prob:	λ	$.5 - \lambda$	$.5 - \lambda$	λ

where $\lambda \in (0, 0.5)$. Find the conditional probabilities of income's being in each state at time 2 given each state at time 1.

(c) Denote by q the price of a one-period bond which pays 1 unit of income in period 2 in exchange for q units in period 1. Thus its return is is $r = (1/q) - 1$. From now on assume that $\alpha = 1$ and suppose that $\lambda = .4$. Using your answers to parts (a) and (b) solve for q in each state at time 1 and hence for the average bond price.

(d) Now suppose that $\lambda = 0.25$, rather than 0.4. Again find the average bond price.

(e) What can one conclude, from these two experiments, about the effects of the persistence in business cycles on the average level and the variability of the interest rate on bonds?

Answer

(a) $c_1^{-\alpha} = \mathrm{E}_1\beta(1+r)c_2^{-\alpha}$ so $y_1^{-\alpha} = \mathrm{E}_1.9(1+r)y_2^{-\alpha}$. Note that $(1+r)$ must be inside the E operator!

(b) In each case 2λ that stay and $2(.5-\lambda)$ that go. So iid if $\lambda = 0.25$.

(c) With $\alpha = 1$:

$$q_L = \mathrm{E}_L.90 \cdot y_1/y_2 = 0.90(0.8)[2l/.8 + 2(.5-l)/1.2] = .84$$

$$q_H = \mathrm{E}_H.90 \cdot y_1/y_2 = 0.90(1.2)[2l/1.2 + 2(.5-l)/.8] = .99$$

Thus if $\lambda = 0.4$ we find $q_L = .84$ and $q_H = .99$. Thus the values for the interest rate are .19 and .01. The average bond price is .915.

(d) If $\lambda = 0.25$ we find $q_L = .75$ and $q_H = 1.125$. Thus the values for the interest rate are .34 and $-.11$. The average bond price is .9375.

(e) More persistence lowers the average interest rate and lowers the variance of the interest rate too. Notice that the average interest rate in each case cannot be found from the average bond price.

11. This question studies the predictions of a simple, economic theory for the some properties of international macroeconomic fluctuations. Consider a two-period, world, one-good, endowment economy, with two countries, denoted A and B. Preferences are given by:

$$U = \ln(c_{1A}) + \beta \mathrm{E}_1 \ln(c_{2A}),$$

in country A and similarly in country B. Endowments are distributed independently and identically over time. At any time,

$$y_A = .5 + \frac{\eta}{2} - \epsilon$$

$$y_B = .5 + \frac{\eta}{2} + \epsilon$$

where η is a mean-zero, random shock common to both countries incomes, and ϵ is another mean-zero shock which has opposite effects in the two countries. The shocks ϵ and η have no covariance.

(a) Solve for a competitive equilibrium by forming an equal-weight planning problem.

(b) Find the correlation between c_A and c_B.

(c) Find the correlation between the trade balance (the excess of output over domestic consumption) and output *i.e.* find out whether the trade balance is predicted to be procyclical or countercyclical.

(d) How well do the predictions match up with the historical evidence?

Answer

(a) A planning problem, using world constraints to substitute:

$$\max \ln(c_{1A}) + \beta \mathrm{E}_1 \ln(c_{2A}) + \ln(1+\eta - c_{1A}) + \mathrm{E}_1 \beta \ln(1+\eta - c_{2A})$$

gives

$$\frac{1}{c_{1A}} - \frac{1}{1+\eta - c_{1A}} = 0$$
$$\beta \mathrm{E}_1 \frac{1}{c_{2A}} - \beta \mathrm{E}_1 \frac{1}{1+\eta - c_{2A}} = 0$$

The plan turns out to be (though actually this is difficult to show):

$$c_{1A} = .5 + \frac{\eta}{2} \quad c_{1B} = .5 + \frac{\eta}{2}$$

$$\mathrm{E}_1 c_{2A} = .5 \quad \mathrm{E}_1 c_{2B} = .5.$$

To complete the C.E. we need relative prices of the good in the two time periods. Here $p_2/p_1 = \beta c_1 \mathrm{E}_1(1/c_2)$ in either country.

The idea is to see what happens when, realistically, there are world-wide shocks (η) but also country-specific shocks (ϵ).

(b) It is easy to see that $\mathrm{corr}(c_A, c_B) = 1$.

(c) The trade balance is $y - c$. So in country A, for example (because the countries are identical),

$$y_A - c_A = (.5 + \frac{\eta}{2} - \epsilon) - (.5 + \frac{\eta}{2}) = -\epsilon.$$

Then $\mathrm{corr}(y_A - c_A, y_A) = \mathrm{corr}(-\epsilon, .5 + \eta/2 - \epsilon) > 0$.

(d) Both predictions are at odds with the facts. The consumption correlation is below one and the trade balance is countercyclical in many countries.

12. Suppose that nominal interest rates and prices of discount bonds are based on:

$$(1+R_t^S)^{-1} = Q_t^S = \mathrm{E}_t \beta \cdot \frac{c_t P_t}{c_{t+1} P_{t+1}}$$
$$(1+R_t^L)^{-2} = Q_t^L = \mathrm{E}_t \beta^2 \cdot \frac{c_t P_t}{c_{t+2} P_{t+2}},$$

where c is real consumption, P is the general price level, R is a nominal yield, and Q is a bond price. The superscripts denote maturity. Thus Q_t^S is the price of a (short) bond which pays 1 dollar in period $t+1$ and Q_t^L is the price of a (long) bond which pays 1 dollar in period $t+2$.

Suppose that $\mathrm{E}_t(c_t/c_{t+i}) = 1$, for all i. Suppose that P_{t-1}/P_t can take on two values: 0.96 and 0.90, each with probability 0.5. Then suppose that, given a value of P_{t-1}/P_t, then P_t/P_{t+1} can take the same value with probability .8 or can switch to the other value with probability .2. The same conditional probabilities govern the evolution of P_{t+1}/P_{t+2}.

(a) If $\beta = 0.98$, and $P_{t-1}/P_t = 0.96$, solve for the short and long nominal interest rates.

(b) Does the yield curve of nominal interest rates slope up or down?

(c) What is $\mathrm{E}_t r_{t+1}^S$, the expected future short-term interest rate?

Answer

(a) $Q_t^S = \mathrm{E}_t\beta \cdot P_t/P_{t+1} = 0.98 \cdot [0.8 \cdot .96 + .2 \cdot .90] = .92904$

Next, note that the expression involves P_t/P_{t+2} which is $(P_t/P_{t+1}) \cdot (P_{t+1}/P_{t+2})$. Thus tracking the four possibilities would give:

Event	Probability
$.96 \cdot .96$	$.8 \cdot .8$
$.96 \cdot .90$	$.8 \cdot .2$
$.90 \cdot .90$	$.2 \cdot .8$
$.90 \cdot .96$	$.2 \cdot .2$

$Q_t^L = \mathrm{E}_t\beta^2 P_t/P_{t+2} = .98^2[0.64 \cdot .9216 + .16 \cdot .864 + .16 \cdot .81 + .04 \cdot .864] = .98^2 \cdot .892224 =$ 0.85689 where the probabilities are those of the four combinations of events that can occur. Thus $R_t^S = 1/.92904 - 1 = 0.07638 = 7.6\%$. And $R_t^L = 1/\sqrt{.85689} - 1 = 0.08 = 8\%$

(b) The yield curve slopes up. But this has nothing to do with growth or recession; the real side is constant in this example. Here the slope is caused entirely by inflation expectations.

(c) $\mathrm{E}_t r_{t+1}^S = ?$

$$Q_{t+1}^S = \mathrm{E}_{t+1}\beta \cdot \frac{P_{t+1}}{P_{t+2}}$$

If $P_t/P_{t+1} = 0.96$ then $Q_{t+1}^S = .92904$, just as in part (a). If $P_t/P_{t+1} = 0.90$ then $Q_{t+1}^S = .98[.8 \cdot .9 + .2 \cdot .96] = .89376$. Thus,

$$\mathrm{E}_t R_{t+1}^S = \mathrm{E}_t[\frac{1}{Q_{t+1}^S} - 1] = .8[1/.92904 - 1] + .2[1/.89376 - 1] = .0844 = 8.44\%$$

Notice that the long rate is approximately the average of current and expected future short rates.

13. It is often suggested that the behaviour of the stock market can forecast changes in output and income. To see how this might work, suppose that output can take on two values: 2 in a recession (denoted state r) and 4 with strong growth (denoted g). Consider a simple, two-period model in which the economy is currently in recession. Thus $y_1 = 2$ and $y_2 = 2$ with probability 0.7 and $y_2 = 4$ with probability 0.3 *i.e.* these are the conditional probabilities.

Suppose that asset prices satisfy the usual Euler equation, that there is log utility, and that $\beta = .99$. Also consider an endowment economy so that $c_1 = y_1$ and $c_2 = y_2$.

(a) Consider the price of an asset which pays out a dividend of 1 in state g in period 2 and 0 in state r. Find its price, given that we are currently in state r.

(b) Now suppose that the unconditional probabilities of the two states are 0.5 and 0.5. Find the unconditional mean of the stock price.

Answer

(a) Currently in state 1 so $q_1/y_1 = \beta[.3 \cdot 1/4]$ so $q_1 = .1485$.

(b) Now we can show that the conditional probabilities are symmetric (because the unconditional ones are) or that joint probs are .35, .15, .15, .35. So $q_1(g) = y_1(g)\beta[.7 \cdot 1/4] = .693$, so the average price is .42075.

14. It often is argued that a 'productivity slowdown' occurred in North America in the mid 1970s. This question studies the predicted effects on saving and interest rates of an exogenous slowdown in productivity, using the two-period model. Suppose that there are two time periods and that setting aside k units of the one good in period 1 allows one to consume an extra $f(k) + (1 - \delta)k$ units in period 2. We shall assume that $\delta = 1$ and

$$f(k) = \theta_2 k^\alpha,$$

$\alpha \in (0, 1)$, where the depreciation rate is thus 100% (this makes the algebra simpler) and the production function now includes an exogenous, random term θ_2 that affects the productivity of capital (*i.e.* the ratio of output to input). Suppose that the first-period endowment is 1 and the second-period endowment is zero.

Suppose that preferences are described by: $U = \ln(c_1) + \beta \ln(c_2)$.

(a) Define a competitive equilibrium.

(b) Solve for a competitive equilibrium.

(c) Consider two, two-period economies which are identical in all respects except that θ_2 is higher in one than in the other. Which economy will have a higher interest rate? Which economy will have a higher savings rate?

Answer

(a) $\{c_1, c_2, k\}$ and $\{p_1/p_2\}$ such that: households maximize utility taking prices as given; firms maximize profits given prices and the production function; markets clear so that supply equals demand for each good.

(b) Substitute the constraints in the objective function: max

$$\ln(1-k) + \beta \ln(\theta_2 k^\alpha),$$

because $\delta = 1$ and $y_2 = 0$. FOC:

$$\frac{-1}{1-k} + \frac{\beta\theta_2\alpha k^{\alpha-1}}{\theta_2 k^\alpha} = 0.$$

Thus

$$k = \frac{\beta\alpha}{1+\beta\alpha}$$
$$c_1 = 1-k = \frac{1}{1+\beta\alpha}$$
$$c_2 = \theta_2 k^\alpha = \theta_2 [\frac{\beta\alpha}{1+\beta\alpha}]^\alpha$$

Because $\delta = 1$, $f'(k) - 1 = r$. So

$$\alpha\theta_2 \cdot (\frac{\beta\alpha}{1+\beta\alpha})^{\alpha-1} - 1 = r = \frac{p_1}{p_2} - 1$$

Or, equivalently

$$\frac{p_1}{p_2} = \frac{c_2}{\beta c_1} = \theta_2 \alpha^\alpha (\frac{1+\alpha\beta}{\beta})^{1-\alpha}$$

Notice that substituting the constraints in the objective function, just as in the notes, is much the simplest method.

(c) Notice that the economy with higher θ_2 will have a higher interest rate but that the savings rate will be unaffected. This suggests that exogenous productivity changes (at least the way we have thought of them here and with log utility) cannot explain variation over time in savings rates. So there must be another explanation for changes in North American savings rates, at least in this simple framework.

15. Straightforward economic theory suggests that asset prices or returns are related to consumption growth, as we have seen in two-period model economies. Sometimes it is argued that this relationship does not hold empirically because asset prices can be very volatile while consumption growth in OECD-type economies typically is quite stable. In this question we shall see whether the basic theory is consistent with those facts.

Consider a two-period economy with a single, nonstorable good with endowments y_1 and y_2. In equilibrium $c_1 = y_1$ and $c_2 = y_2$. Suppose that lifetime, expected utility is given by

$$\mathrm{E}U = \frac{c_1^{1-\gamma}}{1-\gamma} + \beta \mathrm{E} \frac{c_2^{1-\gamma}}{1-\gamma}$$

with $\gamma > 0$.

(a) Write the Euler equation linking asset returns to the expected, intertemporal, marginal rate of substitution between c_1 and c_2.

(b) Consider the price q_1 of a riskless asset such as a bond, which pays 1 in any state in period 2. To make things simple, suppose that c_1 and c_2 can take on only two values, 1 and 2, each with probability 0.5 and that c_1 and c_2 are independently distributed. Suppose that $\beta = 1$. Also assume that $\gamma = 2$. Find the variance of the interest rate on the bond.

(c) Now suppose that $\gamma = 4$. Again find the variance of the interest rate. Is it larger than the variance of the gross consumption growth rate (c_2/c_1)?

Answer

(a) $u'(c) = c^{-\gamma}$, so $c_1^{-\gamma} = \mathrm{E}_1 \beta (1+r) c_2^{-\gamma}$. This form is general because one has left $1+r$ inside the E, to allow for cases in which the return is uncertain until period 2 arrives *i.e.* cases other than bonds. Also note that $\mathrm{E}_1 c_2^{-\gamma} \neq 1/\mathrm{E}_1 c_2^{\gamma}$, and $[\mathrm{E}_1 c_2^{-\gamma}]^{-1/\gamma} \neq \mathrm{E}_1 c_2$

(b) For a one-period riskless bond:

$$q_1 = \beta \mathrm{E}_1 (\frac{c_2}{c_1})^{-\gamma}$$

Suppose that $c_1 = 1$. Then

$$q_1 = \beta \cdot [.5(1)^{-2} + .5(2)^{-2}] = 0.625$$

Suppose that $c_1 = 2$. Then

$$q_1 = \beta 4 \cdot [.5(1)^{-2} + .5(2)^{-2}] = 2.5$$

Thus $r = 1/q - 1 = \{0.6, \, -0.6\}$. The mean of the interest rate is thus zero because the two states are equally likely. Thus its variance is just the average squared value which is

$$var(r) = 0.5(0.6)^2 + 0.5(-0.6)^2 = 0.36.$$

Make sure you know the definition of a variance!

(c) Now suppose that $\gamma = 4$. If $c_1 = 1$ then

$$q_1 = \beta [0.5(1)^{-4} + 0.5(2)^{-4}] = 0.53125$$

If $c_1 = 2$ then

$$q_1 = 2^4 \cdot [0.53125] = 8.5$$

Thus $r = 1/q - 1 = \{.882, -0.882\}$. The mean is 0. The variance is .77.

While the numbers used in the computational example are unrealistic, the results show that a given variance of consumption can be associated with a much larger variance of interest rates, if γ is large, which means that risk aversion is large or intertemporal substitution small. Notice that the variance of the gross consumption growth rate (*i.e.* 1 plus the growth rate) is given as follows: It can take on 4 values: 1 w.p. .5, 2 w.p. .25 and .5 w.p. .25. Thus

$$\begin{aligned} \mathrm{E}(c_2/c_1) &= 1.125 \\ var(c_2/c_1) &= 0.5(1-1.125)^2 + 0.25(2-1.125)^2 + 0.25(.5-1.125)^2 = 0.296 \end{aligned}$$

Thus the fact that the variance of interest rates is much larger than the variance of consumption growth is certainly consistent with straightforward theory.

16. Typically the interest rate charged by banks which lend to Latin American governments is quoted relative to some other rate such as the U.S. prime rate or LIBOR (the London interbank offer rate). In this question we shall study how the two rates might be related, using a two-period model. Suppose that all interest rates satisfy the following equation:

$$c_1^{-1} = .89 \cdot \mathrm{E}_1(1+r)c_2^{-1},$$

where c is aggregate consumption in a lending country and r is any return. Suppose that c_t is independently and identically distributed and can take on two values: 1.1 and .9, each with probability 0.5.

(a) First let us solve for the 'safe' return (say, the U.S. prime rate). Find the probability density function (*i.e.* a value in each state in period 1) for the return on an asset which pays one unit in period 2 in all states.

(b) Now let us study the market interest rate on a sovereign debt issue. Suppose that this asset pays one unit in period 2 if $c_2 = 1.1$. But if $c_2 = .9$ (say, there is a world-wide recession) then with probability .25 the country is unable to meet its repayment obligation and instead will pay .5 of a unit; while with probability .75 it will continue to pay one unit even in a recession. Solve for the probability density function of the interest rate on this debt.

(c) Thus find the mean interest premium (*i.e.* the mean difference between the two rates).

(d) More generally, is the Euler equation given above a good guide to understanding asset prices empirically?

Answer

(a) Note that $q_1/c_1 = \mathrm{E}_1\beta/c_2$. Thus in state H:

$$q_1 = 1.1 \cdot .89(\frac{.5}{1.1} + \frac{.5}{.9}) = .9888888$$

In state L:

$$q_1 = .9 \cdot .89(\frac{.5}{1.1} + \frac{.5}{.9}) = .80909.$$

Thus $r = .011$ w.p. .5 and $r = .23$ w.p. .5. This is the density *i.e.* a list of what can happen and with what probability.

(b) Now in state H

$$q_1 = 1.1 \cdot .89[\frac{.5}{1.1} + \frac{.5 \cdot (.25 \cdot .5 + .75 \cdot 1)}{.9}] = .92090$$

In state L:

$$q_1 = .7534$$

The expected payoff is 0.9375, so in state H the expected return is 0.018 and in state L it is 0.244.

(c) Thus the mean expected premium is 0.014 or 1.4%. Notice also that the premium is higher if we are in a recession. Slightly different answers are possible depending on when one rounds off the numbers.

(d) Various examples can be invoked: LDC debt on secondary markets, the time series evidence on consumption growth and interest rates, the term structure of interest rates (which would require non-iid consumption growth), the premium on equities as opposed to discount bonds.

17. One challenge for consumption-based theories of asset-pricing is to explain that equities (stocks) tend to have higher returns than bonds. This question links each type of return to aggregate, real activity. Consider a two-period, competitive, endowment economy with a representative consumer whose Euler equation is:

$$\frac{1}{c_1} = E_1 \beta (1 + r_i) \frac{1}{c_2},$$

where r_i is the return on any asset. Let $\beta = 0.9$. Suppose that $c_1 = 1$ and let $c_2 = 1.1$ with probability 0.5 and 0.9 with probability 0.5.

(a) Find the price q and the return r on a riskless bond which pays 1 in each state in period 2.

(b) Now find the price $\bar{q}$ and the expected return $\bar{r}$ on an equity which pays c_2 (*i.e.* here it pays 1.1 when $c_2 = 1.1$ and it pays 0.9 when $c_2 = 0.9$), so that it is a claim to the aggregate consumption stream.

(c) Two empirically unrealistic features of our results so far are the low equity premium $\bar{r} - r$ and the high riskfree rate r. Show the effect on both of a larger variance of c_2. (Hint: Do not change the probabilities, but raise the large value and lower the small value, leaving the mean unchanged.)

Answer

(a) $q = .9 \cdot [.5/1.1 + .5/.9] = .90909$ so that $r = .10$.

(b) Clearly $\bar{q} = 0.9$ so the return is either 0.222 or 0, and the expected return is $\bar{r} = .111$.

(c) This increase in variance does lower r (because of the precautionary saving motive) and hence raises the premium.

18. This question uses some simple economic theory to describe patterns in the returns on sovereign debt. Suppose that there are two time periods. Asset returns r satisfy

$$\frac{1}{c_1} = E_1 \beta (1+r) \frac{1}{c_2}$$

where $\beta = 0.9$.

Imagine a government which issues one-period, real bonds for price q_1 in period one. The bonds are promises to pay one unit of consumption in period two. Consumption can take on values 1 and 2 in each time period. Given the value of consumption in period one, consumption in period two takes the same value with probability 0.8 and switches to the other value with probability 0.2. Moreover, investors believe that when $c_2 = 1$ the borrowing government will default (payoff zero) with probability λ. An economist argues that the default probability should affect the average interest rate but not the variability in the interest rate over time. We shall study that prediction.

(a) Solve for the debt price, when $c_1 = 1$ and $\lambda = .5$.

(b) Solve for the debt price, when $c_1 = 2$ and $\lambda = .5$.

(c) The two states are equally likely unconditionally. Solve for the mean and variance of the expected return on sovereign debt.

(d) How does a higher default probability λ affect the mean and variance of the return?

Answer

(a)

$$q_1(1) = .9(1)[.8(1-\lambda)/1 + .2/2] = .9[.9 - .8\lambda]$$

so when $\lambda = .5$, $q_1(1) = .45$.

(b)

$$q_1(2) = .9(2)[.8/2 + .2(1-\lambda)/1] = 1.8[.6 - .2\lambda]$$

so when $\lambda = .5$, $q_1(2) = .9$.

(c) Remember not to use the mean bond price. In state 2, the expected return is zero, since the expected payoff is 0.9. In state 1 the expected return is 0.3333.

(d) Suppose that $\lambda = .6$. Repeat the steps above.

19. When we study business cycles we need to consider production economies in which both capital and labour are supplied elastically. This question sets the groundwork. Consider a two-period economy with no uncertainty. A representative agent has lifetime utility function:

$$U = \ln c_1 + \ln(1-n) + \beta \ln c_2$$

where n is labour supply in the first period. The endowments of the one good are y_1 in period 1 and 0 in period 2. There is 100 percent depreciation ($\delta = 1$). The production function produces output in period 2 from capital and labour in period 1 according to the production function $f(k, n) = k^{\alpha} n^{1-\alpha}$.

(a) Define a competitive equilibrium. (Hint: Do not forget to include the real wage.)

(b) Solve for a competitive equilibrium.

Answer

(a) A competitive equilibrium is an allocation c_1, c_2, k, n and a price system p_1, p_2, w such that

▷ households choose c_1, k, c_2, and n to maximize utility taking wages and prices as given.

▷ Firms choose k and n to maximize profits, taking wages and prices as given.

▷ Goods and factor markets clear so that:

$$n \leq 1$$

$$c_1 + k = y_1$$

$$c_2 = f(k, n) = k^{\alpha} n^{1-\alpha}$$

(b) Find the allocation by substitution of constraints in U. This gives:

$$k = \frac{\beta\alpha}{1+\beta\alpha} y_1$$

$$n = \frac{(1-\alpha)\beta}{1+(1-\alpha)\beta}$$

Then these results can be substituted in:

$$c_1 = y_1 - k$$

$$c_2 = k^{\alpha} n^{1-\alpha}$$

Then the two relative prices are given by:

$$\frac{1}{c_1} = \frac{\beta\, p_1}{p_2\, c_2}$$

$$w = k^{\alpha}(1-\alpha)n^{-\alpha}$$

20. This question uses a model of two countries with two time periods to make some predictions about international correlations in income and consumption. Suppose that there is one good. Each country receives a non-storable endowment of it in each time period. The first country has endowments y_1, y_2 in the two time periods, and a typical consumer there seeks to maximize

$$U = \ln(c_1) + \beta \ln(c_2),$$

where c denotes consumption. The other country receives endowments x_1, x_2. A typical consumer there seeks to maximize

$$W = \ln(d_1) + \beta \ln(d_2),$$

where d is consumption. We shall suppose that we observe a competitive equilibrium.

(a) Find a competitive equilibrium by solving the following Pareto optimum problem:

$$\max \ \ \lambda \cdot U + (1-\lambda) \cdot W$$

subject to resource constraints. Solve for consumptions and relative prices in terms of endowments and λ.

(b) Suppose $y_1 = 2$, $y_2 = 1$, $x_1 = 1$, and $x_2 = 2$. Let $\beta = 0.9$. Find the value of λ such that the Pareto optimum satisfies all the conditions of a competitive equilibrium.

(c) With the same values, solve for the trade balance in both countries and time periods.

(d) Show that there is intertemporal external balance.

(e) Are any qualitative predictions of this model inconsistent with empirical evidence?

Answer

(a) $c_1 = \lambda(y_1 + x_1)$, $c_2 = \lambda(y_2 + x_2)$. $d_1 = (1-\lambda)(y_1 + x_1)$, $d_2 = (1-\lambda)(y_2 + x_2)$.

$$p_1/p_2 = 1 + r = \frac{y_2 + x_2}{\beta(y_1 + x_1)}$$

(b) To satisfy the budget constraints, $\lambda = 0.5087$.

(c) In the first country $s_1 = y_1 - c_1 = .4739$, and $s_2 = y_2 - c_2 = -.5261$. Exactly the opposite in the other country.

(d) World endowments are constant so $1 + r = 1/.9$. Balance requires that

$$s_1 + \frac{s_2}{1+r} = 0$$

which holds with these values.

(e) The predictions of a procyclical trade balance and perfect consumption correlations are not supported empirically.

21. One of the central predictions of financial economics is that assets with riskier payoffs should have higher returns. To study this idea simply, take a two-period, competitive exchange economy with a representative agent who has non-storable endowment y in period 1, and faces a distribution of risky income for period 2 given by: State 1: $y + D$ with probability 0.5; State 2: $y - D$ with probability 0.5. Assume the agent has preferences

$$u = \ln c_1 + E_1 \beta \ln c_2.$$

In this economy describe the way in which we can derive the prices of the following assets;

(a) A bond which pays off 1 unit of the consumption good in each state of the world in period 2. Define this price as q_b.

(b) A stock which has dividend payment identical to the realized endowment in period 2 (*i.e.* $y + D$ in state 1, $y - D$ in state 2). Define this price as q_s.

(c) Assume that $y = 1$. Show that $q_b > q_s$.

Answer

(a)

$$q_b = y\beta(\frac{0.5}{y-D} + \frac{0.5}{y+D})$$

(b)

$$q_s = y\beta(.5 + .5)$$

(c) Clearly (if $y = 1$) $q_b > q_s$ if

$$\frac{1}{1-D} + \frac{1}{1+D} > 2.$$

This can be shown by some numerical examples or using the quadratic inequality.

22. As statistical background to this question, note that for a random variable x if

$$\Pr\{x = a + b\} = \Pr\{x = a - b\} = 0.5$$

then

$$E\left(\frac{1}{x}\right) = \frac{a}{a^2 - b^2}.$$

This question studies a model of asset-pricing, and the factors which contribute to the 'equity premium', namely the excess of equity returns over bond returns. Consider an economy in which $c_t = 1 + \sigma$ with probability 0.5 and $c_t = 1 - \sigma$ with probability 0.5 in each time period. Thus consumption is iid. The utility function of the representative consumer is

$$U_0 = E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

(a) State (do not derive) an expression for the price of a one-period discount bond in a competitive equilibrium, denoted p_t^b. This bond pays 1 in all states in period $t + 1$.

(b) Solve for this bond price, in terms of variables and parameters which are known at time t.

(c) What is the effect of an increase in σ on the average bond price? What is the economic reason for this effect?

(d) Now consider an equity claim with price given by

$$p_t^s = \beta c_t E_t(\frac{y_{t+1}}{c_{t+1}}).$$

Suppose also that $y_{t+1} = c_{t+1}$. Find $E(R_t^s)$ and $E(R_t^b)$, the average, gross returns on the stock (equity) and the one-period bond.

(e) Does an increase in the variance of consumption (represented by an increase in σ) affect the equity premium?

Answer

(a)

$$p_t^b = E_t(\frac{\beta c_t}{c_{t+1}})$$

(b)

$$p_t^b = \beta c_t \cdot (\frac{1}{1 - \sigma^2})$$

(c) The average bond price is

$$E(p_t^b) = \frac{\beta}{1 - \sigma^2}$$

so an increase in σ raises the average price; the rationale stems from precautionary saving (as discussed in the lecture notes).

(d) $p_t^s = \beta c_t$ so

$$E(R_t^s) = (\frac{1}{\beta(1 - \sigma^2)})$$

and

$$E(R_t^b) = \frac{1}{\beta}.$$

(e) Yes. Clearly it raises the premium.

23. The term structure of interest rates was sharply upward sloping in Canada in late 1999. In this question, we see what theory tells us about the forecasts that may be embodied in that shape. Imagine there are three periods, and two types of bonds. One bond has a price Qs_1 in period 1 and pays \$1 in all states in period 2. The other bond has a price Ql_1 and pays \$1 in all states in period 3. Suppose investors have log utility and discount factor 0.9. Denote nominal consumption in the three time periods by C_1, C_2, and C_3.

(a) Write expressions for the two bond prices in terms of consumption.

(b) The growth rate of real consumption is

$$\frac{c_t}{c_{t-1}} = 1 + \eta_t$$

and the gross inflation rate is

$$\frac{p_t}{p_{t-1}} = 1 + \pi_t$$

for $t = 2, 3$. Does an upward-sloping term structure tell us anything about these rates?

(c) With many time periods, how could we test the predictions in part (b) using information on the interest rates on these two types of bonds? Is there any support for the theory?

Answer

(a)

$$Qs_1 = E_1 0.9 \frac{C_1}{C_2}$$

$$Ql_1 = E_1 0.81 \frac{C_1}{C_3}$$

(b) It tells us one of them is expected to increase.

(c) Linear regressions could be used with growth in consumption or prices regressed on lagged yield differentials. There is some success in predicting recessions, and in predicting growth changes about one year ahead.

24. Sometimes bond prices involve a default risk premium which varies over time. An explanation for this variation might be that investors believe the probability of default depends on the state of the economy. This question explores this issue in a two-period, endowment economy.

(a) Suppose that a representative agent has time-separable, log utility,

$$U = \ln(c_1) + \beta \ln(c_2),$$

and that $c_t = y_t$, $t = 1, 2$. Write an expression for q_1, the price of a one-period discount bond that pays 1 in period 2 in all states.

(b) Suppose that $y_1 = 1.00$ and that $y_2 = 1.10$ with probability 0.5 and $y_2 = 1.05$ with probability 0.5. Suppose that $\beta = 0.98$. Solve for q_1, and for the corresponding interest rate, r_1.

(c) Suppose that another asset also pays 1.00 when $y_2 = 1.10$. When $y_2 = 1.05$ this asset pays 1.00 with probability $1 - \pi$ and 0.75 with probability π. Thus π is the probability of a partial default. If we observe that the price of this asset is $q_1' = 0.88$ what is the implied value of π?

(d) What is the expected return on this second asset?

(e) Sometimes risk is modelled using a *mean preserving spread*, which increases variance without changing mean. Suppose an asset's payoff, denoted x, and y_2 are independent. Imagine adding risk to each of these random variables, by increasing the variance and holding the mean constant. Which of these risks has the greatest effect on the asset price?

Answer

(a)

$$q_1 = E_1 \beta c_1 / c_2.$$

(b) $q_1 = 0.912$, $r_1 = 9.63\%$

(c)

$$0.88 = 0.98[0.5 \cdot \frac{1}{1.10} \cdot 1.00 + 0.5 \cdot \frac{1}{1.05} \cdot (1 - \pi + 0.75\pi)]$$

so $\pi = 0.275$.

(d) The asset pays 1 with probability $0.5 + 0.5 \cdot 0.725$ and 0.75 with probability $0.5 \cdot 0.275$. Its price is 0.88 so its expected return is 9.72%

(e) The x-risk has no effect, by linearity. The y-risk raises the price and lowers the return. The intuition comes from Jensen's inequality and precautionary saving. This question shows how a certain *covariance* matters for risk premiums.

25. Financial economists usually argue that persistent differences in rates of return across assets reflect the assets' differing risk. In macroeconomics we model attitudes to risk – and therefore asset prices – using the expected utility model. For example, imagine a two-period economy, with periods denoted 1 and 2. The within-period utility function is:

$$u(c_i) = \frac{c_i^{1-\gamma}}{1-\gamma},$$

for $i = 1, 2$ and $\gamma > 0$. The discount factor is $\beta = 0.98$, while $\gamma = 2.0$. In the first period, $c_1 = 1.00$. In the second period, $c_2 = 1.03$ with probability 0.75 and $c_2 = 0.99$ with probability 0.25.

In this economy we will consider a one-period bond, which pays 1 in all states in period 2. There also is a one-period stock, which pays c_2 (its payout is equal to consumption) in period 2.

(a) Solve for both the price q_1 and rate of return r_1 on the bond.

(b) Solve for both the price $\overline{q}_1$ and expected return $E(\overline{r}_1)$ on the stock.

(c) What is the expected equity premium, $E(\overline{r}_1) - r_1$?

(d) What would be the effect on the predicted equity premium if the value of γ were higher? What is the economic interpretation of this effect?

Answer

(a) $q_1 = 0.94276$ and $r_1 = 0.0607$

(b) $\overline{q}_1 = 0.96106$ and $E(\overline{r}_1) = 0.0613$.

(c) The expected equity premium is approximately zero. (0.0006)

(d) Usually we would predict higher values of γ would lead to higher premiums. As people become more risk averse, a larger premium is required to compensate them for bearing more risk. We could check that is this case using a numerical example or some algebra.

26. Models of intertemporal budgetting have implications for asset prices, including the prices of bonds of different maturities. Imagine a three-period model, with utility function:

$$EU = \ln(c_1) + E_1 0.98 \ln(c_2) + E_1 0.98^2 \ln(c_3).$$

(a) State an expression for the price q_1^S in period 1 of a one-period, real, discount bond which pays 1 in all states in period 2.

(b) State an expression for the price q_1^L in period 1 of a two-period, real, discount bond which pays 1 in all states in period 3.

(c) Suppose that each period c takes the value 1.02 or 0.98. In the next period the value stays the same with probability 0.8 and changes to the other value with probability 0.2. Suppose that $c_1 = 1.02$. Find the values of q_1^S and q_1^L.

(d) Find the values of r_1^S and r_1^L, the yields on the two bonds. Also find $E_1 r_2^S$, the forecast of next-period's short interest rate. (*Hint:* Find possible values for q_2^S first.) Does the slope of the yield curve reflect the forecasted change in the short rate?

Answer

(a)

$$q_1^S = E_1 0.98 c_1 / c_2$$

(b)

$$q_1^L = E_1 0.98^2 c_1 / c_3$$

(c)

$$q_1^S = 0.98 \cdot 1.02 \cdot (0.8/1.02 + 0.2/0.98) = 0.9879$$

$$q_1^L = 0.98^2 \cdot 1.02 \cdot (0.68/1.02 + 0.32/0.98) = 0.9729$$

(d) The yield on the short bond is $r_1^s = 0.0122$ or 1.22 percent. The yield on the long bond is $r_1^L = 0.0138$ or 1.38 percent.

Next, if $c_2 = 1.02$ then $r_2^S = 0.0122$. If $c_2 = 0.98$ then $q_2^S = 0.9723$ and so $r_2^S = 0.0285$. Thus the forecast for next period's short-term interest rate is

$$E_1 r_2^S = 0.8 \cdot 0.0122 + 0.2 \cdot 0.0285 = 0.0155.$$

The upward-sloping yield curve does forecast a rise in the short term rate. And notice that the long rate is the average of current and expected future short term rates.

27. Two-period models also have been used to describe the current account. To see how this works, suppose that there are two countries. One country has endowments y_1 and y_2 of a nonstorable good. Its utility function is:

$$U(c_1, c_2) = \ln(c_1) + 0.9 \ln(c_2).$$

Meanwhile, the other country has endowments of the same good x_1 and x_2 and utility function

$$U(d_1, d_2) = \ln(d_1) + 0.9 \ln(d_2).$$

Denote the interest rate by r.

(a) Define a competitive equilibrium for this problem.

(b) Suppose that $y_1 = x_1 = 1$, $y_2 = 1.5$, and $x_2 = 0.5$. Solve for a competitive equilibrium. [Hint: Set this up as a Pareto optimum problem of maximizing $\lambda U(c_1, c_2) + (1-\lambda)U(d_1, d_2)$ and then find the values of λ and r that satisfy the budget constraints and the Euler equations.]

(c) Now imagine a slightly different situation in which the discount factor for the first country is 0.7 instead of 0.9. What is the effect on the current account behaviour, compared to that in part (b)?

Answer

(a) A C.E. is a set of consumptions in each country and an interest rate, such that each agent maximizes utility subject to a budget constraint such as

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$

and similarly in the other country (taking r as given), and the market clears in each period:

$$c_1 + d_1 = y_1 + x_1$$

$$c_2 + d_2 = y_2 + x_2.$$

(b) The Pareto problem gives $c_1 = c_2 = 2\lambda$. Then the Euler equation gives $r = 0.111$ and then the budget constraint gives $c_1 = 1.236$ and $d_1 = 0.763$.

(c) Now suppose the discount factor in the first country is 0.7 rather than 0.9. You would expect this greater impatience to lead to a decline in c_2 and a rise in c_1 and so a larger first-period current account deficit. The answer is: $c_1 = 1.27$, $c_2 = 1.15$, $d_1 = 0.73$, $d_2 = 0.85$, and $r = 0.2935$.

28. Imagine using the utility-based asset pricing model to study bond prices. Suppose that the period utility function is logarithmic, and there is a constant discount factor β. A one-period bond pays 1 next year and has price q_t^1 this year, while a two-period bond pays 1 in two years and has price q_t^2 this year. For now, assume that there is no inflation.

(a) What is the relationship between the two bond prices and the forward price?

(b) Find the theory's prediction for the average difference between the forward price and the expected future short price, $E_t q_{t+1}^1$. Can this be used to test the theory?

(c) Now suppose there is random inflation π_t. Suppose that bond market participants learn new information which causes them to revise up their estimate of the inflation rate from 1997 to 1998. Describe the effect on the interest rate on a two-year nominal bond in 1996. Which reacts more to the news, this interest rate or the forward rate?

Answer (a)

$$f_t^1 = \frac{q_t^2}{q_t^1}$$

(b)

$$f_t^1 = E_t q_{t+1}^1 + \frac{\beta^2}{q_t^1} \cdot cov_t\left(\frac{c_t}{c_{t+1}}, \frac{c_{t+1}}{c_{t+2}}\right).$$

Take unconditional expectations of both sides: there will be on average a positive difference between the forward and future spot rates if on average there is a positive covariance between successive consumption growth rates. That can be tested by looking at the autocorrelation in the consumption growth rate.

(c)

$$\left(\frac{1}{1+R_t^2}\right)^2 = Q_t^2 = E_1\left(\frac{1}{1+r_t^2}\right)^2\left(\frac{1}{1+\pi_{t+1}}\right)\left(\frac{1}{1+\pi_{t+2}}\right).$$

So of course an increase in π_{t+2} leads to an increase in R_t^2. This reacts less than the forward interest rate; the logic is that the long rate is roughly the average of current and expected future short rates.

29. Consider a two-country, general equilibrium, endowment economy, with no uncertainty. There are two time periods. In country 1 the endowments are $y_1 = 1$ and $y_2 = 1$. The utility function is

$$U = \ln(c_1) + 0.9\ln(c_2).$$

Meanwhile, in country 2 the endowments are $x_1 = 2$ and $x_2 = 2$, and the utility function is

$$U = \ln(d_1) + 0.9\ln(d_2).$$

We shall study the predictions for competitve equilibria.

(a) What is the consumption growth rate in each country? What is the world real interest rate?

(b) Suppose instead that $y_1 = 2$. How much does country 1 export in the first period?

(c) What does this model imply about the procyclicality or countercyclicality of the trade balance?

Answer (a) The consumption growth rate in each country is zero, as they each consume their endowment, which does not grow. The interest rate therefore is the discount rate: 0.1111. These answers can be read off directly from the information given.

(b) I get $\lambda = 0.4122$ and $c_1 = 4\lambda = 1.6488$ so they export 0.3512. Remember that r changes from part (a) to part (b)!

(c) The model implies that the trade balance is pro-cyclical, while the empirical evidence of this is relatively weak.

30. Imagine an endowment economy, in which all agents are identical. Consumption in adjacent time periods satisfies

$$\frac{1}{c_t} = E_t \frac{\beta(1 + r_{t+1})}{c_{t+1}},$$

where $1 + r_{t+1}$ is the gross return on any investment from period t to period $t + 1$.

(a) State an expression for the price, q_t, of a one-period, riskless, discount bond.

(b) As an alternative to buying a one-period bond, an investor is considering buying foreign currency at price e_t, then using that to buy a foreign bond, with price q_t^*. When the foreign bond matures it pays one unit of foreign currency, which the investor will convert to her home currency at the prevailing exchange rate, e_{t+1}. What does the CCAPM predict about the relationship between foreign and domestic bond prices and current and future exchange rates? Use a covariance decomposition to clarify your result.

(c) How could the investor avoid the risk that the domestic currency will appreciate while she is holding the foreign bond? What theory do we have to predict the price of avoiding this risk?

Answer (a)

$$q_t = E_t \frac{\beta c_t}{c_{t+1}}$$

(b)

$$\frac{q_t^*}{e_t} = E_t \beta \frac{c_t}{c_{t+1}} \cdot \frac{1}{e_{t+1}},$$

since the investor is the same as in part (a) and so has the same risk-adjusted discount factor or IMRS. Thus

$$q_t^* = q_t E_t \frac{e_t}{e_{t+1}} + cov_t[\frac{\beta c_t}{c_{t+1}}, \frac{e_t}{e_{t+1}}].$$

This is uip, plus a risk premium.

(c) The risk can be avoided by purchase of a forward contract. Arbitrage gives us the price:

$$q_t^* = q_t \frac{e_t}{f_t},$$

which is cip.

31. This question studies whether the observed, cyclical properties of consumption, labour supply, and real wages can be consistent with simple models of household planning. Suppose that there are two time periods, and a typical household's goals is to maximize

$$u(c_1, n_1) + \beta E_1 u(c_2, n_2)$$

subject to

$$c_1 + s = w_1 n_1$$
$$c_2 = s(1+r) + w_2 n_2,$$

taking wages and the interest rate as given.

(a) Comment on whether in actual data consumption, labour supply, and real wages are procyclical, acyclical, or countercyclical.

(b) Suppose that the period utility function in the model is:

$$u(c, n) = \ln(c) + \ln(1-n).$$

Find the first-order condition linking c_1, n_1, and w_1. Is this link between the variables consistent with your answer in part (a)?

(c) As an alternative, suppose that the period utility function is:

$$u(c, n) = \ln(c - \gamma n).$$

Find the first-order condition linking c_1, n_1, and w_1. Is this link between the variables consistent with your answer in part (a)?

Answer

(a) Roughly speaking, consumption and labour supply are procyclical while real wages are acyclical.

(b) Each period

$$w = \frac{c}{1-n}.$$

For w not to vary with the cycle, leisure must be procyclical or labour supply countercyclical; so this does not fit the facts.

(b) Now each period:

$$w = \frac{-u_n}{u_c} = \frac{\gamma(c-\gamma n)}{c-\gamma n} = \gamma$$

which does better at fitting this fact of acyclicality of the real wage, and is consistent with the joint cyclicality of consumption and employment.

32. This question uses a quadratic utility function to study the effect of taxes on consumption. Suppose that there are two time periods, denoted 1 and 2. The utility function is:

$$EU = (a - bc_t^2) + E_1 \frac{1}{1+r}(a - bc_{t+1}^2),$$

so that the discount rate always equals the real interest rate. The budget constraint is:

$$c_1 + E_1 \frac{c_2}{1+r} = y_1(1-t_1) + E_1 \frac{y_2(1-t_2)}{1+r},$$

where y denotes labour income and t denotes a tax rate. Suppose that there is a linear technology in this economy, which determines the constant interest rate r.

(a) Find the Euler equation.

(b) Find the decision rule for c_1.

(c) Suppose that y and r are constant. In period 1 the tax authority announces the following rule for setting taxes in period 2:

$$t_2 = \lambda t_1 + \epsilon_2,$$

where ϵ_2 is an unpredictable shock. Find the statistical relationship between c_1 and t_1. What does the Lucas critique say about this relationship?

(d) In general, can we evaluate the effects of tax changes using decision rules?

Answer. (a)

$$c_1 = E_1 c_2$$

(b)

$$c_1 = \frac{1+r}{2+r}\left[y_1(1-t_1) + E_1 \frac{y_2(1-t_2)}{1+r}\right].$$

(c) With this policy rule, the relationship is:

$$c_1 = \frac{1+r}{2+r}\left[y(1-t_1) + \frac{y(1-\lambda t_1)}{1+r}\right].$$

The effect of t_1 on c_1 depends on λ, so we need a forecasting rule. The coefficient linking c_1 and t_1 is predicted to change if the parameter in the policy rule, λ, changes. This shift can be used to test for the importance of the Lucas critique.

(d) In general we need general equilibrium, and not simply a decision rule, to evaluate the effects of a policy change. See if you can explain why.

33. This question uses the consumption-based asset-pricing model to predict the effects on the real interest rate of the mean and variance of consumption growth. Suppose that a representative agent has time-separable, logarithmic utility, and discount factor β. From one period to the next, consumption evolves this way:

$$\frac{c_{t+1}}{c_t} = \begin{array}{ll} (1+\lambda)(1-\epsilon) & \text{w.p. } 0.5 \\ & \\ (1+\lambda)(1+\epsilon) & \text{w.p. } 0.5 \end{array}$$

where ϵ and λ are small positive numbers. There is no inflation.

(a) State the Euler equation which gives the real interest rate on a one-period discount bond.

(b) Solve for this interest rate using the stochastic process for gross consumption growth.

(c) What are the effects of the mean and variance of consumption growth on the interest rate? What is the economic rationale for each effect?

(d) What is the shape of the yield curve in this model? In general, what properties of consumption growth are needed to explain an upward-sloping real yield curve?

Answer (a)

$$\frac{1}{1+r_t} = E_t \beta \frac{c_t}{c_{t+1}}.$$

(b)

$$1+r = \frac{(1+\lambda)(1-\epsilon^2)}{\beta}$$

(c) An increase in λ raises r. With faster growth there is less need to save. An increase in ϵ lowers r. With more volatile consumption growth, saving rises when $u''' > 0$ (precautionary saving).

(d) In this model consumption growth is independently and identically distributed, so the real yield curve is flat. I think positive persistence produces an upward-sloping yield curve.

34. Consider a competitive, endowment economy with time periods indexed by t and endowments $\{y_t\}$. A representative agent has preferences given by:

$$U = E\sum_{t=0}^{\infty} \beta^t \ln(c_t).$$

Call the real return on asset i from t to $t+1$ r_{it+1}, and the price level p_t. Future endowments, price levels, and asset payoffs may be uncertain.

(a) State the Euler equation linking any adjacent consumption quantities c_t and c_{t+1}

(b) Consider a one-period asset which costs Q_t dollars in period t and pays 1 dollar in period $t+1$ in all states of the world. Find an expression for this asset's price.

(c) The government plans to introduce a 'real' bond that pays a guaranteed real return. Explain how you could use information in $\{Q_t\}$, consumption growth, and inflation to recommend a fair price for this new asset.

Answer (a)

$$\frac{1}{c_t} = E_t\beta\frac{1+r_{it+1}}{c_{t+1}}.$$

(b)

$$Q_t = E_t\beta\frac{c_t}{c_{t+1}}\frac{p_t}{p_{t+1}}$$

(c) Use a covariance decomposition:

$$q_t = Q_t - cov\Big(\frac{\beta c_t}{c_{t+1}}, \frac{p_t}{p_{t+1}}\Big)\cdot\Big(E_t\frac{p_t}{p_{t+1}}\Big)^{-1}$$

We do not know the price level, so we cannot simply use Q/P. But alternately, we could use the forecast of real consumption, with β.

35. To study business cycles and tax policy, we need to consider variations in labour inputs. This question studies a two-period production economy with a variable labour supply. There is no uncertainty. The utility function is:

$$U = \ln(c_1) + \beta\ln(c_2) + \beta\ln(1-n_2),$$

so that people may supply labour n_2 in the second period. Households have an endowment y in the first period, but no endowment in the second period. However, by setting aside units of this single good through a production function they may gain access to goods in the second period. If an amount k is set aside then

$$k^\alpha n_2^{1-\alpha}$$

units are produced in period two (the depreciation rate is 100 percent).

(a) Define a competitive equilibrium.

(b) Solve for a competitive equilibrium.

Answer (a) A competitive equilibrium is a set of quantities c_1, c_2, k, and n_2 and prices w and r such that (i) households maximize utility subject to their budget constraint and labour endowment; (ii) firms maximize profits; and (iii) goods and labour markets clear each period.

(b) The easiest way to find the competitive equilibrium is to set up an unconstrained planning problem:

$$\max \quad \ln(y-k) + \beta \ln(k^\alpha n_2^{1-\alpha}) + \beta \ln(1-n_2)$$

The first order conditions give:

$$n_2 = \frac{1-\alpha}{2-\alpha},$$

and

$$k = \frac{\beta\alpha y}{1+\beta\alpha}.$$

Then

$$c_1 = \frac{y}{1+\beta\alpha}.$$

Finally, c_2 can be found from the production function, then r and w from the household's Euler equations or the firm's demands.

36. (Ricardian equivalence 'counterexample'). Consider the following variation of our two-period economy. There are two agents, 1 and 2, each with utility function $u(x,z) = \ln x + \beta \ln z$. Agent 1 has endowment $\{y, 0\}$ and pays taxes t_1 in the first period, none in the second. Agent 2 has endowment $\{0, y\}$ and pays taxes t_2 in the second period, none in the first. The government spends $g < y$ each period.

(a) What are the budget constraints of the two private agents and the government?

(b) Derive the competitive equilibrium for this economy for arbitrary tax policies satisfying the government budget constraint. What does this tell you about the Ricardian equivalence theorem?

37. A crude characterization of some of the econometric 'tests' of the Ricardian equivalence theorem is the following (see, for example, Bernheim's survey in the *1987 NBER*

Macroeconomics Annual). Since the timing of taxes is irrelevant to private decisions, including consumption, the theory is said to imply a zero correlation between consumption and taxes. (We could, that is, regress consumption on taxes and test the hypothesis that the coefficient of taxes is zero.) An alternative hypothesis is that consumption is lower when taxes are high (a negative correlation). We will use the exchange economy with constant government spending ($g_1 = g_2 = g$) and log preferences to examine a similar prediction in our theoretical economy.

(a) Use the equivalence class defined by the Ricardian equivalence theorem to construct numerical examples in which consumption is (i) positively and (ii) negatively 'correlated' with taxes. [Hint: If equilibrium consumption is high/low, construct examples in which taxes are low/high and high/low, respectively.]

(b) Comment on the relation of the proposed test to the theorem.

(c) How does your answer change if we replace the word taxes with deficits?

38. (Difficult) Consider the production economy with log preferences, $u(c_1, c_2) = \ln c_1 + \beta \ln c_2$, endowment $y_1 = y, y_2 = 0$, and technology, $f(k) = k^\alpha$, for $0 < a < 1$. Government spending g takes place only in the first period and is financed by a proportional tax on the gross return to investment, $f(k)$. Thus aftertax profits on investment are

$$PR = (1 - \tau)p_2 f(k) - p_1 k.$$

(a) Define a competitive equilibrium for this economy.

(b) Compute the equilibrium for a fixed level of g.

(c) What levels of g and τ give the private agent the highest utility? Why?

(d) How does your answer to (c) change if utility depends on government spending in the following way:

$$u(c_1, c_2, g) = \ln c_1 + \beta \ln c_2 + \gamma \ln\ g?$$

Comment.

39. Consider a two-period economy with one agent and one government. The agent has rational expectations and knows the structure of the economy.

The government spends g_1 in period 1 and $\lambda g_1 + \epsilon$ in period two, where ϵ is a shock (to both the government and the agent) such that $E_1 \epsilon = 0$. The government raises taxes t_1 in period 1 and, after observing ϵ, t_2 in period 2. Budgets balance in expected present-value terms.

Output arrives exogenously in known amounts y_1 and y_2 respectively in the two periods. The national income identity is

$$y_i = c_i + g_i \quad i = 1, 2$$

Suppose that the agent's preferences are given by

$$ac_1 - bc_1^2 + \beta E_1(ac_2 - bc_2^2)$$

where a and b are coefficients such that $u' > 0$, $u'' < 0$, and $\beta\,\varepsilon(0,1)$. Both the agent and the government have access to a storage technology by which the good can be transferred between periods while paying net interest rate r. Assume that $\beta = 1/(1+r)$.

(a) Use the preferences and budget constraint of the agent to find c_1 and c_2.

(b) Show that the decision rule for c_1 depends on the parameter of the government spending rule, namely λ. Explain the sign of $\partial c_1/\partial\lambda$. Briefly relate this result to Lucas's critique of econometric policy evaluation.

(c) Show that this economy exhibits Ricardian equivalence *i.e.* that the timing of tax collections does not affect c_1 and c_2.

40. This question studies the effects of fiscal policy on the trade balance and national savings. We begin by studying a two-period, world, exchange (endowment) economy, with no governments and a single good. There are two countries, denoted A and B. Country A has incomes y_1 and y_2 and consumptions c_1 and c_2 in the two periods. Country B has incomes x_1 and x_2 and consumptions d_1 and d_2. Suppose that people in country A maximize $\ln(c_1) + \beta\ln(c_2)$ and that the utility function in country B is similar.

(a) Write the budget constraint for each country. Also write the world market clearing conditions in each period.

(b) If $y_1 = x_1 = 1$ and $y_2 = x_2 = 1$, then solve for c_1, c_2, d_1, d_2, the interest rate, and the trade balance in each country and time period, in a competitive equilibrium.

(c) Now suppose that there is a government in country A. It spends g_1 in period 1 and g_2 in period 2 and raises lump-sum taxes in country A in each time period. Suppose that $g_1 = g_2 = 0.25$. Suppose that the government balances its budget in each time period. Solve for a competitive equilibrium.

(d) Does a change in the timing of taxes affect any component of the competitive equilibrium?

(e) Describe the effects of a change in first-period government spending on the competitive equilibrium, by setting $g_1 = 0.5$

(f) What additional features in this theoretical model might give rise to more interesting trade balance dynamics?

Answer

(a) Budget constraints are:

$$c_1 + c_2/(1+r) = y_1 + y_2/(1+r)$$
$$d_1 + d_2/(1+r) = x_1 + x_2/(1+r).$$

Market clearing gives

$$c_1 + d_1 = y_1 + x_1$$
$$c_2 + d_2 = y_2 + x_2$$

(b) Obviously each country consumes its endowment so: $c_i = d_i = 1$, so that the trade balance is zero in both time periods, and $r = 0.11111$, if $\beta = 0.9$, say. With two countries, the simplest way to solve for a C.E. is to find a P.O., because there are no externalities here. Notice that the P.O. problem involves no prices. Here the general problem is:

$$\text{Max } \lambda[\ln(c_1) + \beta \ln(c_2)] + (1-\lambda)[\ln(d_1) + \beta \ln(d_2)],$$

subject to

$$c_1 + d_1 = y_1 + x_1 - g_1$$

$$c_2 + d_2 = y_2 + x_2 - g_2.$$

This gives:

$$c_1 = \lambda(y_1 + x_1 - g_1)$$

$$c_2 = \lambda(y_2 + x_2 - g_2)$$

$$d_1 = (1-\lambda)(y_1 + x_1 - g_1)$$

$$d_2 = (1-\lambda)(y_2 + x_2 - g_2)$$

We then can return to the usual conditions for the C.E. in order to find r and λ. From the Euler equation of either country's consumer:

$$(1+r) = \frac{(y_2 + x_2 - g_2)}{\beta(y_1 + x_1 - g_1)}$$

The budget constraint in the first country is:

$$c_1(1+r) + c_2 = y_1(1+r) + y_2 - [t_1(1+r) + t_2].$$

Substituting in our answers for c_1 and c_2 and r above will give λ. In part (b) of this question we find $c_1 = 1$, $c_2 = 1$, $d_1 = 1$, and $d_2 = 1$. Also $r = 0.111$. Here $\lambda = 0.5$.

(c) $g_1 = g_2 = .25$. The same method gives $c_1 = .75$, $c_2 = .75$, $d_1 = 1$, and $d_2 = 1$. The interest rate is unchanged. Here $\lambda = 0.428$. In this case consumers in the first country have become poorer. The value of λ reflects this, and the specific P.O. that corresponds to the C.E. is tilted towards the second country.

(d) A change in tax timing has no effect on the competitive equilibrium.

(e) $g_1 = 0.5$ and $g_2 = 0.25$. Market clearing now gives $c_1 + d_1 = 1.5$ and $c_2 + d_2 = 1.75$. In this case the interest rate is

$$r = \frac{1.75}{\beta 1.5} - 1 = 0.296$$

if $\beta = 0.9$, say. We can use this rate, with our expressions for consumption as a share of world private endowment, in the budget constraint, to find that:

$$\lambda(1.5)(1.296) + \lambda(1.75) = 1(1.296) + 1 - [0.5(1.296) + .25]$$

so that $\lambda = 0.378$.

(f) There is trade in this case. In the first country consumers in period 1 have income 0.5 and consumption 0.567. In the second period they have income 0.75 and consumption 0.6615. Thus they have a trade surplus of -0.067 in the first period and 0.0885 in the second period (the surplus in the other country will be exactly the opposite). The external budget constraint for that country is

$$tb_1 + (\frac{tb_2}{1+r}) = 0.$$

These numbers satisfy that constraint.

41. Much of our understanding of the dynamic effects of fiscal policy comes from the study of dramatic episodes in fiscal history, such as wars. Consider a two-period, exchange economy with a certain, non-storable endowment $y_1 = 2$ and $y_2 = 2$. Suppose that consumers budget to maximize lifetime utility given by

$$U = E_0[\ln(c_1) + .9log(c_2)],$$

where E_0 means that plans are made at the beginning of period 1. Finally, suppose that the government spends g_1 in period 1 and g_2 in period 2. Let $g_2 = 0.5$ with certainty. Let $g_1 = 0.5$ with probability $1 - \pi$ and $g_1 = 1$ with probability π. The idea is simply that 0.5 is the normal level of public spending but that a one-year war or other emergency may occur in the first year with probability π. Note that $y = c + g$ in each period. The government can levy lump sum taxes $\{t_1, t_2\}$.

(a) Show that the higher is π the higher is the expected or average value of the competitive equilibrium interest rate on investments carried from period 1 to period 2.

(b) Suppose that the government faces the usual, two-period budget constraint. It sets $t_1 = E_0 g_1$, then it adjusts the tax in period 2 to balance the two-period budget, given the outcome for public spending in period 1. Solve for t_1 and for the two possible values of t_2.

(c) Under this same financing scheme, find the the correlation between the budget deficit and the interest rate (if you cannot find it exactly then state its sign). Does this result cast doubt on some tests of Ricardian equivalence?

Answer

(a)

$$E_0(1/c_1) = 0.9E_0(1+r)(1/1.5)$$
$$(2/3) + \pi/3 = 0.9(2/3)(1+r)$$

so that if π rises so does r

(b) $t_1 = \pi + (1-\pi)0.5 = \pi/2 + 1/2$

Then

$$s = \pi/2 + 1/2 - 1 \quad \text{w.p. } \pi$$
$$s = \pi/2 + 1/2 - 0.5 \quad \text{w.p. } 1-\pi$$

so $E(s) = 0$.

Also one can show that

$$r = 0.6667 \quad \text{w.p. } \pi$$
$$r = 0.111 \quad \text{w.p. } 1-\pi$$

And that

$$t_2 = 0.5 - (\pi/2 + 1/2 - 1)(1.6667) \quad \text{w.p. } \pi$$
$$t_2 = 0.5 - (\pi/2 + 1/2 - 0.5)(1.1111) \quad \text{w.p. } 1-\pi$$

(c) Thus the correlation between the $-s$ and r is positive. So it looks non-equivalent (as if tax timing matters for r). But not if one controls for g_1. This was a difficult question. In this question there are two possible states (war and peace) in the first period. In state P $c_1 = 1.5$. Also we know that $c_2 = 1.5$. Thus from the Euler equation $r = 0.111$. In state W $c_1 = 1$ and then again $c_2 = 1.5$. Thus $r = 0.6667$. Thus the average interest rate is

$$E(r) = (1-\pi)(.111) + \pi(.6667)$$

Now in state P the government spends $g_1 = .5$ while in state W it spends $g_1 = 1$. Tax revenue is set to balance average spending:

$$t_1 = (.5)(1-\pi) + (1.0)(\pi)$$

Then, in the second period, revenue must be set so that the government's intertemporal constraint holds:

$$t_2 = (1+r)(g_1) + g_2 - t_1(1+r)$$

If state P occurs in the first period then

$$t_2 = (1.111.)(0.5) + 0.5 - t_1(1.111).$$

If state W occurs in the first period then

$$t_2 = (1.667)(1.5) + 0.5 - t_1(1.667).$$

We may need a specific value for π in order to find the correlation coefficient between s_1 and r, but it will be negative, for both of these variables are affected by g_1.

42. Consider a two-period economy in which there is a constant, aggregate endowment $y = 3$ in each period, which is nonstorable. The interest rate satisfies

$$\frac{1}{c_1} = E_1 0.98(1+r)\frac{1}{c_2}.$$

The government spends g_1 in period 1 and g_2 in period 2, and raises lump-sum taxes t_1 and t_2. Suppose that $t_1 = 1$. In state W the government sells one-period discount bonds in order to finance its deficit. There are two equiprobable states of the world in period 1. In state P $g_1 = 1$. In state W $g_1 = 1.5$. Taxes in period 2 are set to satisfy the government's budget constraint. Suppose that $g_2 = 1$. The equilibrium is competitive.

(a) Solve for c_1, c_2, and r in each state.

(b) Find the two possible values for t_2.

(c) Find the correlation between r and s_1, the government's budget surplus in the first period.

(d) Does this example suggest any problems in testing for Ricardian equivalence?

Answer

(a) In state P $c_1 = 2$, $c_2 = 2$, and $r = .0204$. In state W $c_1 = 1.5$, $c_2 = 2$, and $r = .3605$.

(b) In state P $t_2 = 1$. In state W $t_2 = -s(1+r) + g_2 = .5(1+r) + 1 = 1.68$.

(c) $E(s) = -0.25$, $E(r) = .1902$. Also $std(s) = 0.25$, and $std(r) = .17$. Then $cov(s,r) = -0.0425$. Thus $corr(s,r) = -1.0$. Thus higher deficits are associated with higher interest rates.

(d) In this example there is Ricardian equivalence. It seems that one could control for government spending in a linear regression and find a zero partial correlation between the interest rate and the deficit. But is the relationship linear?

43. Consider a two-period, competitive, exchange economy with a representative agent. The endowment in the first period is $y_1 = 3$. The endowment in the second period is $y_2 = 3.5$ with probability 0.5 and $y_2 = 2.5$ with probability 0.5. The utility function is

$$EU = \frac{-1}{2c_1^2} - E_1\frac{0.8}{2c_2^2}.$$

(a) Solve for the price and return on a one-period riskless bond.

(b) Solve for the price and expected return of an equity which pays 1 if $y_2 = 3.5$ and 0 otherwise.

(c) Now suppose that there is a government which raises revenue by lump-sum taxes. In the first period it spends $g_1 = 1$. In the second period it spends $g_2 = 1.25$ if $y_2 = 3.5$ and $g_2 = 0.75$ if $y_2 = 2.5$. Solve for the return on the bond and the expected return on the equity.

(d) In the first period the government raises lump-sum taxes of $t_1 = 0.86$. It is considering two ways to finance its deficit. In scheme S it sells riskless bonds to finance its deficit. In scheme R it sells the equity claim in an amount sufficient to finance its deficit. Under which scheme will second-period taxes, t_2, be lower on average?

Answer

(a)

$$q_1 = 0.8 \cdot 3^3[0.5/3.5^3 + 0.5/2.5^3] = 0.943$$

The return is 0.0604.

(b)

$$\overline{q} = 0.8 \cdot 3^3[0.5/3.5^3 + 0] = 0.25.$$

The price is 0.25 and the expected payoff is 0.5 so the expected return is 1.

(c) Now the bond price is:

$$q_1 = 0.8 \cdot 2^3[0.5/2.25^3 + 0.5/1.75^3] = 0.87.$$

So the return is 0.15. The equity price is

$$\overline{q} = 0.8 \cdot 2^3[0.5/2.25^3] = 0.28.$$

So its expected return is 0.786. Notice that smoothing the consumption stream has lowered this equity premium.

(d) The deficit is 0.14. We know that the return on debt is 0.15 so this raises t_2 by $(1.15)0.14 = 0.161$. Thus t_2 will be $1.25 + 0.161$ or $0.75 + 0.161$ for an average of 1.161. Under scheme R the government will sell one-half equity claim, because the price of a claim is 0.28. Then its taxes will be $1.25 + 0.5$ or 0.75 for an average of 1.25. Hence average taxes are higher under the equity finance scheme. Clearly to smooth taxes and have lower average taxes the government should choose scheme S. You might also ask yourself whether Ricardian equivalence holds in this economy and whether the tax arrangement affects welfare.

44. Imagine a two-period endowment economy, inhabited by two agents and a government. The government consumes $g = 1$ in each time period. Agent A has endowments $\{4, 0\}$ (*i.e.* 4 in period 1 and 0 in period 2) while agent B has endowments $\{0, 4\}$. They both have log utility with a discount factor of 0.9. The government raises lump-sum taxes in each time period.

(a) Suppose that the government balances its budget in each time period. Solve for a competitive equilibrium.

(b) Now suppose that the government reduces taxes in the first period, and raises them in the second period to balance its budget in present-value terms. Does Ricardian Equivalence hold in this economy?

(c) Are there other types of heterogeneity across the population under which a change in the timing of lump-sum taxes affects aggregate consumption and interest rates?

Answer

(a) It is clear that aggregate consumption is 3 in each period, and the interest rate is 0.1112.

(b) The change in tax timing affects the distribution of income, but not aggregates in this economy. This can be shown with a social-welfare planning problem for example.

(c) In the actual economy aggregate consumption might be affected by the change in the distribution of after-tax income. In that case interest rates will be affected too. Other ways in which heterogeneity may lead to non-equivalence include liquidity constraints and differring ages.

45. Imagine a two-period, endowment economy. Private sector endowments of a non-storable good are y_1 in the first period and y_2 in the second period. Government spending is g_1 in the first period and g_2 in the second period. Both y_2 and g_2 may be uncertain. Lump-sum taxes are available to the government in each period. Utility is given by

$$EU = \ln(c_1) + E_1 \beta \ln(c_2).$$

(a) Define a competitive equilibrium for this economy.

(b) The government might finance a deficit by issuing one-period riskless debt. Find an expression for the price of such debt.

(c) What is the effect on the riskless interest rate of an increase in the variance of future government spending?

(d) Show that the government can lower the average rate of interest it must pay on debt if the payoff on debt has a covariance with government spending in the second period.

Answer

(a) The private sector maximizes utility subject to its two-period budget constraint, taking prices as given. The government satisfies its budget constraint. Markets clear in each period: $y_i = c_i + g_i$.

(b)

$$q = E_1 \beta (\frac{y_1 - g_1}{y_2 - g_2}).$$

(c) Unless there also is a positive covariance with y_2 (which we can rule out the way the question is phrased), an increase in the variance of g_2 will lower r due to precautionary saving.

(d) Use a covariance decomposition

$$q' = E_1\beta(\frac{y_1 - g_1}{y_2 - g_2})E_1\text{payoff} + cov.$$

A positive covariance between the payoff and g_2 provides some consumption insurance and so raises the asset price.

46. The 'twin deficits' hypothesis suggests that an increase in the public sector budget deficit leads to an increase in the current account deficit. To examine this, consider a small open economy, with two time periods. The government can borrow at interest rate r which is given in the world economy. Its budget constraint is:

$$4 = t_1 + \frac{t_2}{1+r}$$

where 4 is the present value of government spending.

Households maximize

$$U = \ln(c_1) + \beta\ln(c_2),$$

subject to

$$c_1 + \frac{c_2}{1+r'} = 10 - t_1 - \frac{t_2}{1+r'}.$$

Here r' is the interest rate at which the private sector can borrow, and it too is determined in the world economy.

(a) State the Euler equation for the household.

(b) Solve for the consumption function for c_1.

(c) Under what conditions does Ricardian equivalence hold?

(d) How can we test Ricardian equivalence in a small open economy? What is the evidence from such tests?

Answer

(a)

$$\frac{1}{c_1} = \beta(1+r')\frac{1}{c_2}$$

(b)

$$c_1 = (\frac{1}{1+\beta})(10 - t_1 - \frac{t_2}{1+r'})$$

(c) If $r' = r$. Otherwise a tax cut can reduce national saving, given by $y - g - c$.

(d) Some tests for small open economies use a linear regression of the current account on the budget deficit and government spending. The coefficient on government spending is expected to be negative. The coefficient on the budget deficit is zero under Ricardian equivalence, and positive otherwise. Some evidence finds both are insignificant ...

47. Consider a small, open economy, that takes the world real interest rate r as given. There are two time periods. The country has non-storable endowments y_1 and y_2. Government spending is given by g_1 and g_2. The government collects lump-sum taxes t_1 and t_2. There is no uncertainty. The private sector's goals can be summarized by a utility function:

$$U = \frac{c_1^{1-\alpha}}{1-\alpha} + \left(\frac{1}{1+r}\right)\left(\frac{c_2^{1-\alpha}}{1-\alpha}\right).$$

(a) What is the economy's intertemporal budget constraint?

(b) Find the effect on the current account of an increase in the world interest rate.

(c) Find the effect on the current account of a tax deferral.

(d) How could one test for Ricardian equivalence in a small open economy?

Answer (a) For goods market clearing:

$$y_i = c_i + g_i + x_i,$$

where x_i is exports. The national budget constraint is:

$$x_1 + \frac{x_2}{1+r} = 0.$$

(b) The market clearing constraint is:

$$c_1 + \frac{c_2}{1+r} = y_1 - g_1 + \frac{y_2 - g_2}{1+r},$$

even though this is an endowment economy, because there is effectively a reversible storage technology at rate r. The Euler equation is:

$$c_1^{\alpha} = c_2^{\alpha}.$$

Thus

$$c_1 = \frac{1+r}{2+r}\left[y_1 - g_1 + \frac{y_2 - g_2}{1+r}\right].$$

See Romer section 7.4: the effect depends on whether the country lends or borrows initially. If the country initially has a non-positive current account then saving rises, and the current account rises; otherwise the effect is ambiguous.

(c) There is no effect on national saving, because Ricardian equivalence holds.

(d) Typical tests would regress the current account deficit on controls for government spending, the world interest rate, and then the government budget deficit, to test for 'twin deficits.'

48. This question studies the effect of fiscal policy on interest rates. Imagine a two-period economy, in which all households have identical preferences:

$$EU = \frac{c_1^{1-\alpha}}{1-\alpha} + \beta E_1 \frac{c_2^{1-\alpha}}{1-\alpha}, \quad \alpha > 0$$

and receive nonstorable endowments y_1 and y_2. The economy is competitive. The government levies lump sum taxes to finance government spending g_1 and g_2.

(a) What is the effect of uncertainty about g_2 on the interest rate on a one-period riskless bond?

(b) Will the interest rate be lower if a procyclical spending policy ($cov(y_2, g_2) > 0$) or a countercyclical spending policy is expected?

(c) If the government cuts taxes in period 1 then suppose it could raise them in period 2 with probability π or instead cut g_2 with probability $1-\pi$. Will this tax cut affect interest rates? Would tests for Ricardian equivalence find support for it?

Answer

(a) With $\alpha > 0$, $u''' > 0$ so there is a precautionary saving motive. The increase in the forecast variance of g_2 will raise saving and lower r.

(b) The procyclical policy reduces the variance of $y_2 - g_2$ and so leads to a rise in r.

(c) If the forecast of g_2 falls then r rises. Tests for Ricardian equivalence control for g, so they should correctly indicate that the deficit *per se* does not affect the interest rate.