Econ 250 Mid-Term Test 2
23 November, 2011

Instructions: You may use a hand calculator. Do not hand in the question sheet. Answer all four questions in the answer booklet provided. Show your work. Formulas and tables are provided at the end of the question pages.

1. The following table displays the joint probability distribution of X and Y.

	$X = 0$	$X = 1$	$X = 2$
$Y = 1$	0.09	0.08	0.23
$Y = 2$	0.16	0.14	0.05
$Y = 3$	0.07	0.16	0.02

   (a) Find the conditional probability function of X, given $Y = 1$.
   (b) Are X and Y independent? Why?
   (c) Let $W = XY$. Find μ_W, the mean of W.

2. Suppose that a population is uniformly distributed between 0 and 12. A random sample of size n = 36 is obtained.
   (a) What are the mean and standard deviation of the sampling distribution for the sample means?
   (b) What is the probability that the sample mean is between 5.93 and 6.05?
   (c) With probability 0.9 the sample mean is greater than what value?

3. Let X_1, X_2, X_3, and X_4 be a random sample of observations from a population with mean μ and variance σ^2. Consider the following two point estimators of μ:
   $\widehat{\theta}_1 = 0.10X_1 + 0.20X_2 + 0.40X_3 + 0.30X_4$, and
   $\widehat{\theta}_2 = 0.25X_1 + 0.25X_2 + 0.30X_3 + 0.20X_4$
   (a) Show that both estimators are unbiased.
   (b) Which estimator is more efficient, $\widehat{\theta}_1$ or $\widehat{\theta}_2$? Explain in detail.

4. A random sample of eight salespersons that attended a motivational course on sales techniques was monitored in the three months before and the three months after the course. The table shows the values of sales (in thousands of dollars) generated by these eight salespersons in the two periods. Assume that the population distributions are normal.

Salesperson	Before Course	After Course
1	215	240
2	280	290
3	205	195
4	330	334
5	170	190
6	198	184
7	204	210
8	275	290

   (a) Find an 80% confidence interval for the difference between the two population means.
   (b) Based on the confidence interval in part (a), is there evidence that the population means are different?
   (c) Now suppose that we know the population standard deviation of the difference between sales values after the course and before the course is 10 thousand dollars. And we want to find a 95% confidence interval for the difference between the two population means of width at most 4 thousand dollars. How large a sample is needed to achieve such an interval?

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1. The following table displays the joint probability distribution of X and Y.

	$X=0$	$X=1$	$X=2$
$Y=1$	0.09	0.08	0.23
$Y=2$	0.16	0.14	0.05
$Y=3$	0.07	0.16	0.02

   (a) (9 Marks) Find the conditional probability function of X, given $Y=1$.
   $P(Y=1)=0.09+0.08+0.23=0.4$
   $P(X=0|Y=1)=\frac{P(X=0,Y=1)}{P(Y=1)}=\frac{0.09}{0.4}=0.225$
   $P(X=1|Y=1)=\frac{P(X=1,Y=1)}{P(Y=1)}=\frac{0.08}{0.4}=0.2$
   $P(X=2|Y=1)=\frac{P(X=2,Y=1)}{P(Y=1)}=\frac{0.23}{0.40}=0.575$

   (b) (8 Marks) Are X and Y independent? Why?
   $P(X=0)=0.09+0.16+0.07=0.32$
   Since $P(X=0)P(Y=1)=0.32\times 0.4=0.128\neq 0.09=P(X=0,Y=1)$, X and Y are not independent.

   (c) (8 Marks) Let $W=XY$. Find μ_W, the mean of W.
   $\mu_W=0\times0.09+1\times0.08+2\times0.23+0\times0.16+2\times0.14+4\times0.05+0\times0.07+3\times0.16+6\times0.02=1.62$

2. Suppose that a population is uniformly distributed between 0 and 12. A random sample of size n = 36 is obtained.

   (a) (9 Marks) What are the mean and standard deviation of the sampling distribution for the sample means?
   Let X denote the population
   $E(X)=\frac{0+12}{2}=6$
   $Var(X)=\frac{(12-0)^2}{12}=12$
   $\mu_{\overline{X}}=E(\overline{X})=E(X)=6$
   $\sigma^2_{\overline{X}}=\frac{Var(X)}{n}=\frac{12}{36}=0.3333$
   $\sigma_{\overline{X}}=\sqrt{0.3333}=0.5773$

   (b) (8 Marks) What is the probability that the sample mean is between 5.93 and 6.05?
   Since $n=36>25$, the central limit theorem applies.
   $P(5.93<\overline{X}<6.05)=P(\frac{5.93-\mu_{\overline{X}}}{\sigma_{\overline{X}}}<\frac{\overline{X}-\mu_{\overline{X}}}{\sigma_{\overline{X}}}<\frac{6.05-\mu_{\overline{X}}}{\sigma_{\overline{X}}})=P(\frac{5.93-6}{0.5773}<Z<\frac{6.05-6}{0.5773})$
   $=P(-0.12<Z<0.09)=P(Z<0.09)-(1-P(Z<0.12))=0.5359-(1-0.5478)=0.0837$

   (c) (8 Marks) With probability 0.9 the sample mean is greater than what value?
   $P(\frac{\overline{X}-\mu_{\overline{X}}}{\sigma_{\overline{X}}}>-Z_{0.9})=0.9$
   $\overline{X}>\mu_{\overline{X}}-Z_{0.9}\sigma_{\overline{X}}=6-1.28\times0.5773=5.2611$

3. Let X_1, X_2, X_3, and X_4 be a random sample of observations from a population with mean μ and variance σ^2. Consider the following two point estimators of μ:
   $\widehat{\theta}_1=0.10X_1+0.20X_2+0.40X_3+0.30X_4$, and
   $\widehat{\theta}_2=0.25X_1+0.25X_2+0.30X_3+0.20X_4$

(a) (12 Marks) Show that both estimators are unbiased.
$E(\widehat{\theta}_1) = E(0.10X_1 + 0.20X_2 + 0.40X_3 + 0.30X_4) = 0.10E(X_1) + 0.20E(X_2) + 0.40E(X_3) + 0.30E(X_4) = 0.10\mu + 0.20\mu + 0.40\mu + 0.30\mu = \mu$
$E(\widehat{\theta}_2) = E(0.25X_1 + 0.25X_2 + 0.30X_3 + 0.20X_4) = 0.25E(X_1) + 0.25E(X_2) + 0.30E(X_3) + 0.20E(X_4) = 0.25\mu + 0.25\mu + 0.30\mu + 0.20\mu = \mu$
Therefore bothe the estimators are unbiased.

(b) (13 Marks) Which estimator is more efficient, $\widehat{\theta}_1$ or $\widehat{\theta}_2$? Explain in detail.
$Var(\widehat{\theta}_1) = Var(0.10X_1+0.20X_2+0.40X_3+0.30X_4) = 0.10^2Var(X_1)+0.20^2Var(X_2)+0.40^2Var(X_3)+0.30^2Var(X_4)$
$= 0.10^2\sigma^2 + 0.20^2\sigma^2 + 0.40^2\sigma^2 + 0.30^2\sigma^2 = 0.3\sigma^2$
$Var(\widehat{\theta}_2) = Var(0.25X_1+0.25X_2+0.30X_3+0.20X_4) = 0.25^2Var(X_1)+0.25^2Var(X_2)+0.30^2Var(X_3)+0.20^2Var(X_4)$
$= 0.25^2\sigma^2 + 0.25^2\sigma^2 + 0.30^2\sigma^2 + 0.20^2\sigma^2 = 0.255\sigma^2$
Since $Var(\widehat{\theta}_1) > Var(\widehat{\theta}_2)$, $\widehat{\theta}_2$ is more efficient.

4. A random sample of eight salespersons that attended a motivational course on sales techniques was monitored in the three months before and the three months after the course. The table shows the values of sales (in thousands of dollars) generated by these eight salespersons in the two periods. Assume that the population distributions are normal.

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(a) (12 Marks) Find an 80% confidence interval for the difference between the two population means.
$\overline{d} = \frac{1}{8}\left[(240-215) + (290-280) + (195-205) + (334-330) + (190-170) + (184-198) + (210-204) + (290-\right.$
$= \frac{1}{8}(25 + 10 - 10 + 4 + 20 - 14 + 6 + 15) = 7$
$s_d^2 = \frac{1}{8-1}\left[(25-7)^2 + (10-7)^2 + (-10-7)^2 + (4-7)^2 + (20-7)^2 + (-14-7)^2 + (6-7)^2 + (15-7)^2\right]$
$= \frac{1}{7}(324 + 9 + 289 + 9 + 169 + 441 + 1 + 64) = 186.5714$
$s_d = \sqrt{186.5714} = 13.6591$
The 80% confidence interval for the difference between the two population means is
$\overline{d} \pm t_{n-1,\alpha/2}\frac{s_d}{\sqrt{n}}$
$7 \pm t_{7,0.1}\frac{13.6591}{\sqrt{8}}$
$[0.1667, 13.8333]$

(b) (5 Marks) Based on the confidence interval in part (a), is there evidence that the population means are different?
The confidence interval in part (a) only includes positive values, indicating a higher population mean after the course.

(c) (8 Marks) Now suppose that we know the population standard deviation of the difference between sales values after the course and before the course is 10 thousand dollars. And we want to find a 95% confidence interval for the difference between the two population means of width at most 4 thousand dollars. How large a sample is needed to achieve such an interval?
$ME = Z_{\alpha/2}\frac{\sigma_d}{\sqrt{n}}$
$Z_{\alpha/2}\frac{\sigma_d}{\sqrt{n}} \leq 2$
$n \geq \frac{Z_{\alpha/2}^2\sigma_d^2}{4} = \frac{1.96^2 \times 10^2}{4} = 96.04$
The sample size should be at least 97.