Queen's University
Faculty of Arts and Sciences
Department of Economics
Winter 2008

Economics 250 : Introduction to Statistics

Midterm Exam I (make-up):
Time allowed: 120 minutes.

Instructions:

READ CAREFULLY. Calculators are permitted. At the end of the exam are several formulae. Answers are to be written in the examination booklet. Do not hand in the question sheet. You are to answer **ALL** questions. **SHOW ALL YOUR WORK**. Most of the marks are awarded for showing how a calculation is done and not for the actual calculation itself!

There are a total of 100 possible marks to be obtained. Answer all 9 questions (marks are indicated)

Do all 9 questions

1. **(10 marks)** The final grades in Economics of 14 students are recorded in the following,

   Grades X: 96, 78, 89, 61, 75, 95, 66, 79, 83, 71, 85, 74, 69, 75

   (a) Find the median grades ($\widetilde{X}$).
   (b) Find the mean grades ($\overline{X}$).
   (c) Find the MAD of grades.
   (d) Construct the relative frequency table of each class (Hint: You don't need to calculate the number of the class size, just simply choose the number of classes as 4. For example, the first class is 60-70).
   (e) Find the grouped mean and standard deviation.

   Answer:

   a) $\widetilde{X} = 76.5$.

   b) $\overline{X} = 85.5$.

Grades	midpoint	Frequency	% of frequency	$m_j * f_j$	$(m_j - \overline{X}^2)f_j$
60-70	65	3	21.43%	195	495.37
70-80	75	6	42.86%	450	48.74
80-90	85	3	21.43%	255	153.37
90-100	95	2	14.29%	190	588.25
		14	1.0	$\overline{X} = 77.85$	$s^2 = 98.9$

c) $MAD_X = 10.57$.

d), e)

2. **(10 marks)** If $\sum_{j=1}^{6} X_j = -4$, $\sum_{j=1}^{6} X_j^2 = 10$ and $\sum_{j=1}^{6} X_j Y_j = 5$, calculate

   (a) $\sum_{j=1}^{6}(2X_j + 3)$.

   (b) $\sum_{j=1}^{6} X_j(X_j - 1)$.

   (c) $\sum_{j=1}^{6}(X_j - 5)^2$.

   (d) $\sum_{j=1}^{6}(X_j + Y_j)(X_j - Y_j)$.

   Answer:

   a) $\sum_{j=1}^{6}(2X_j + 3) = 10$.

   b) $\sum_{j=1}^{6} X_j(X_j - 1) = 14$.

   c) $\sum_{j=1}^{6}(X_j - 5)^2 = 200$

   d) $\sum_{j=1}^{6}(X_j + Y_j)(X_j - Y_j) = \sum_{j=1}^{6}(X_j^2 - Y_j^2) = \sum_{j=1}^{6} X_j^2 - \sum_{j=1}^{6} Y_j^2 = 10 - 5 = 5$

3. **(10 marks)** The *symmetric difference* of two sets A and B is denoted $A \triangle B$ and is defined as follows:

$$A \triangle B = (A \cap \bar{B}) \cup (B \cap \bar{A})$$

   If A and B are two events, show using Venn diagrams that

$$P(A \triangle B) = P(A \cup B) - P(A \cap B)$$

4. **(10 marks)** Two distinguishable fair dice are thrown.

   (a) What is the sample space for this random experiment (i.e. what are all the possible outcomes)?

(b) What is the probability that a 2 appears?
(c) What is the probability that sum of the numbers is even?

Answer
(a) The sample space consists of 36 elements of the type (a, b) where a is the number on the first die and b is the number on the second die. That is,

$$S = \{(1,1),(2,1),(3,1),(4,1),(5,1),(6,1),(1,2),\ldots,(6,6)\}$$

Notice that because the dice are distinguishable, the outcomes (2, 1) and (1, 2) are different.

(b) The outcomes for this event are $(1,2),(2,2),(3,2),(4,2),(5,2),(6,2)$ and $(2,1),(2,2),(2,3),(2,4),$ There are therefore 12-1 = 11 outcomes in this event (we counted (2,2) twice. Therefore the P(2) = 11/36

(c) The sum of two odd numbers is even and the sum of two even numbers is even. The sum of an odd and even number is not even! Therefore, we are looking for all outcomes in which the two numbers are either both odd or even. The number of ways of getting two odd numbers is just (3)(3) = 9. Similarly, the number of ways of getter two even number is just (3)(3) = 9. Thus, the probability is (9+9) /18 = 1/2

Note, the outcomes for this events are
$(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(1,3),(3,1),(1,5),$
$(5,1),(3,5),(5,3),(2,4),(4,2),(2,6),(6,2),(4,6),(6,4)$

5. **(10 marks)** One bag contains 4 white balls and 2 black balls; another contains 3 white balls and 5 black balls. If one ball is drawn from each bag, find the probability that

   (a) both are white.

   (b) both are black.

   (c) one is white and one is black.

Answer: Let $W_1 = event\ white\ ball\ from\ first\ bag$, $W_2 = event\ white\ ball\ from\ second\ bag.$

a)

$$P(W_1W_2) = P(W_1)P(W_2) = (\frac{4}{4+2})(\frac{3}{3+5}) = \frac{1}{4}$$

b)

$$P(\overline{W}_1\overline{W}_2) = P(\overline{W}_1)P(\overline{W}_2) = (\frac{2}{4+2})(\frac{5}{3+5}) = \frac{5}{24}$$

c) want to know $W_1\overline{W}_2 + \overline{W}_1W_2$. Since event $W_1\overline{W}_2$ and $\overline{W}_1W_2$ are mutually exclusive, we have

$$\begin{aligned} P(W_1\overline{W}_2 + \overline{W}_1W_2) &= P(W_1\overline{W}_2) + P(\overline{W}_1W_2) \\ &= P(W_1)P(\overline{W}_2) + P(\overline{W}_)P(W_2) \\ &= (\frac{4}{6})(\frac{5}{8}) + (\frac{2}{6})(\frac{3}{8}) \\ &= \frac{13}{24} \end{aligned}$$

6. **(15 marks)** Given the following set of data for the variable X,

$$X: \quad 10,\ 14,\ 14,\ 14,\ 16,\ 16,$$

(a) Find the mean and variance of X.

(b) Determine the values which result from making the transformation $Z = (X - \overline{x})/s_x$.

(c) Show numerically that the distribution of the transformed variables Z is a unit distribution. That is, show that $\overline{z} =$ and $s_z^2 = 1$.

Answer:

a) We have

$$\overline{x} = \frac{10 + 14 + \ldots + 16}{6} = 14$$

$$\begin{aligned} s_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2 &= \frac{1}{6}[(10-14)^2 + (14-14)^2 + \ldots + (16-14)^2] \\ &= \frac{1}{6}(16 + 0 + \ldots + 4) = 4. \end{aligned}$$

$$s_x = 2$$

b) Under the transformation

$$Z = \frac{X - \overline{x}}{s_x} = \frac{X - 14}{2},$$

the values of Z become Z: -2, 0, 0, 0, 1, 1.

c) Next,

$$\overline{z} = \frac{1}{n}\sum_{i=1}^{n} z_i = \frac{1}{6}(-2 + \ldots + 1) = 0$$

and

$$\begin{aligned} s_z^2 = \frac{1}{n}\sum_{i=1}^{n}(z_i - \overline{z})^2 &= \frac{1}{6}[(-2-0)^2 + (0-0)^2 + \ldots + (1-0)^2] \\ &= \frac{6}{6} = 1. \end{aligned}$$

7. **(10 marks)** Two fair coins are tossed. A gambler wins two dollars if both coins show heads and otherwise loses a dollar. What is his expected gain? What is the variance of his gains? Would you call the game fair?

Answer: The expected gain is

$$\begin{aligned} &2 \times P(\text{two heads}) + (-1) \times P(\text{not two heads}) \\ &= 2 \times (1/4) + (-1) \times (1 - 1/4) \\ &= 1/2 - 3/4 \\ &= -1/4 \end{aligned}$$

The variance of the expected gains is

$$\begin{aligned} &(2 - (-1/4))^2 \times P(\text{two heads}) + (-1 - (-1/4))^2 \times P(\text{not two heads}) \\ &= 49/16 \times (1/4) + 25/16 \times (1 - 1/4) \\ &= 31/16 \\ &\approx 2 \end{aligned}$$

Since the game has a negative expected gain for the gambler we conclude that it is not fair.

Grade	price $	P(X=x)
A	5.0	0.2
B	4.5	0.4
C	4.0	0.3
D	3.5	0.1
		1.0

8. **(15 marks)** Suppose Brown Lumber Company sells four grades of 1/4-inch den paneling. Further, suppose that the probability that a panel-purchase customer will select a specific grade and the corresponding price are as given in the table below: Suppose the company from which Brown Lumber Company buys paneling is increasing its price by 10 percent; and further more, the city decides to collect a tax of $0.25 for each panel sold. Brown plans to increase the price of paneling accordingly, i.e. $0.25 + 1.1X$.

   (a) Find the expected price per panel prior to the price increase.
   (b) Find the expected price per panel after the price increase.
   (c) Use individual new prices to find the expected price per panel after price increase.
   (d) What can you conclude from (b) and (c).

   Assume in each case that relevant probabilities will be unchanged by the price increases.

Answer:

a) $E(X) = 5(0.2) + 4.5(0.4) + 4(0.3) + 3.5(0.1) = 4.35$

b) $E(0.25 + 1.1X) = 0.25 + 1.1E(X) = 5.035$.

c) We first determine each of the individual prices after the increase:

Grade	price $	P(X=x)
A	0.25+1.1(5.0)=5.75	0.2
B	0.25+1.1(4.5)=5.2	0.4
C	0.25+1.1(4.0)=4.65	0.3
D	0.25+1.1(3.5)=4.1	0.1
		1.0

Then, $E(X) = 5.75(0.2) + 5.2(0.4) + 4.65(0.3) + 4.1(0.1) = 5.035$.

d) The results of (b) and (c) are identical.

9. **(10 marks)** Roads A, B and C are the only escape routes from Kingston Penitentiary. Prison records show that, of the prisoners who tried to escape, 30% used road A, 50% used road B, and 20% used road C. These records also show that 80% of those who tried to escape via A, 75% of those who tried to escape via B, and 92% of those who tried to escape via C were captured. What is the probability that a prisoner who succeeded in escaping used road C?

   (Hint: if X, Y are any two events then $P(X|Y) = 1 - P(\bar{X}|Y)$)

Answer:

We are looking for $P(\text{C was used}|\text{Escaped})$. Using Bayes' rule we know that:

$$P(\text{C was used}|\text{Escaped}) = \frac{P(\text{C was used} \cap \text{Escaped})P(\text{C was used})}{P(\text{Escaped})}$$

We know $P(\text{C was used})$ is simply 20%. Also, using the hint we know that $P(\text{Escaped} \cap \text{C was used}) = 1 - P(\text{Caught} \cap \text{C was used}) = 0.08$.

Now,

$$\begin{aligned} P(E) &= P(E|A)P(A) + P(E|B)P(B) + P(E|C)P(C) \\ &= (0.2)(0.3) + (0.25)(0.5) + (0.08)(0.2) \\ &= 0.201 \end{aligned}$$

Thus, $P(\text{C was used}|\text{Escaped}) = (0.08)(0.2)/(0.201) \approx 0.0796$.

Formula Sheet

Statistics Formulas

Notation

- All summations are for $i = 1, \dots n$ unless otherwise stated
- $\sim$ means 'distributed as'

Population Mean

$$\mu \ = \ \frac{1}{N} \sum_{i=1}^{N} x_i$$

Sample Mean

$$\bar{X} \ = \ \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Mean Absolute Deviations (MAD)

$$MAD = \frac{1}{n} \sum_{i=1}^{n} \mid X_i - \overline{X} \mid$$

Population Variance

$$\begin{aligned} \sigma^2 \quad &= \quad \frac{1}{N} \sum_{i=1}^{N} [x_i \ - \ \mu]^2 \\ &= \quad \frac{1}{N} \sum_{i=1}^{N} {x_i}^2 - \mu^2 \end{aligned}$$

Sample Variance

$$s^2 \quad = \tfrac{1}{n-1} \textstyle\sum_{i=1}^{n} [x_i \; - \; \bar{X}]^2$$

Alternatively

$$s^2 = \tfrac{1}{n-1} [\textstyle\sum_{i=1}^{n} x_i^2 \; - \; n\bar{X}^2]$$
$$s^2 = \frac{\sum_{i=1}^{n} x_i^2 - \frac{(\sum X_i)^2}{n}}{n-1}$$

Grouped Data (with k classes)

$$\bar{X} = \frac{1}{n} \sum_{j=1}^{k} \nu_j f_j \quad \textit{where } \nu_j \textit{ is the class mark for class } j$$
$$s^2 = \frac{1}{n-1} \sum_{j=1}^{k} f_j (\nu_j - \bar{X})^2$$

Probability Theory

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (\textit{additive law})$$
$$P(A \cap B) = P(B) P(A \mid B) \quad (\textit{multiplicative law})$$

If E_i is mutually exclusive and exhaustive events for $i = 1, \ldots n$, then

$$P(A) = \sum_{i}^{n} P(A \cap E_i) = \sum_{i}^{n} P(E_i) P(A \mid E_i)$$
$$P(E_i \quad \mid \quad A) = \frac{P(E_i) P(A \mid E_i)}{P(A)} \quad (\textit{Bayes's Theorem})$$

Counting Formulae

$$_NP_R = \frac{N!}{(N-R)!}$$
$$_NC_R = \binom{N}{R} = \frac{N!}{(N-R)!R!}$$

Random Variables

Let X be a discrete random variable, then:

$$\begin{aligned}
\mu_x &= E[X] = \sum_x xP(X=x) \\
\sigma_x^2 &= V[X] = E[(x-\mu_x)^2] = \sum_x (x-\mu_x)^2 P(X=x) \\
\sigma_x^2 &= E[X^2] - (E[X])^2
\end{aligned}$$

If X and Y are independent random variables and a, b, and c are constants, then:

$$\begin{aligned}
E[a+bX+cY] &= a + b\mu_x + c\mu_y \\
V[a+bX+cY] &= b^2\sigma_x^2 + c^2\sigma_y^2
\end{aligned}$$

Coefficient of Variation (CV)

$$\begin{aligned}
CV &= \frac{\sigma}{\mu} \times 100\% \; for\ population \\
CV &= \frac{s}{\bar{X}} \times 100\% \; for\ sample
\end{aligned}$$