4. DYNAMIC EQUILIBRIUM MODELS II: GROWTH

> *Knowledge is the only instrument of production that is not subject to diminishing returns.*
>
> – John Maurice Clark (1884–1963)

So far the dynamic economies we have studied have not been growing, and yet economic growth is one of the central subjects of macroeconomics. So in this section we shall study two simple but general models in which many issues concerning growth can be examined.

(a) Overlapping Generations

In the next type of dynamic economy we shall study, individuals again live for only two periods. The complication now is that at any time there are two generations or cohorts alive. If our generation is young at time t and old at time $t+1$ then we overlap with an older generation which is young at $t-1$ and old at t and with a subsequent younger generation. The model is appealing for two sorts of reasons. First, it allows us to study life-cycle saving. Individual behaviour when young typically differs from that when old and the model captures that. And there are a number of policy issues involved in having a heterogeneous population, and the two types of agents alive at any time in this model allow study of those. Second, it provides examples of competitive equilibria which are not Pareto optimal *i.e.* which are not efficient.

I'll just sketch the basic model, due to Samuelson and Diamond. Because agents live for two periods, the household planning problem is very similar to that in section 2:

$$\max U = u(c_{1t}) + \beta u(c_{2t+1})$$

subject to

$$\begin{aligned} c_{1t} + s_t &= w_t \\ c_{2t+1} &= (1 + r_{t+1})s_t \end{aligned}$$

The notation is obvious except that there are now two subscripts on consumption. The first one gives the age of the consumer, the second one gives the date (because the economy itself goes on forever). Agents work (inelastically) when young, receiving wage w and when old live off their savings, leaving nothing behind.

Notice that we can solve this problem by replacing s and observing that $c_{2t+1} - (1 + r_{t+1})(w_t - c_{1t}) = 0$. Call the Lagrange multiplier on this constraint λ. As usual we assume conditions on the curvature of u. Then foc include:

$$\begin{aligned} u'(c_{1t}) + \lambda(1 + r_{t+1}) &= 0 \\ \beta u'(c_{2t+1}) + \lambda &= 0 \end{aligned}$$

These combine to give:

$$u'(c_{1t}) = \beta(1 + r_{t+1})u'(c_{2t+1})$$

This is called an Euler equation. We have already dealt with examples in section 2 and we shall see variations later. So far I have used no specific functional form.

Let us be even more specific and return to log utility. Then the Euler equation is:

$$\frac{1}{c_{1t}} = \frac{\beta(1 + r_{t+1})}{c_{2t+1}}$$

Substituting this in the consolidated budget constraint gives:

$$\beta(1 + r_{t+1})c_{1t} = (1 + r_{t+1})(w_t - c_{1t}) = (1 + r_{t+1})w_t - (1 + r_{t+1})c_{1t}$$

$$c_{1t} = \frac{w_t}{1 + \beta}$$

much like earlier results for the two-period economy. Savings are given by

$$s_t = w_t - c_{1t} = \frac{w_t \beta}{1 + \beta}$$

an increasing function of wage income. With these specific preferences savings does not depend on the interest rate (the income and substitution effect offset exactly). Note that this is saving by young agents – there is no representative agent in this model. Notice also that aggregate consumption is given by $c_{1t} + c_{2t}$.

So much for household behaviour, now let us turn to firms. Firms are price takers. They maximise profits and have production function $F(K, N)$ which has the usual properties. Here K is capital and N is the number of young workers. If all generations are the same size and we normalise to one we can write $F(K, N) = NF(K/N, 1) = Nf(k)$ because there are constant returns to scale. This is called the production function in *intensive* form, and k is the capital-labour ratio. We've used K for the aggregate capital stock so as to reserve k for the per capita capital stock.

The firm's problem is static. From the first-order conditions for profit maximisation:

$$F_N = f(k) - kf'(k) = w \quad \text{[use the chain rule]}$$

$$f'(k) = r. \quad [\text{with } \delta = 0]$$

The capital simply embodies the savings of the current old generation.

To solve for a competitive equilibrium we next find conditions for equilibrium in the goods and factor markets. For the goods market to clear:

$$k_{t+1} - k_t = s_t(w_t) - k_t$$

which simply holds that savings equals investment. The right hand side is the change in the capital stock and the left hand side is saving of the young minus dissaving of the old.

There is no depreciation (though that could be added easily). The capital stock increases only if the amount set aside or saved by the young exceeds the amount set aside last period by the current old, who withdraw their savings in this period.

Since there is only one type of good this gives equilibrium in the capital market too, combined with $f'(k_t) = r_t$. Meanwhile, in the labour market, labour is supplied inelastically so the wage rate can be read off from the firm's demand curve for labour, which is given above. One could add choices over leisure and hence an elastic labour supply, just as in two-period models.

To make the example even clearer let us take a Cobb-Douglas functional form for the production function, combined with logarithmic utility. Then we have:

$$\begin{aligned} s_t &= \frac{w_t \beta}{1+\beta} \\ k_t^{\alpha} - k_t \alpha k_t^{\alpha-1} &= k_t^{\alpha}(1-\alpha) = w_t \\ \alpha k_t^{\alpha-1} &= r_t \\ k_{t+1} &= s_t(w_t). \end{aligned}$$

As in many macroeconomic models we have seen already the model can be written as a system of equations. Here the system is derived from optimization problems and is dynamic.

Thus by substitution:

$$k_{t+1} = \frac{\beta}{1+\beta} \cdot k_t^{\alpha}(1-\alpha)$$

If we can solve this difference equation for k_t then clearly we will have a complete solution since we can then read off w_t, r_t, and s_t (and hence the consumptions). It is a general property of OLG models that they give rise to a nonlinear difference equation in an endogenous variable (capital or the price level, say). To study this equation is to study equilibria of the model.

This example is linear in logs; so one could take logs and then solve a linear difference equation and then exponentiate the solution, since this example is non-stochastic. An even simpler approach would be to assign numbers for β and α and then take various starting values and simulate paths for k on a computer or hand calculator.

We can see directly whether or not a steady state exists. A steady state is simply a situation in which $k_{t+1} = k_t$. Take the derived equation of motion for the capital stock and remove the time subscripts:

$$k = \frac{\beta}{1+\beta} \cdot k^{\alpha}(1-\alpha)$$

or

$$k^* = \left[\frac{\beta(1-\alpha)}{1+\beta}\right]^{\frac{1}{1-\alpha}}$$

You can check whether this has positive solutions for k for given values of the technology parameter α and the preference parameter β. The phase diagram for k_{t+1} and k_t can tell us about the uniqueness and stability of any steady-state equilibria that exist. In this example it is easy to show that $1 > dk_{t+1}/dk_t > 0$ evaluated at k^*. Next, see if you can find the steady-state values for other endogenous variables: savings, consumption of young and old, the wage rate, and the interest rate.

In the example we have worked out there is no growth. When N is growing it turns out that K grows at the same rate so that k again may have a steady-state value. Consumption also will grow at the same rate as population. This is our first example of a *balanced growth path.* To see this, suppose that:

$$N_t = (1+\eta)N_{t-1}.$$

Then s_t is the saving of an individual, so the market clearing condition is:

$$K_{t+1} = L_t \cdot s_t,$$

so that

$$(1+\eta)k_{t+1} = s_t.$$

Thus K also grows at rate η. So does output.

So far, we've found the steady state, and we could use this to compare economies with different parameters, for example with different population growth rates. But we also can use the model to study *transitional dynamics*, defined as the paths of the variables after a shock to the steady state. For example, imagine a change in the population growth rate, η, or the loss of part of the capital stock due to a natural disaster. The model predicts the complete subsequent time path for the economy. The exercises contain a number of examples of applications like these. We'll defer many other applications (including the possibility of over-saving) to the next sub-section, since they also can be studied with the Solow-Swan model.

Fruitful additions to the basic OLG model include: uncertainty, bequests, money (as an additional vehicle for savings), and a government. Many of the macroeconomic issues we discuss in the remainder of the course could be studied in OLG environments. Some of the exercises deal with these additions, and so you might find them easier after you've completed sections 4 and 5 of the course.

(b) Solow-Swan Growth

The next dynamic model we shall study again involves growth. Like the OLG model its competitive equilibrium gives rise to a first-order difference (or, in continuous time, differential) equation in the capital stock. This equation turns out to be quite easy to study. Unlike the OLG model all individuals are identical. The model is real and has fully employed resources.

The model was developed in the 1950s separately by Solow and Swan, and based on earlier work by Domar, Harrod, and Kaldor. It was designed to be consistent with some rough facts concerning growth, such as approximate constancy of the capital-labour ratio and the absence of trends in factor returns.

The consumption behaviour underlying the simplest Solow-Swan one-sector growth model is very simple. There is again a single good (for both consumption and investment). The model is based on three assumptions:

• *Assumptions*

(1) the labour force input grows exogenously at rate η: $N_t = N_0 exp(\eta t)$. We could do this in discrete time just as easily, but differentiation is simpler. Labour is supplied inelastically.

(2) investment and savings are fixed fractions of output: $S = sY$, so that $I = S = dK/dt$ (an ordinary time derivative) $= sY$. This is one of the main differences from the OLG model. By assuming a constant savings rate we avoid having to solve an infinite-horizon consumption planning problem (in the OLG case that plan is simple to find because the horizon is just two periods). Of course, savings rates are not constant and hence more sophisticated versions of the model will not feature this simplification.

(3) Y or $Q = F(K, L)$, an hd1 function *i.e.* CRS. To start with, I'll work with cases in which $\delta = 0$, so that there is no depreciation. Thus $q = Q/L = F(K/L, 1) = f(k)$ which is called the production function in intensive form. Then $f'(k) > 0$, $f''(k) < 0$.
e.g. Cobb-Douglas: $q = k^\alpha$

Before we put these three assumptions together, a few words on notation. N will represent the labour force. Later on, we'll use L to denote the *effective* labour force or labour input: N adjusted for changes in labour productivity. For now, though, $L = N$. Next, lower-case letters will denote division by the labour input. So $k \equiv K/L$ and so on. The exception to this rule is s, which is the saving rate and *not* per capita savings. Finally, recall from section 1 of the course that dots denote log time derivatives, continuous-time growth rates.

• *Equilibrium*

An equilibrium is defined as a situation in which factor (capital and labour) markets clear and the goods market clears. A sufficient and necessary condition for equilibrium is $S = I$ (which is similar to the market-clearing condition in the OLG model). Remember that there is only one good and that labour is supplied inelastically.

This implies

$$\dot{k} = \dot{K} - \dot{L} = \frac{sQ}{K} - \eta = \frac{sf(k)}{k} - \eta$$

This is a differential equation, the fundamental equation of the growth model. This non-linear, first-order differential equation is easy to work with. We could do the same thing in discrete time or with some randomness in the population growth rate.

• *Steady-State Equilibrium*

In a steady state, quantities are growing at constant rates (which in some cases are zero). This is another example of *balanced growth*, because a number of variables may be growing at the same rate. In steady-state equilibrium $\dot{k} = 0$ so

$$q^* = f(k^*) = \frac{\eta k^*}{s}$$

where stars denote steady-state values. In (k, q) space note that there is a ray from the origin with slope η/s, which gives the steady state. Show that this is stable, by drawing the phase diagram: $\dot{k}$ v. k. Also see if you can show why the Inada conditions are adopted.

Once we have found k^*, values for the other endogenous variables follow from that (just as in the case of the real OLG model earlier). We can study both steady states and transitions, and we can do so in the general case, using a diagram, or using the Cobb-Douglas example.

- *Solution given k*

Recall from section 2 that in a competitive economy:

$$\pi = f'(k) = r + \delta$$

just the slope of the intensive production function, and since $Q = Lf(k)$:

$$w = \frac{\partial Q}{\partial L} = Lf'(k)(-K/L^2) + f(k) = f(k) - kf'(k)$$

It is easy to show that factor shares exhaust output:

$$Lw = Lf(k) - Lkf'(k) = Q - Kf'(k) = Q - K\pi.$$

To get the shares, divide by Q. Try illustrating these shares graphically. Note that relative shares are constant:

$$\frac{\pi K}{Q} = \frac{\pi k}{q} = \frac{f'(k)k}{q}.$$

As an exercise, show that for f Cobb-Douglas the factor shares are α and $1 - \alpha$.

What about savings and investment and consumption? Clearly

$$S = sQ = \frac{Q\eta k}{q} = \eta K$$

using the growth equilibrium. So $Q = C + S$, so $q = c + sq$, so

$$C = Lf(k) - \eta K \quad \text{and} \quad c = f(k) - \eta k$$

This means savings per capita $= (S/L) = \eta k$ is constant since η is constant and $\dot{k} = 0$ in equilibrium. Thus S grows at rate η, Q grows at rate η, so C grows at rate η, and c is constant. As an exercise, solve for these variables in the Cobb-Douglas example.

- *Depreciation*

An obvious modification is to allow for the depreciation of the capital stock. It is time to do this explicitly. Now $dK/dt = I_{\text{net}} = I_{\text{gross}} - \delta K = S - \delta K$ so K depreciates at rate δ. Divide by K:

$$\dot{K} = \frac{S}{K} - \delta$$

So the fundamental equation of the Solow-Swan growth model becomes:

$$\dot{k} = \dot{K} - \dot{L} = \frac{sQ}{K} - \delta - \eta = \frac{sf(k)}{k} - (\delta + \eta)$$

With this modification the analysis proceeds as before. I'll leave this as an (important) exercise.

- *Predictions*

We next study the properties of equilibrium. Be careful to remember which parameters or ratios are fixed and which are endogenous. For example, δ, η, and s are fixed, while w, c, π, and r are things we must solve for. We shall first study steady-state growth paths, then transitional dynamics (between steady states, in response to shocks), and then cross-country implications.

Let us compare two, steady-state growth paths. Imagine they differ only because of different savings rates. With higher s the ray from the origin is shallower. In steady-state equilibrium, everything grows at rate η; that result does not depend on s. But the economy with higher s has higher K and Q, a higher capital/output ratio, and a lower rate of profit (slope of $f(k^*)$).

We may also study the *transitional dynamics*. I refer to thought experiments of a different kind now. Suppose that in a single economy s rises to a new constant level. What happens on the way to the new steady-state equilibrium? En route K and Q grow faster than rate η, so only in transition does a higher savings rate lead to higher growth rate.

As a second thought experiment, deduce the predictions when K is suddenly halved, with no change in s or η. The model predicts that the rate of return to capital in Japan in 1946 must have been extremely high. It also is consistent with the rapid postwar growth of Japan, West Germany, France, and the USSR and with the optimism about development which followed the Marshall Plan. But most less-developed countries are not like developed countries whose physical capital stocks have been partly destroyed.

Exercise: Consider the Chinese policy of reducing η. Find the predicted effects on the steady-state levels q, w, r, c. Describe the effects on growth, both in the steady state and in the transition to it.

So far we have used the model to predict the effects (on levels, steady-state growth rates, and transitional growth rates) of some changes over time in a single country, in η or s, for example. The competitive equilibrium here also yields some interesting predictions for cross-country properties of growth rates and for the interaction between countries.

Exercise: Suppose that that Mexico and the United States have the same technology but that Mexico has a higher population growth rate. Under the basic neoclassical model and autarky which country grows faster? Which country has higher real wages? If the borders are opened to movements of capital and labour what would you expect to happen to relative wage, interest, and growth rates?

Suppose that we apply the one-sector Solow-Swan model to the two countries, with parameters as follows:

Mexico	**U.S.A.**
$s = .15$	$s = .15$
$\delta = 0$	$\delta = 0$
$\eta = 0.05$	$\eta = 0.01$

Thus the only difference is in population growth rates.

Then it is simple to see by algebra or a diagram that, if the countries have the same production function, the U.S.A. has higher real wages and a lower real interest rate than does Mexico. If there is movement of capital (south) *or* labour (north) across the border then (though this may take some time) the prediction is that wage and interest rates will be equalized. This exercise should suggest why U.S. capitalists and Mexican labourers might prefer integration to autarky.

Observe that relative factor prices depend only on k, the capital-labour ratio. In the model, labour tends to move from our fictional Mexico to the United States so that k is equal across the two countries. But there is an alternative mechanism, which is to move capital to Mexico. Large and continuing movements of labour across national boundaries often are associated with political controversy (for example in South Africa, Israel, France, Germany). One reason for American interest in free trade with Mexico may be to improve incentives for capital to move there (because firms would then be able to export back to the U.S.) and thus reduce incentives for labour to move to the United States.

The model under autarky (*i.e.* with borders closed) predicts that Mexico will grow faster than the U.S., though it will be less capital intensive and will have lower output and consumption per capita. With borders open to factor movements the growth rates will be equalized (think of one country's ray rotating down and another's rotating up, along a production function). In both countries, output will increase at a rate between 0.01 and 0.05.

However, with or without trade the model predicts the same growth rate for per capita income in each country, namely zero. That is because $\dot{Q} = \eta$ in this model. So the model is not as successful in this respect, since it cannot mimic growth in per capita income, which is observed in some countries.

This exercise may seem odd because it assumes that the two countries have access to the same technology. In a few pages we shall allow for differing rates of productivity growth and see how the predictions change. Meanwhile, let us summarize some of the predictions from the growth model:

▷ high population growth is associated with low output and consumption per capita (in autarky or for the world economy)

▷ high savings rates are associated with high output and consumption per capita (in autarky or for the world economy) and with fast growth in transitions

▷ labour flows from poor to rich countries

▷ capital flows from rich to poor countries

▷ high population growth causes high output growth

▷ there is no steady-state growth in output per capita

▷ the growth rate of output per capita is the same across countries

I have listed these in order, starting with the most successful predictions and ending with the ones that are wildly rejected. We'll revisit them once we study productivity growth, but first we digress to consider some normative questions.

• *Optimality and dynamic efficiency*

It's natural to ask about the optimality properties of the competitive equilibrium. In doing this, we'll come across the concepts of *dynamic efficiency* and *golden rules*. In the Solow-Swan model, to ask whether the growth path is optimal or not is the same as asking whether the economy is saving the right amount or not. Is s too high or too low?

Consider a preliminary definition of optimality as involving maximum steady-state consumption per capita. Recall that consumption per capita is $c = f(k) - (\eta + \delta)k$. To maximize c:

$$\frac{dc}{dk} = f'(k) - \eta - \delta = 0$$

and note that the second-order conditions for a maximum are satisfied since $f''(k) < 0$. This gives the *golden rule* consumption level of the capital stock, defined implicitly by:

$$f'(k_g) = \eta + \delta$$

at which the rate of profit equals the rate of population growth plus the depreciation rate. At k_g the function f is parallel to a ray from the origin with slope $\eta + \delta$. Again let me remind you that k and r are endogenous. Imagine varying s so that the usual ray from the origin intersects $f(k)$ at k_g. Notice that k_g can be greater than or less than k^*.

The golden rule can be expressed as

$$r = \eta,$$

where η is exogenous and r is endogenous. If $r > \eta$ then the economy is undercapitalized or the savings rate is too low. If $r < \eta$ then the economy is overcapitalized or the savings rate is too high. Recall that there is an inverse relationship between r and k.

The idea here is that an arbitrary saving rate won't necessarily maximize steady-state consumption per capita. It could be too low or, suprisingly, too high. The golden rule tells us which in direction the saving rate should change, if we accept the optimality criterion. This brings us to the concept of *dynamic efficiency.* Recall that Pareto efficiency holds if no one can be made better off without someone else being made worse off. Dynamic efficiency simply applies this idea to successive generations. We shall show that an equilibrium with $k^* > k_g$, that is too high a savings rate, that is $r < \eta$, is dynamically inefficient. In the steady state each generation consumes $c^* < c_g$ by definition. This period, we should consume an extra amount $k^* - k_g$, then consumption per capita would be higher in each successive period. What would a dynamically inefficient economy look like? If $r < \eta$ then income grows faster than debts do. You could borrow money today and never pay off the debt, yet the debt-income ratio would fall over time.

In contrast, if $k^* < k_g$ the economy is not dynamically inefficient. Why? Because to increase future consumption a sacrifice is demanded of the current generation, they must raise the savings rate.

It is easy to set up examples in which the competitive equilibrium is dynamically inefficient *i.e.* is overcapitalised relative to the golden rule equilibrium so that $k^* > k_g$. Notice that the golden rule equilibrium is efficient since it maximises consumption per capita and treats all generations equally. But, just as in static economies, there are many efficient allocations with $k < k_g$ (with different weights on different people) and different social welfare functions will pinpoint different ones.

We have seen that a society would choose to accumulate k_g if it weighted subsequent generations equally. So you can imagine that a society that discounted the utilities of future generation would have a steady-state capital-labour ratio less than k_g. Suppose that the social welfare function involves a constant discount factor θ. Then here is the maximisation problem that the planner would consider:

$$\max \int_0^\infty exp(-\theta t) \cdot [f(k(t) - (\eta + \delta)k(t)]dt$$

subject to

$$\dot{k} = \frac{sf(k)}{k} - \eta - \delta.$$

One of the first-order conditions is the modified golden rule:

$$f'(k_o) = \eta + \delta + \theta,$$

so k_o will be less than k_g. The golden rule arises in the special case in which $\theta = 0$.

This modified golden rule allows for discounting of future utilities, but it still assumes that there is a constant savings rate, and then tries to select the best one. If the savings rate is not constant, and instead there is intertemporal budgetting based on a period utility function $u(c_t)$ then at the optimum:

$$f'(k_R) + \frac{du'(c)/dt}{u'(c)} = \eta + \delta + \theta$$

which is called the Ramsey rule. This looks a bit unfamiliar, so let us also look at the discrete-time version:

$$u'(c_t) = \frac{[1 + f'(k_t) - \delta]u'(c_{t+1})}{(1+\theta)(1+\eta)}$$

See if you can explain the intuition behind this equation. We have derived this type of Euler equation (for competitive equilibria) in the case of a two-period horizon.

- *Technical Progress*

Now it is time to make the model more realistic by modifying it so that output and the capital stock may grow faster than the labour input in the steady state *i.e.* by allowing productivity growth or technical change.

The simplest way to allow for productivity growth is to assume that it affects labour and that it occurs at exogenous rate λ. The idea is that labour input cannot be measured simply by counting persons. Growth in productivity of workers could reflect education (*e.g.* in courses other than this one) or vaccination programs. Then we simply adopt a change of variables. For L above read *effective* labour input, or labour measured in efficiency units. Call N population or labour supply in hours. We know that

$$N_t = N_0 exp(\eta t),$$

and

$$L_t = N_t exp(\lambda t) = N_0 exp[(\eta + \lambda)t].$$

so

$$Q(t) = F(K(t), L(t)) = F(K(t), N(t) exp \lambda t)$$

So effective labour inputs grow at rate $\eta + \lambda$. In the one-sector neoclassical growth model so do capital and output. But now the model is consistent with the rough fact that output and capital grow faster than population in many developed economies (*e.g.* Quebec, Japan, Sweden). The difference between $\dot{K}$, $\dot{Q}$ on the one hand and $\dot{N}$ on the other is given by the rate of growth of labour productivity.

To solve the model, let us use lower case letters to denote division by L and *not* by N. Thus $k = K/L$ and $q = Q/L$. The idea here is that these lower-case variables will continue to be trendless; but they no longer represent *per capita* measures.

A few moments reflection should convince you that the basic equation of the growth model gives:

$$f(k^*) = \frac{(\eta+\lambda+\delta)k^*}{s},$$

which yields k^*. In this economy N grows at rate η, but L, which enters the production function, grows at rate $\eta+\lambda$, so that K and Q also grow at rate $\eta+\lambda$.

Use of the national income identities can help you to avoid mistakes in figuring out growth rates. One version of the identity is

$$Q = C + S.$$

Thus C and S both grow at rate $\eta+\lambda$ (if they didn't then one of them would eventually account for all of output). Thus per capita consumption C/N (which is not c!) grows at rate λ.

The other national income identity is

$$Q = wN + \pi K.$$

Q grows at rate $\eta+\lambda$. K grows at rate $\eta+\lambda$. Thus π grows at rate zero. In a steady-state growth equilibrium it is constant. Also, N grows at rate η. Thus w must grow at rate λ (otherwise the labour share of output would vanish or become arbitrarily large). But that is exactly what one would expect: growth in labour productivity leads to growth in real wages.

To show all this I shall use the Cobb-Douglas example. Here $Q = F(K,L) = K^\alpha L^{1-\alpha}$, or $q = k^\alpha$. Thus we know that $f(k) = k^\alpha = (\eta+\lambda)k/s$, or that (if $\delta = 0$)

$$k^* = [\frac{\eta+\lambda}{s}]^{\frac{1}{\alpha-1}}.$$

Next, we can work out wages and profits. With zero depreciation,

$$r = \pi = F_K = \alpha K^{\alpha-1}L^{1-\alpha} = \alpha k^{\alpha-1},$$

so that

$$r^* = \alpha[\frac{\eta+\lambda}{s}]$$

And the real wage rate is

$$\begin{aligned} w = F_N &= K^\alpha(1-\alpha)N^{-\alpha}exp(\lambda t)^{1-\alpha} \\ &= K^\alpha(1-\alpha)N^{-\alpha}exp(\lambda t)^{-\alpha}exp(\lambda t) \\ &= (1-\alpha)k^\alpha exp(\lambda t), \end{aligned}$$

which grows at rate λ. The complete solution is:

$$w^* = (1-\alpha)\big(\frac{\eta+\lambda}{s}\big)^{\frac{\alpha}{\alpha-1}}exp(\lambda t).$$

Notice that the real wage is the derivative of the production function with respect to N not L. We could continue and describe consumption and per capita consumption as well.

This change allows us to mimic growth in labour productivity. Another fact associated with growth is that some countries grow much faster than others. In the basic model growth is at the rate of population growth. We know that population growth is lower in Japan than in Kenya so that model obviously cannot explain cross-country differences in growth rates. Can the model do so once we allow for technical progress? The surprising answer is: no. If there is any tendency for technical progress to be known to all (or for capital and labour to move) then the model predicts that growth rates should tend to be equalised (*adjusted for population growth*). There is not much evidence of that tendency. We shall see all this explicitly in a moment, but first let me pose an extension to the exercise suggested earlier.

Let us return to our two-country example. Suppose that now the U.S.A. has a higher rate of labour-augmenting technical progress than does Mexico. The parameters are:

Mexico	**U.S.A.**
$s = .15$	$s = .15$
$\delta = 0$	$\delta = 0$
$\eta = 0.05$	$\eta = 0.01$
$\lambda = 0.00$	$\lambda = 0.03$

My choice of parameter values, used in our earlier results, means that the U.S.A. will grow at a rate of 4% while Mexico will grow at a rate of 5%. But per capita income will grow at 3% in the U.S. and zero in Mexico. So we have 'corrected' one of the implausible features of the original model. Now there can be growth in per capita income, and countries with relatively fast population growth do not necessarily have relatively fast output growth. Of course this correction adds exogenous labour productivity growth at rate λ to exogenous population growth at rate η; ultimately both rates are endogenous.

Notice that r_{US} will be lower than r_{MEX} under autarky, and that real wages will grow more rapidly in the U.S.A. than in Mexico. Of course, if λ_{US} were higher than 0.03 you can see that r_{US} could well exceed the rate in Mexico so that *both* factor prices would be higher in the U.S.A.: but either way there will be an incentive for labour to move from Mexico to the United States.

You might suspect that our addition of labour-augmenting technical progress, and hence real wage growth, in our fictional U.S.A. would make it much more difficult to resist the economic impetus to immigration from Mexico. In fact, this addition makes that easier. All that is required is that the knowledge that brings about growth at rate λ be freely shared. If it cannot be kept secret then Mexico also begin to grow at rate $\eta + \lambda = 0.05 + 0.03$, and Mexican per capita income will grow at rate 0.03. This can happen even without movements of capital and labour. There will still be some differences between the two economies, in the absence of movements of labour and capital, but growth rates in per capita income will be equal across countries.

This last thought experiment shows us that there are still problems with this model. It again predicts the equalization of growth rates in per capita income: convergence. But in history there is not much evidence that growth rates tend to be equalized across countries. Explaining persistent differences in growth rates is one of the aims of modern theories of *endogenous* growth.

(c) Application: Convergence

We've seen that the Solow-Swan model predicts the convergence of growth rates across countries or regions. There are two versions of the convergence hypothesis. First, absolute or *unconditional* convergence implies that poor economies tend to grow faster per capita than rich ones. Usually this is tested by plotting per capita growth rates, against the initial level of real GDP per capita. When this plotting is done for a large set of countries, there is no evidence of a downward-sloping line through the dots, as unconditional convergence would imply.

If countries have different parameters and if they are not integrated, then the model predicts *conditional* convergence. In other words, each country converges to its steady state, but a country with a higher savings rate will have higher output per capita. The evidence for conditional convergence is stronger. For example, within OECD countries or U.S. states – which have similar parameters – there seems to be convergence.

Two pitfalls in testing for convergence are the *sample selection* problem and the *regression fallacy*. To understand the sample selection problem, imagine you chose OECD countries and regressed their grwoth since 1870 on their level of output per capita in 1870. A negative coefficient in this regression would be taken as evidence of convergence. However, this amounts to choosing countries which are similar (in eligibility to join the OECD) at the end of the time period, in other words countries which have converged! If instead you included countries like Uruguay (which was as prosperous as Australia in 1870 but has had slower growth since) then you would find much less evidence of convergence.

The regression fallacy, also called Galton's fallacy, is simply a reminder that extreme values may occur by chance. Suppose that growth rates were randomly distributed across countries during each decade. Imagine that you looked at growth rates in one decade and then in a second decade. Countries that had high growth in the first decade would tend to slow down in the second, and vice versa. But this would not be evidence of convergence to a common growth rate or level.

Why doesn't unconditional convergence occur? This is a key question in growth theory. In the simple model we have studied, the mechanism for convergence is the flow of factors between countries. So a related question is: Why doesn't capital flow from rich to poor countries? Lucas *AER(P)* (1990) has outlined several possible explanations, including differences in human capital and capital market imperfections. You should read his short article.

Tests of convergence are examples of attempts to explain the diversity of growth rates across countries. In convergence models, the only explanatory variable is the initial level of output per capita. More generally, economists have included a large number of regressors to try to explain the diversity. Explanatory variables that seem to be resilient are: the initial level, investment, and education.

One problem is that many different regressors may be correlated so that it is difficult to identify what accounts for differences across countries. There also may be some nonsense correlations. Here is an example. Suppose the dependent variable is g, the growth rate of real per capita income between 1960 and 1990. A regression for 95 non-OPEC, non-communist countries gives:

$$g = 103.9 + 80.3D_b - 43.0D_c,$$

where the coefficient on D_b has a t-value of 2.01 and that on D_c has a t-value of 1.03. D_b and D_c are dummy variables which respectively indicate countries where baseball and cricket are played. Clearly, switching from cricket to baseball in 1960 would have raised growth by 123% over the following thirty years.

(d) Growth Accounting

Our modification of the Solow-Swan model illustrates an example of technical progress, which is labour augmenting. I'll next illustrate neutral technical progress. Suppose that the production function is

$$Q(t) = Z(t)F(K(t), L(t))$$

Then the multiplicative term $Z(t)$ can grow, enhancing the productivity of both factors. An example might be air-conditioning, which makes possible tall glass buildings, air travel, and Houston, Texas. Decomposing output growth into growth in inputs and various kinds of technical progress is called *growth accounting.*

Analytically, growth accounting means taking the log of the production function, then differentiating with respect to time. Suppose that the production function is Cobb-Douglas and that there is both neutral and labour-augmenting technical progress:

$$Q(t) = Z(t) \cdot K(t)^{\alpha}[(N(t)exp(\lambda t)]^{1-\alpha}.$$

Remember that dotted variables are derivatives with respect to time of logs. Suppose that $Z(t) = exp(\zeta t)$, so that ζ is the rate of neutral technical progress. Then

$$\dot{Q} = \zeta + \alpha\dot{K} + (1-\alpha)\dot{N} + (1-\alpha)\lambda$$

Notice for the record that if $Q = F(K, N)$ then differentiating with respect to time gives:

$$\frac{dQ}{dt} = F_K\frac{dK}{dt} + F_L\frac{dL}{dt}$$

and then dividing by Q gives

$$\dot{Q} = \epsilon_K\dot{K} + \epsilon_L\dot{L}$$

where the ϵ's are the elasticities. These hold for any hd1 function ($\epsilon_K + \epsilon_L = 1$). In the Cobb-Douglas case the elasticity is also the factor share.

We can isolate the terms that are directly observable, so that

$$\dot{Q} - \alpha\dot{K} - (1-\alpha)\dot{N} = \text{sr}$$

This accounting is general in that it applies even *out of steady states*; it uses only the production function and does not assume that growth rates are constant. This difference is sometimes called total factor productivity (TFP) or the *Solow residual* for the time period in question. It is called a residual since it involves growth which is not accounted for by the growth in inputs. In our example the residual is accounted for by the two types of technical progress, and

$$\text{sr} = \zeta + (1-\alpha)\lambda.$$

Suppose that $\dot{Q} = 0.03$, $\dot{K} = 0.03$, $\dot{N} = 0.01$, and $\alpha = 0.3$. Then the residual growth rate is sr $= 0.014$. One possibility is that there is neutral technical progress of 1.4% per unit of time. Another is that there is only labour augmenting progress at rate 2% per unit of time. If the rate of technical progress were purely capital-augmenting it would have to be even faster (because of the smaller capital share α) to account for the Solow residual.

We'll next see some applications of growth accounting, to the 1970s productivity slowdown in Canada and other industrialized countries, and then to some east Asian economies. For Canada, the accounting looks something like this:

Canada

Time Period	1962-1973	1974-1979	1980-1986
Output Growth	5.4%	4.2%	2.7%
Employment Growth	2.8	2.8	1.4
Capital Accumulation	0.6	0.4	0.8
TFP Growth	2.0	1.0	0.5

As you can see, average output growth has declined. And the decline is due to slower employment growth and slower growth in TFP. Note that the input row entries are weighted by factor shares, so that the columns add up. For example, in the third column $1.4 = (1-\alpha)\dot{N}$.

How do these figures compare to those from the past? Historical data may involve some large measurement errors, but here is some earlier evidence (source: R. Pomfret *The Economic Development of Canada* p. 66):

Canada

Time Period	1891-1910	1910-1926	1926-1956
Output Growth	3.38%	2.46%	3.89%
Employment Growth	1.82	0.98	0.58
Capital Accumulation	0.81	0.31	0.61
TFP Growth	0.75	1.16	2.70

You can see that the productivity slowdown must be put in perspective.

It would be interesting to compare these results to those for other countries. Unfortunately, this accounting cannot be done for many countries, because of problems with data. But the OECD countries have standard data and the results for some of them are as follows:

United States

Time Period	1962-1973	1974-1979	1980-1986
Output Growth	3.8%	2.8%	2.2%
TFP Growth	1.5	-0.1	0.0

Japan

Time Period	1962-1973	1974-1979	1980-1986
Output Growth	9.7	3.8	3.8
TFP Growth	6.1	1.8	1.7

West Germany

Time Period	1962-1973	1974-1979	1980-1986
Output Growth	4.6	2.4	1.6
TFP Growth	2.8	1.8	0.8

Sweden

Time Period	1962-1973	1974-1979	1980-1986
Output Growth	2.4	1.8	0.8
TFP Growth	1.4	0.8	0.1

Notice that the productivity slowdown seemed to occur in a number of countries during the mid 1970s.

There are numerous explanations for the productivity slowdown. The main contenders seem to be: (a) measurement problems (from quality improvements, changes in utilization rates, or sectoral shifts); (b) the oil price shock; and (c) technological depletion.

Explanation (a) (measurement problems) has two versions. Some economists argue that unmeasured improvements in output quality have led to the underestimation of productivity growth. Others suggest that declines in input quality (due to a decline in schooling perhaps) have had the same effect. It is clear that if we underestimate $\dot{Q}$ or overestimate $\dot{L}$ we will underestimate tfp growth.

The second version of the measurement story concerns sectoral shifts. The story is that there has been a shift from manufacturing (where productivity growth is high) to services (where it is low). So even if producitivity growth had not declined in manufacturing there would be an aggregate decline. There are two problems with this story, though. First, productivity growth is not necessarily slower in services. Consider the tremendous productivity growth in airline reservation systems or banking, for example. It probably isn't accurate to think of manufacturing as being robotics and services as being opera.

Second, within manufacturing productivity growth has declined. A good study of this for Canadian, U.S., and Japanese industries is that by Denny *et al* in the *CJE* 1992.

The other problem with both versions concerns the timing. Why did measurement problems suddenly become worse in 1973 in all countries? That brings us to explanation (b) (oil price shock). This explanation matches the timing and the international nature of the slowdown. But there does not seem to be much evidence that the prices of capital goods fell, as one would expect if they were made obsolete by a rise in the price of oil. Also, productivity growth was not restored in the mid-1980s when the price of oil fell sharply.

The third explanation (c) (technological depletion) suggests that there has been a slowdown in new inventions since 1973. Data on patents and R and D spending don't really support this however. Instead, perhaps the information revolution that began in the 1970s first caused a decline in productivity growth but will eventualy lead to faster productivity growth. A different possibility may be that it was the 1945-1973 period that was the anomaly. Perhaps there was a backlog of inventions and ideas that had not been brought to market because of depression and war, and that made for extraordinary productivity growth during that period. Looking at growth rates before 1945 provides some support for this view. But a complete explanation of the post-1973 slowdown remains elusive.

Looking at the figures for the OECD, it is tempting to view productivity growth as an international phenomenon, within industries. There are two difficulties with this generalization, though. First, there is evidence within the set of OECD countries that there is a significant country-specific component to productivity growth (see Costello, *JPE* 1993). In other words, Canadian productivity growth is not slower than Japanese productivity growth simply because we have a smaller semi-conductor industry. Even within industries, there seem to be significant differences in productivity growth across countries.

Second, some countries may have serious measurement problems. For example, it often is argued that countries like Hong Kong and Singapore have very high productivity growth. But Alwyn Young has shown that in fact most of their rapid growth can be accounted for by $\dot{K}$ and $\dot{L}$, in particular due to increases in labour force participation. As we've seen, increases in the saving rate or in the rate of labour force participation – unlike ongoing technical progress – cannot lead to ongoing output growth. So growth accounting which reveals these as the sources of output growth in some Asian countries can help forecast future output growth rates.

The same kind of forecasting may be possible for North America. Here, the slowdown in productivity has been offset by an increase in the labour force participation of women and also by an increase in participation in post-secondary education. But again, we know that the effects of these changes are transitional. That lesson leads to the prediction that output growth will decline in the future, unless there is a resurgence on productivity growth.

One of the most important tasks in studying growth should be to explain the dramatic cross-country differences in TFP. Lucas suggests a revealing transformation of these numbers in which one calculates the number of years it will take for output (or perhaps per capita output) to double, if the growth rate is constant. Note that

$$y(1+\lambda)^n = 2y$$

so that

$$n = \frac{\ln 2}{\lambda} = \frac{.69}{\lambda},$$

because $\ln(1+\lambda) = \lambda$ approximately. As an example, suppose that growth rates in the 1980's continued. Then Sweden's output will double every 86 years while Japan's output will double every 18 years. Of course, even more dramatic differences would be evident if poorer countries were included in this accounting.

(e) Endogenous Growth

An objection to the Solow-Swan growth model is that the growth rate is explained entirely by exogenous population growth or by exogenous technical progress. That is unsatisfactory because it leaves the fundamental causes of growth unexplained (and focuses on how things fit together as growth occurs). Moreover, we do seem to have some evidence on what causes growth. For example, countries with relatively high savings rates grow relatively fast. Making growth depend on savings makes the model one of *endogenous* growth.

The basic idea in some theories of endogenous growth is that positive externalities can prevent the levelling off feared by the classical economists, as accumulation proceeds. You can think of endogenous growth theories as seeking to explain the sources of productivity growth λ.

To see a simple example, retain the Solow assumption that the savings rate, s, is a constant, though it can differ across countries. Let the production function be:

$$Y = AK^{\alpha}(NH)^{1-\alpha},$$

where A is a scaling factor, N is the population, and H is a labour-augmenting technical progress which (importantly) is taken as given by the firm.

For simplicity, there is no depreciation and the population is constant at $N = 1$ ($\eta = 0$). The variable H can be thought of as the stock of knowledge, which makes a given population more productive. We shall assume that in equilibrium $H = K$, so that the stock of knowledge is proportional to the capital stock. The idea is that as the economy accumulates capital it also accumulates knowledge automatically. This *learning-by-doing* externality is sometimes called a *spillover*.

Exercise:

(a) Solve for the balanced or steady-state growth rate.

(b) Show that along this path the real interest rate is constant and the real wage is growing. Are those features sensible empirically?

(c) Suppose now that there are many such economies and that $H = \overline{K}$, where $\overline{K}$ is the average capital stock across countries. Show that the model predicts that a small country should grow faster than a large one with the same savings rate.

Answer

(a) To find the growth equilibrium we again set savings equal to investment. Thus

$$\frac{dK/dt}{K} = \dot{K} = \frac{S}{K} = \frac{sY}{K}$$

with no depreciation. In equilibrium $Y = AK^{\alpha}N^{1-\alpha}H^{1-\alpha} = AK$ so $\dot{K} = sA$, and the higher the savings rate the faster is growth. For obvious reasons, this model is sometimes called the AK model. Notice that there are aggregate constant returns to scale to capital, which keeps growth going.

(b) Now recall that H is an externality of accumulation, taken as given by the firm in its investment decisions. Thus

$$\begin{aligned} r &= \alpha AK^{\alpha-1}(NH)^{1-\alpha} = \alpha A \\ w &= (1-\alpha)AK^{\alpha}(N)^{-\alpha}H^{1-\alpha} = (1-\alpha)AK. \end{aligned}$$

So r is constant while w grows at rate s.

(c) $\dot{K} = sY/K = sAK^{\alpha}\overline{K}^{1-\alpha}/K = sA(\overline{K}/K)^{1-\alpha}$. So a country with a smaller K will have a faster growth rate (implying convergence). One could work out the differences in r and w across countries and then look at the implied flows after that.

The growth in this economy is driven by an *externality*. Accumulating capital also makes labour more productive, although firms do not take that into account in making their investment decisions. But one thing we know about externalities is that they affect the welfare theorems of microeconomics. In this case the competitive rate of growth will be inefficiently low, relative to the socially optimal rate. One possible way to increase growth would be to subsidize the accumulation of human capital.

Of course, not all externalities from economic growth are positive. Ongoing growth also has some negative effects which are not priced or accounted for in investment and savings decisions. The most obvious example is the greenhouse effect. If negative externalities outweigh positive ones then that suggests that the competitive growth rate is too high rather than too low. It is straightforward to build that effect into the models we have outlined so far. For example, imagine that production or consumption requires a second type of capital, provided by the environment. Then imagine some negative feedback from capital accumulation, an output flow, or population growth to this environmental capital. If private sector decisions do not take this externality into account then competitive growth will be too fast. Growth accounting also would have to be modified.

This model of learning-by-doing is not the only possible model of endogenous growth. A second category of endogenous growth models explicitly study the growth of ideas and techniques in a research and development sector. These are two-sector growth models.

Further Reading

The OLG model is introduced by David Romer *Advanced Macroeconomics* (1996), chapter 2, part B and by Olivier Blanchard and Stanley Fischer *Lectures on Macroeconomics* (1989), chapter 3.1. For more detailed applications see George McCandless and Neil Wallace's *Introduction to Dynamic Macroeconomic Theory: An Overlapping Generations Approach* (1991) which is comprehensive but not technical.

The Solow-Swan model is introduced by Romer, chapter 1, and in a fine book by Charles Jones: *Introduction to Economic Growth* (1998). M.A. students should read chapter 1-3 and 8, while Ph.D. students should read chapters 1-9. Ph.D. students also should read Romer chapter 2, part A, Blanchard and Fischer chapter 2, and parts of Robert Barro and Xavier Sala-i-Martin's *Economic Growth* (1995).

For empirical evidence on the Solow-Swan model see Greg Mankiw, David Romer, and David Weil's 'A contribution to the empirics of economic growth' *Quarterly Journal of Economics* (1992) 407-437. Lucas's point on the tendency for models to display convergence is made in his Schwartz Lecture at Northwestern University (1987), 'On the mechanics of economic development.' He gives a thought-provoking list of explanations for non-convergence in 'Why doesn't capital flow from rich to poor countries? *American Economic Review* (P) (990) 92-96. See also Barro, Mankiw, and Sala-i-Martin's 'Capital mobility in neoclassical models of economic growth,' *American Economic Review* (1995) 103-115.

There is an enormous literature on convergence and cross-country growth regressions. A fine review of empirical evidence is given by James Brander in 'Comparative economic growth' *Canadian Journal of Economics* (1992) 792-818. For Canadian evidence see Serge Coulombe and Frank Lee's 'Is there convergence among Canadian provinces?' *Canadian Journal of Economics* (1995) 886-898. On the methods used to study convergence see J. Bradford DeLong's 'Productivity, grwoth, convergence, and welfare: comment' *American Economic Review* (1988) 1138-1154, and Ross Levine and David Renelt's 'A sensitivity analysis of cross-country growth regressions' *American Economic Review* (1992) 942-963. Also see Lance Pritchett's 'Divergence: Big Time,' *Journal of Economic Perspectives* (1997) Summer, 3-17. The regression fallacy is described by Milton Friedman in 'Do old fallacies never die? *Journal of Economic Literature* (1992) and Danny Quah in 'Galton's fallacy and tests of the convergence hypothesis' *Scandinavian Journal of Economics* (1993) 427-443.

Alwyn Young's applications of growth accounting are found in 'A tale of two cities' *NBER Macroeconomics Annual* (1992), 'Lessons from the East Asian NICs: a contrarian view' *European Economic Review* (1994), and 'The tyranny of numbers: confronting the statistical realities of the East Asian growth experience' *Quarterly Journal of Economics* (1995).

On endogenous growth models, see Jones, chapter 8, Romer, chapter 3, and also Gene Grossman and Elhanan Helpman's 'Endogenous innovation in the theory of economic growth,' *Journal of Economic Perspectives* Winter (1994) 23-44. A comprehensive book on endogenous growth is *Endogenous Growth Theory* (1998) by Philippe Aghion and Peter Howitt.

Exercises

Overlapping Generations

1. Consider an overlapping generations economy in which agents live for two periods. An individual born at time t solves the problem:

$$\max U = \ln c_{1t} + \beta \ln c_{2t+1}$$

Individuals work only when young, supplying labour inelastically and earning a wage w_t. The individual budget constraint is:

$$c_{1t} + s_t = w_t; \quad c_{2t+1} = (1 + r_t)s_t$$

Population N_t grows at rate η so that $N_t = (1 + \eta)N_{t-1}$.

A firm hires labour and capital and pays them competitively. The production function in intensive form is:

$$f(k_t) = k_t^{\alpha}$$

where $\alpha \in (0, 1)$ and there is no depreciation. Capital at time $t + 1$ is equal to the saving of the young at time t.

(a) Find an individual agent's consumption when young and when old, given factor prices.

(b) Find the steady-state capital-labour ratio k^*.

(c) Is the steady-state unique and stable?

(d) Find the steady-state interest rate r^*. If $\alpha = 0.3$ and $\beta = 0.9$ find the population growth rates for which the economy is dynamically inefficient.

(e) Suppose that there is a one-time positive shock ϵ to population so that

$$\ldots N_t = (1 + \eta)N_{t-1}, \quad N_{t+1} = (1 + \eta)N_t + \epsilon, \quad N_{t+2} = (1 + \eta)N_{t+1}, \ldots$$

Describe the effects on interest rates.

Answer

(a) The Euler equation is $c_2 = \beta(1+r)c_1$. Substitute in the budget: $c_1(1+r)+c_2 = w(1+r)$ or $c_1(1 + r) + \beta(1 + r)c_1 = w(1 + r)$. Thus $c_1 = w/(1 + \beta)$, $c2 = \beta(1 + r)w/(1 + \beta)$, and $s = w - c_1 = \beta w/(1 + \beta)$. Notice I have omitted the t subscript in the steady state. This takes prices w and r as given.

(b) $K_{t+1} = N_t s_t$ and $(1 + \eta)k_{t+1} = s_t$. Then

$$w_t = f(k_t) - k_t f'(k_t) = k_t^{\alpha} - k_t \alpha k_t^{\alpha-1} = k_t^{\alpha}(1 - \alpha)$$

$$s_t = \frac{\beta w}{1 + \beta} = \frac{\beta k_t^{\alpha}(1 - \alpha)}{1 + \beta}$$

Thus

$$k_{t+1} = \frac{\beta k_t^{\alpha}(1-\alpha)}{(1+\beta)(1+\eta)}$$

In a steady state

$$k^* = \frac{\beta k^{*\alpha}(1-\alpha)}{(1+\beta)(1+\eta)}$$

$$k^* = \Big(\frac{\beta(1-\alpha)}{(1+\beta)(1+\eta)}\Big)^{\frac{1}{1-\alpha}}$$

(c) Stability: check on slopes. Inada conditions for uniqueness.

(d) $r^* = f'(k^*) = \alpha k^{*\alpha-1} = \alpha(\beta(1-\alpha)/[(1+\beta)(1+\eta))^{-1} = \alpha(1+\beta)(1+\eta)/\beta(1-\alpha)$

For dynamic inefficiency the economy is overcapitalized so that $f'(k) = r < n$. Then $r^* = .3(1.9)(1+\eta)/.9(.7) < \eta$ or $\eta > 10$

This implies that the competitive equilibrium tends to be overcapitalized only at phenomenally high population growth rates. I am not sure what this means but I'll think about it.

(e) If there is a baby boom at $t+1$ then the interest rate will rise, since:

$$r_{t+1} = f'(k_{t+1}) = f'(\frac{K_{t+1}}{N_{t+1}}) = f'(\frac{N_t s_t}{N_{t+1}})$$

Then as aggregate savings rises the capital stock will rise in subsequent periods until k and r return to steady-state values. In discussing the transitional dynamics for the OLG model it is easy to see how to track transitions from one steady state to another after a shock. Recall that:

$$\begin{aligned} K_{t+1} &= N_t s_t = N_t \frac{\beta}{1+\beta} w_t \\ &= N_t \frac{\beta}{1+\beta}(1-\alpha)K_t^{\alpha} N_t^{-\alpha} \\ &= \frac{\beta}{1+\beta}(1-\alpha)K_t^{\alpha} N_t^{1-\alpha} \end{aligned}$$

Then imagine the following exogenous sequence of N's: $1, ..., 1, 1+\epsilon, 1, ...1$. You can see by running this sequence through the difference equation that K will gradually adjust.

From the fundamental equation (which is *always* the way to find transitional dynamics) we then can find other variables. For example, the real wage will drop at the time of the shock, because there are many young agents. Then in the period after the shock it will switch to being above the steady-state value because of the large value of capital saved by that same generation. This will lead to high wages for the next generation, who in turn will save more than in the steady state. So w drops, then rises, then *gradually* falls back to its steady-state value.

2. Consider an overlapping generations economy in which agents live for two periods and there is no population growth. There is no production and agents receive a nonstorable endowment w_1 when young and w_2 when old. These two endowments are constant over time. Lifetime preferences are for an agent young at time t are given by:

$$u = \ln(c_{1t}) + \beta \ln(c_{2t+1})$$

where the first subscript denotes age and the second one denotes the date. Suppose that the single good has a price of p_t in period t, p_{t+1} in period $t+1$, and so on.

(a) Write the agent's lifetime budget constraint (using prices, rather than an interest rate).

(b) Solve for demand (consumption) functions.

(c) Use the market clearing condition to find a difference equation in the price.

(d) Characterize competitive equilibria.

3. OLG structures are helpful guides to some fiscal policy questions. Consider an economy with a constant population of two-period-lived agents. They receive a constant, non-storable endowment y when young and must save to finance consumption when old. They behave competitively and seek to maximize:

$$U = \ln(c_{1t}) + \beta \ln(c_{2t+1})$$

subject to

$$c_{1t} + s_t = y,$$

$$c_{2t+1} = s_t(1 + r_{t+1}).$$

(a) Solve for optimal consumption when young (c_1) and when old (c_2) in the steady state.

(b) The government proposes to levy a lump-sum tax τ on the old so that

$$c_2 = s(1 + r) - \tau.$$

One economist argues that the tax will affect only the old. A second economist suggests that the new tax will serve to lower interest rates, because young agents will save more to maintain their consumption when old. Show (in the steady state) that both economists are wrong.

Answer

(a) $c_1 = \frac{1}{1+\beta} y$; $c_2 = \frac{\beta}{1+\beta}(1 + r)y$.

(b) The consolidated budget constraint becomes:

$$c_1 + \frac{c_2 + \tau}{1 + r} = y$$

$$c_1 + \frac{c_2}{1+r} = y - \frac{\tau}{1+r}$$

Then

$$c_1 = \frac{1}{1+\beta}(y - \frac{\tau}{1+r})$$

$$c_2 = \frac{\beta}{1+\beta}(1+r)(y - \frac{\tau}{1+r})$$

The first economist is wrong because consumption by the young falls; they bear some of the tax. The second economist is wrong because the ratio c_2/c_1 is unchanged and so r is unchanged.

4. Consider an overlapping generations economy with production. There is no uncertainty. Households live for two periods and maximize utility given by

$$U = \ln(c_{1t}) + \beta \ln(c_{2t+1})$$

subject to a budget constraint

$$c_{1t} + s_t = w_t$$

$$c_{2t+1} = (1 + r_{t+1})s_t$$

where the subscript notation is standard. Households work when young for wage w and do not work when old.

A competitive firm faces Cobb-Douglas technology in that output is given by

$$\theta\, K_t^{\alpha} H_t^{1-\alpha} N_t^{1-\alpha}$$

where K is capital, N is labouring population, and H is an externality. In equilibrium $N_t = 1$ (there is no population growth) and $H_t = K_t$, though the firm does not take this into account in maximizing profits. The firm maximizes profits, taking factor prices as given. The depreciation rate is zero.

(a) Solve for optimal savings, given factor prices.

(b) Find the first-order conditions for profit maximization.

(c) The market clearing condition is

$$K_{t+1} = s_t.$$

Combine this with your answers to parts (a) and (b) to find a difference equation in the capital stock in a competitive equilibrium.

(d) Suppose that $\alpha = 0.3$, $\beta = 0.9$, and $\theta = 3.1$. What is the average growth rate of output? What is the level of the real interest rate?

Answer

(a)

$$s_t = w_t\, \beta/(1+\beta)$$

(b)

$$w = \theta(1-\alpha)K^{\alpha}H^{1-\alpha}N^{-\alpha}$$
$$r = \alpha\theta K^{\alpha-1}H^{1-\alpha}N^{1-\alpha}$$

(c)

$$K_{t+1} = (\theta(1-\alpha)\beta/(1+\beta))K_t$$

(d) The growth rate is 2.8 percent. The real interest rate is $\alpha\theta = 0.93$.

5. Some economists have argued that rapid growth in Singapore and Hong Kong has occurred because of increases in labour-force participation. In this question, we use the real OLG model to see what some of the indicators of such an increase might be.

Consider an OLG model in which agents live for two periods. When young they work for a wage w_t, save s_t, and consume c_{1t}. When old they consume c_{2t+1}. They maximize

$$U = \ln(c_{1t}) + 0.9log(c_{2t+1}),$$

subject to

$$c_{1t} + s_t = w_t n_t$$
$$c_{2t+1} = (1 + r_{t+1})s_t.$$

Let the number of young agents arriving each period be 1. Suppose that each works half-time so that $n_t = 0.5$ is labour supply. The production function is

$$y_t = K_t^{0.5} n_t^{0.5},$$

there is no depreciation, and the market-clearing condition is:

$$K_{t+1} = s_t.$$

(a) Find savings as a function of the real wage.

(b) Solve for numerical values for w and r in a steady-state equilibrium.

(c) Suppose that labour-force participation rises unexpectedly to $n_t = 1.0$. Find the steady-state effects on y, w, and r.

(d) The increase in labour supply is unexpected in the sense that old agents, when young last period, did not expect that the generation following them would supply more labour. Describe what happens to w and r in the transition to the new steady state.

Answer

(a)

$$s = \frac{\beta(wn)}{1+\beta} = 0.236 \cdot w$$

(b) $s_t = 0.236 w_t$. Also $K_{t+1} = s_t$. From the two marginal products:

$$w = 0.707 K^{1/2}$$

$$r = 0.3535 K^{-1/2}$$

In a steady state:

$$K = 0.167 K^{1/2}$$

Thus $K^* = 0.028$. So $w = 0.118$ and $r = 2.11$.

(c) If now $n = 1$ then the system in part (b) becomes

$$s = 0.474 w$$

and $K_{t+1} = s_t$ along with

$$w = 0.5 K^{1/2}$$

$$r = 0.5 K^{-1/2}.$$

Thus in the steady state $K^* = 0.056$. Also, $w^* = 0.118$ and $r = 2.11$, so these are unchanged. Output y doubles to $y = 0.237$.

Suppose the population is 1 and the participation rate is ρ, a fraction. Per capita saving is s_t so aggregate saving is ρs_t. Thus:

$$\begin{aligned} K_{t+1} &= \rho s_t \\ &= \rho \frac{\beta}{1+\beta} w_t \\ &= \rho \frac{\beta}{1+\beta}(1-\alpha) K_t^{\alpha} \rho^{-\alpha} \end{aligned}$$

The steady-state value is

$$K^* = \rho \Big(\frac{\beta}{1+\beta}(1-\alpha) \Big)^{\frac{1}{1-\alpha}}$$

so in the steady state ρ simply determines the scale of the market sector of the economy. Changes in ρ thus have no long-run effect on w or r, though of course they effect per capita consumption. In the labour market, a decrease in ρ shifts the vertical labour supply curve to the left, so that wages rise in the short run. But as aggregate saving has fallen the capital stock falls and the labour demand curve shifts down so that the long-run real wage is unchanged.

(d) In the short run, the capital-labour ratio has fallen and so w falls and r rises. In the current period y rises from 0.118 to 0.167. The real wage falls and the real interest rate rises. So both c_2 and c_1 rise and income distribution is unchanged (a consequence of the 0.5 exponents). After one generation, savings rise and so the accumulation of capital lowers r and raises w to their original values. This simple exercise might give us some clues to look for if an increase in output is really due to an increase in labour supply.

6. Consider an overlapping-generations model in which agents live for two periods. They supply labour inelastically when young, and have logarithmic utility. Thus they save

$$s_t = \frac{\beta w_t}{1+\beta},$$

where w_t is the real wage and β is the discount factor. The technology is Cobb-Douglas with parameter α and there is no depreciation or population growth.

(a) Solve for the steady-state capital labour ratio, k^*.

(b) Solve for the steady-state real wage w^* and the steady-state interest rate r^*.

(c) Does the consumption of individual agents satisfy the usual consumption-based Euler equation? Does aggregate consumption satisfy this equation?

Answer

(a) The capital-labour ratio is

$$k^* = \left(\frac{\beta(1-\alpha)}{1+\beta}\right)^{\frac{1}{1-\alpha}}$$

(b) The interest rate is

$$r^* = \alpha k^{\alpha-1} = \frac{\alpha(1+\beta)}{\beta(1-\alpha)}$$

The real wage is

$$w^* = (1-\alpha)k^{\alpha} = (1-\alpha)\left(\frac{\beta(1-\alpha)}{1+\beta}\right)^{\frac{\alpha}{1-\alpha}}$$

(c) It is easy to show that individual consumption does satisfy the equation. Here

$$c_{2t+1} = \frac{(1+r)w\beta}{1+\beta}; \quad c_{1t} = \frac{w}{1+\beta}$$

so

$$(1+r) = \frac{c_{2t+1}}{\beta c_{1t}}$$

But aggregate consumption does not: $C_{t+1} = C_t$ and $1+r \neq 1/\beta$.

7. This question shows how productivity shocks may lead to business cycles, using the overlapping generations model. Suppose that population is constant, and people live for two periods. Each person supplies one unit of labour inelastically when young. An individual born at time t earns a wage w_t, and seeks to maximize

$$\ln(c_{1t}) + E_t \frac{\ln(c_{2t+1})}{1+\theta},$$

subject to the budget constraint

$$c_{2t+1} = (1 + r_t)(w_t - c_{1t}).$$

Output is given by

$$Y_t = A_t k_t^{\alpha} N_t^{1-\alpha}.$$

Y_t is net output, and capital depreciates fully after one period. $N_t = 1$. The productivity shock A_t evolves as follows:

$$A_t = e^{\gamma} A_{t-1} \epsilon_t,$$

where ϵ_t is independently and identically distributed with log mean zero.

(a) What proportion of wage income is saved?

(b) Find a linear difference equation describing the evolution of the log of the capital stock.

(c) Find a linear difference equation describing the evolution of the output growth rate.

(d) So far we have studied this model as if the productivity term A_t evolves exogenously. Suppose instead that

$$A_t = k_t^{1-\alpha} \epsilon_t$$

where ϵ_t is independently and identically distributed and $E(\ln(\epsilon_t)) = 0$. Suppose that this represents an external effect of capital accumulation, so that firms and households do not take into account that adding to capital also adds to productivity. What is the average growth rate of output?

Answer

(a)

$$s_t = \frac{w_t}{2+\theta}$$

(b) Using $k_{t+1} = s_t$,

$$k_{t+1} = \frac{(1-\alpha)A_t k_t^{\alpha}}{2+\theta}$$

$$\ln(k_{t+1}) = \ln\left(\frac{1-\alpha}{2+\theta}\right) + \alpha \ln k_t + \ln A_t$$

(c)

$$\Delta \ln(y_t) = \gamma + \alpha \Delta \ln(y_{t-1}) + \ln(\epsilon_t)$$

(d) Once again

$$k_{t+1} = \frac{(1-\alpha)A_t k_t^{\alpha}}{2+\theta}$$

and so using the fact that

$$A_t = k_t^{1-\alpha}\epsilon_t$$

gives

$$k_{t+1} = \frac{(1-\alpha)k_t\epsilon_t}{2+\theta}.$$

Meanwhile,

$$\ln(y_t) = \ln(k_t) + \ln(\epsilon_t),$$

so the output growth rate on average is

$$\ln\Big(\frac{1-\alpha}{2+\theta}\Big).$$

This is an example of endogenous growth. Notice that we could calculate moments of output growth, consumption growth, wages, and so on in both versions of the model.

8. Japan, 1945. Consider the OLG model with two-period lives and no uncertainty. Each period a constant number of young agents arrives, and they supply labour inelastically. For simplicity, the size of the population is normalized to $n_t = 1$. The utility function is:

$$U = \ln(c_{1t}) + \ln(c_{2t+1}),$$

in standard notation. Old agents live off their savings. There is no depreciation of capital. The production function is:

$$y_t = k_t^{0.5} n_t^{0.5}.$$

Let t count five-year periods: 1940, 1945, 1950, and so on.

(a) Find the steady-state capital stock, k^*, real wage, and real interest rate.

(b) Find the steady-state consumption of the young (c_1), consumption of the old (c_2), and lifetime utility.

(c) Suppose that the economy is initially in the steady state in but that in 1945 there is a sudden and unexpected loss of half the capital stock, so that now $k_0 = 0.5k^*$. Find values for w, r, c_1, and c_2 for 1945, 1950, and 1955.

(d) Which generation is made worst off by this calamity?

Answer (a) $k^* = 0.0625$ $w = 0.125$; $r = 2.0$; (these satisfy $y = w + rk$);

(b) $c_1 = 0.0625$; $c_2 = 0.1875$; (these satisfy $c_1 + c_2 = y$). In general, $y = k^{.5}$, $w = y/2$, $c_1 = s = y/4$, $r = 1/2y$, $c_2 = (1+r)s$. Lifetime utility is:

$$\ln(0.0625) + \ln(0.1875) = -4.44$$

(c) In 1945: $k = 0.03125$, $y = 0.176$; $w = 0.088$; $r = 2.84$; $c_1 = 0.044$; $c_2 = 0.12$. The old lost half their capital, but they are partly compensated by the high real interest rate. Notice that outside the steady state c_2 no longer equals $y - c_1$!

Remember that $k_{t+1} = s_t = w_t/2$. In 1950: $k = 0.044$, $y = 0.209$; $w = 0.104$; $r = 2.39$; $c_1 = 0.052$; $c_2 = 0.149$.

In 1955: $k = 0.052$, $y = 0.228$; $w = 0.114$; $r = 2.19$; $c_1 = 0.057$; $c_2 = 0.165$.

(d) The generation born in 1940 has utility $\ln(.0625) + \ln(.12) = -4.89$. The generation born in 1945 has utility $\ln(.044) + \ln(.149) = -5.02$. This generation is worst off. The generation born in 1950 has utility $\ln(.052) + \ln(.165) = -4.75$.

9. Japan has recently experienced a decline in the growth rate of its population. This question uses the OLG model to predict some of the economic effects of an unexpected decline in population growth. Suppose that households live two periods and are identical. When young, all households supply labour inelastically and then save out of wage income according to:

$$s_t = \frac{\beta}{1+\beta} w_t,$$

where β is a discount factor. Firms hire capital and labour competitively to maximize profits. There is no depreciation. The production function in intensive form is:

$$y_t = k_t^{\alpha}.$$

Population of young households is denoted N_t, and this grows according to:

$$N_{t+1} = (1+\eta)N_t.$$

(a) Find the steady-state capital labour ratio.

(b) Solve for the steady-state values of the real interest rate (r^*) and output per worker (y^*).

(c) Find the elasticity of r^* and y^* with respect to $1+\eta$. [Hint: Use log differentiation.]

(d) Suppose that the world economy consists of two OLG countries. If η suddenly falls in one country, describe the transitional dynamics in the world economy.

Answer (a)

$$k^* = \left[\frac{\beta(1-\alpha)}{(1+\beta)(1+\eta)}\right]^{\frac{1}{1-\alpha}}$$

(b)

$$y^* = \left[\frac{\beta(1-\alpha)}{(1+\beta)(1+\eta)}\right]^{\frac{\alpha}{1-\alpha}}$$

$$r^* = \alpha\left[\frac{(1+\beta)(1+\eta)}{\beta(1-\alpha)}\right].$$

(c)

$$_{y}\epsilon_{1+\eta} = -\frac{\alpha}{1-\alpha}$$

This should be around -0.5: output per worker rises 0.5 percentage points for each 1 percent fall in the gross population growth rate. Meanwhile,

$$_{r}\epsilon_{1+\eta} = 1,$$

so that the steady-state interest rate falls one percent for each one percent fall in the gross population growth rate.

(d) In Japan, where η falls, r falls and w rises. There will be a tendency for labour to flow into Japan, and for capital to flow out. Output per capita rises to its new value. In the steady state, the world economy grows more slowly, but output per capita is at a higher value.

10. This question examines the relationship between government spending and inflation in a simple overlapping generations model. Suppose that agents live for two periods and have preferences as follows:

$$\ln(c_{1t}) + \ln(c_{2t+1}),$$

where c_{1t} denotes the consumption of a young agent at time t and c_{2t+1} denotes the consumption of an old agent at time $t+1$.

Each generation has nonstorable endowment w_1 when young and w_2 when old. The only way to save is by holding fiat money, denoted m_t. The price level is denoted p_t, and the gross inflation rate is denoted π_t. The government has expenditure g which it finances from the inflation tax, so that

$$g = \frac{m_t - m_{t-1}}{p_t} = \frac{m_t}{p_t} - \frac{m_{t-1}}{p_{t-1}} \cdot \frac{1}{\pi_t}.$$

(a) Write down the budget constraints for an agent, when young and old.

(b) Solve for optimal saving.

(c) Find the market clearing condition.

(d) Describe the steady-state equilibria. In this model, what is the effect on inflation of an increase in government spending?

Answer

(a)

$$c_{1t} + \frac{m_t}{p_t} = w_1$$

$$c_{2t+1} = w_2 + \frac{m_t}{p_{t+1}}$$

(b)

$$s_t = \frac{w_1}{2} - \frac{w_2}{2}\pi_{t+1}$$

(c)

$$\frac{m_t}{p_t} = s_t$$

(d) Substituting the market clearing condition in the government budget constraint gives:

$$g = s_t - \frac{s_{t-1}}{\pi_t}$$

The phase diagram of π_t vs. π_{t+1} shows two equilibria, a stable high-inflation one and an unstable low-inflation one. An increase in g shifts the graph down. In the low-inflation equilibrium an increase in g raises π, but the reverse is true in the high-inflation equilibrium. A simple way to find the steady states is to solve:

$$g = (1 - \frac{1}{\pi})(\frac{w_1}{2} - \frac{w_2}{2}\pi).$$

Bruno and Fischer (*QJE* (1990)) analyze this model in detail and describe how to use policy to attain the low inflation equilibrium and stabilize it.

11. Consider an OLG model in which households work when young and consume when young and old. A person born at time t seeks to maximize:

$$\ln(c_{1t}) + 0.9\ln(c_{2t+1}) + \ln(1 - n_t),$$

where n is labour supply, subject to

$$s_t = w_t n_t - c_{1t}$$
$$c_{2t+1} = s_t(1 + r_{t+1}).$$

There is no population growth or depreciation. Suppose that the production function is subject to random productivity shocks, z_t, that output is

$$q_t = z_t k_t^{0.3} n_t^{0.7},$$

and that z_t is independently distributed and can take on two values, 1.1 or 0.9, each with probability 0.5.

(a) Find the static Euler equation linking consumption and labour supply for young agents.

(b) What fraction of their time do young agents spend working?

(c) Find the law of motion for the logarithm of the capital stock.

(d) Is the real wage pro-cyclical or counter-cyclical?

Answer

(a)

$$\frac{c_1}{(1-n_1)} = w$$

(b)

$$n = \frac{1+\beta}{2+\beta} = 0.655$$

(c)

$$k_{t+1} == 0.376 z_t k_t^{0.3},$$

so

$$\ln(k_{t+1}) = -1.4 + 0.3\ln(k_t) + \ln(z_t).$$

Thus there are two cases, depending on the value of z.

(d) A large z raises output and the marginal product of labour. The supply curve is vertical (with log utility) as we saw in part (b) so the real wage clearly is pro-cyclical.

12. This question studies the effect of risk aversion on savings and interest rates, in a dynamic general equilibrium model. Consider an OLG model with no uncertainty, population growth, or depreciation. The technology is Cobb-Douglas:

$$f(k) = k^{\alpha}.$$

Young agents supply labour inelastically for wage w_t and then invest their savings at rate r_{t+1}. They seek to maximize:

$$\frac{c_{1t}^{1-\sigma}}{1-\sigma} + \frac{c_{2t}^{1-\sigma}}{1-\sigma}.$$

(a) Solve for optimal savings as a proportion of the real wage. What is the effect of a change in the interest rate on the savings rate?

(b) Of course, the interest rate is endogenous. Sketch how you could find the effect of risk aversion on the steady-state real interest rate.

Answer

(a) The Euler equation is:

$$\frac{1}{c_1^{\sigma}} = \frac{(1+r)}{c_2^{\sigma}}.$$

The budget constraint is:

$$c_1 + \frac{c_2}{1+r} = w.$$

We can combine these, because there is no uncertainty, to find consumption of the young:

$$c_1 = w[\frac{1}{1+(1+r)^{\frac{1-\sigma}{\sigma}}}],$$

or

$$s = w\left[\frac{1}{1+(1+r)^{1-\frac{1}{\sigma}}}\right].$$

You can see that an increase in r lowers consumption (raises savings) if $\sigma < 1$ and raises consumption (lowers savings) if $\sigma > 1$.

(b) Market clearing gives $k_{t+1} = s_t$, so in the steady state $k = s$. Also, along the production function r is a decreasign function of k. Using $w = (1-\alpha)k^\alpha$ and $1+r = \alpha k^{\alpha-1}$ gives an implicit equation for the steady-state capital stock:

$$k = (1-\alpha)k^\alpha [\frac{\beta^{\frac{1}{\sigma}}(\alpha k^{\alpha-1})^{\frac{1-\sigma}{\sigma}}}{1+\beta^{\frac{1}{\sigma}}(\alpha k^{\alpha-1})^{\frac{1-\sigma}{\sigma}}}].$$

I would solve this for various values of σ and hence find the effect of σ on k, thence r.

13. The overlapping generations model often is a useful guide to economic issues involving demographic changes and fiscal policy. Imagine that an agent has a non-storable endowment y when young and λy when old, where λ may be greater or less than one. Each agent lives for two periods and seeks to maximize:

$$\ln(c_{1t}) + \beta \ln(c_{2t+1}),$$

in the case of an agent young at time t. There is population growth at rate η, so that for every old agent there are $1+\eta$ young agents.

(a) Solve the saving problem for a typical young agent.

(b) Explain how to find the competitive equilibrium interest rate. How is the interest rate affected by the rate of population growth?

(c) Suppose that $\lambda = 0$ so that agents must save for retirement. Also suppose that there is a government which can levy different lump-sum taxes on the young and the old. Is the interest rate affected by the incidence of taxes?

(d) Can changes in tax timing affect the interest rate and national savings in this economy?

Answer

(a)

$$c_1 = (\frac{1}{1+\beta})[y + \frac{\lambda y}{1+r}]$$

so

$$s = y[\frac{1}{1+\beta}][\beta - \frac{\lambda}{1+r}]$$

(b) Market clearing gives:

$$(1+\eta)c_1 + c_2 = (1+\eta)y + \lambda y.$$

Using the expressions for c_1 and c_2 then gives r. To see the effect of η, set $\lambda = 0$ and notice then that $r = \eta$. Or we could see the effect by total differentiation of the market clearing condition: in general an increase in η raises r.

(c) No. With lump-sum taxes and $\lambda = 0$ the consumption function is:

$$c_1 = \frac{1}{1+\beta}(y - \tau_1),$$

and

$$c_2 = \frac{\beta}{1+\beta}(y - \tau_1) - \tau_2.$$

The market-clearing condition is:

$$(1+\eta)c_1 + c_2 = (1+\eta)y - (1+\eta)\tau_1 - \tau_2.$$

Substitute the consumption functions in the market clearing condition. The τ_2 scancel. Each term contains $y - \tau_1$, which thus cancels. Again $r = \eta$.

(d) Yes.

14. This question uses the overlapping generations model to predict the economic effects of the aging of the Japanese population. Suppose individuals live for two periods, and work when young for wage w. An individual who is young at time t maximizes

$$\ln(c_{1t}) + 0.8log(c_{2t+1}),$$

subject to

$$c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} = w_t.$$

Markets are competitive.

(a) Solve for the optimal savings of an individual.

(b) A young generation is of size n_t and the production function is

$$y_t = k_t^{0.3} n_t^{0.7}.$$

Find the equilibrium difference equation linking k_{t+1} to k_t and the exogenous variable n_t.

(c) Suppose that initially the economy is in a steady state, with $n = 1$. Describe the transitional dynamics when n drops suddenly to 0.5. Can this model explain recent low Japanese rates of interest?

Answer

(a)

$$s_t = \frac{\beta w_t}{1+\beta}$$

(b) Market clearing requires $k_{t+1} = n_t s_t$ (not $k_{t+1} = s_t$) so

$$k_{t+1} = \frac{\beta(1-\alpha)}{1+\beta} k_t^{\alpha} n_t^{1-\alpha}$$

(c) We track the difference equation and see k and y falling. Also w jumps up and r jumps down, before returning to their steady-state levels.

15. Consider the OLG model with production. Individuals live for two periods and have utility function:

$$\ln(c_{1t}) + \beta \ln(c_{2t+1}).$$

They work for wage w_t when young and save to finance their consumption when old. The technology is:

$$K_t^{\alpha}(n_t(1+\lambda)^t)^{1-\alpha}.$$

There is no depreciation. There is no population growth, so $n_t = 1$ for all t. The parameter λ is the rate of labour-augmenting technological progress; the term $(1+\lambda)^t$ indicates that labour productivity grows over time at rate λ.

(a) Find the savings function of young agents.

(b) Define

$$k_t \equiv \frac{K_t}{(1+\lambda)^t}$$

and solve for the steady-state value of k.

(c) Describe the path of output and the real wage along this balanced growth path.

Answer (a)

$$s_t = \frac{\beta}{1+\beta} w_t$$

(b) The fundamental equation gives:

$$K_{t+1} = \frac{\beta}{1+\beta}(1-\alpha)K_t^{\alpha}[(1+\lambda)^t]^{1-\alpha}.$$

Thus:

$$k_{t+1} = \frac{\beta}{1+\beta}\frac{1-\alpha}{1+\lambda}k_t^{\alpha},$$

so

$$k^* = \left[\frac{\beta}{1+\beta}\frac{1-\alpha}{1+\lambda}\right]^{\frac{1}{1-\alpha}}.$$

Notice that we can handle productivity growth in the market-clearing condition in much the same way we handled population growth.

(c) In the steady state the real wage is:

$$w_t = K_t^{\alpha}(1-\alpha)[(1+\lambda)^t]^{1-\alpha} = k^{*\alpha}(1-\alpha)[(1+\lambda)^t].$$

Output is:

$$Y_t = k^{*\alpha}[(1+\lambda)^t].$$

Both grow at the rate of labour productivity growth.

16. This question uses the OLG model with a fixed saving rate to describe some economic transitions. The production function is:

$$y_t = k_t^{0.3}(\rho n_t)^{0.7}0.7,$$

where y is output, k is capital, and n is the number of young agents. There is no population growth, so $n_t = 1$ always, and we do not need to distinguish between upper-case and lower-case variables. The parameter ρ is the participation rate. Suppose that young agents save a fraction θ of their wage income:

$$s_t = \theta w_t.$$

There is no depreciation.

(a) Find the law of motion for the capital stock.

(b) Suppose that initially $\theta = 0.3$ and $\rho = 0.8$. Describe the transitional dynamics if θ falls permanently to 0.2.

(c) Empirically, could you distinguish the effects of a fall in the saving rate from those of a fall in the participation rate?

Answer (a) Differentiating with respect to n:

$$k_{t+1} = 0.7\theta\rho^{0.7}k_t^{0.3}.$$

(b) Initially the steady state is $k = 0.0861$. The steady state after the fall in the saving rate is $k = 0.0482$. The capital stock falls monotonically to the new value. Also, the wage rate falls and the interest rate rises.

(c) Both a fall in θ, the saving rate, and a fall in ρ, the participation rate, lower the steady-state capital-labour ratio. I think they can be distinguished by their differring effects on consumption in the transition, though. Notice the general equilibrium effect of a fall in ρ.

17. This question uses the overlapping generations model to study the macroeconomic effects of a fall in the population growth rate. Suppose that agents have log utility with discount factor 0.8, and supply labour inelastically when young. Their saving function is:

$$s_t = \frac{0.8}{1.8} w_t.$$

The number of young agents, denoted N_t, grows this way:

$$N_t = (1 + \eta) N_{t-1}.$$

Meanwhile, the savings of the previous generation are combined with this generation's labour to produce output according to

$$F(K_t, L_t) = K_t^{0.3} L_t^{0.7}.$$

There is no capital depreciation, but the old consume their capital after it has been used in production.

(a) Find the steady-state values of the capital-labour ratio, the real wage, and the real interest rate, as functions of the population growth rate η.

(b) Suppose that the economy is in a steady state with $\eta = 0.03$. Suppose that η then falls suddenly, permanently, and unexpectedly, to 0.01. Describe the transitional dynamics for the economy.

Answer (a)

$$k_{t+1} = \frac{0.3111}{1 + \eta} k_t^{0.3}$$

so

$$k^* = \frac{0.3111}{1 + \eta}^{1/.7}.$$

Then

$$w = 0.7 k^{0.3}$$

and

$$r = \frac{0.3}{k^{0.7}}.$$

(b) In the initial steady state, $k^* = 0.1808$, $w^* = 0.4190$ and $r^* = 0.9932$. In the final steady state, $k^* = 0.1859$, $w^* = 0.4226$ and $r^* = 0.9739$. See if you can draw the paths for these variables. To fully describe the transition, we also could track consumptions and utility of each generation.

18. This question studies the role of money in the overlapping generations model. Suppose that all generations are of size 1, and each person has lifetime utility

$$\ln(c_{1t}) + 0.9\ln(c_{2t+1}).$$

Each person also has an endowment of 2 when young and 1 when old. There are an infinite number of time periods. There is no physical technology for storage.

(a) What are the values of c_1, c_2, and lifetime utility with no trade between generations?

(b) What value of lifetime utility could persons achieve if consumption were allocated by a utilitarian social planner?

(c) There is no such social planner, but suppose that people begin to trade paper money in nominal amounts denoted by m. The budget constraint when young is

$$c_{1t} + \frac{m_t}{p_t} = 2$$

and when old it is

$$c_{2t+1} = 1 + \frac{m_t}{p_{t+1}}.$$

The stock of paper money is fixed by the government at a constant value. What is the socially optimal inflation rate?

Answer (a) $c_1 = 2$, $c_2 = 1$, and $U = 0.693$.

(b) Now $c_1 = 1.578$, $c_2 = 1.421$, and $U = 0.772$.

(c) The budget constraints can be written as:

$$c_1 + (c_2 - 1)\frac{p_{t+1}}{p_t} = 2.$$

Maximizing utility subject to this constraint gives:

$$\frac{1}{c_1} = \frac{0.9}{c_2}\frac{p_t}{p_{t+1}}$$

Combining these gives the demand functions:

$$c_1 = \frac{1}{1.9}(2 + \frac{p_{t+1}}{p_t})$$

$$c_2 = \frac{0.9}{1.9}(2\frac{p_t}{p_{t+1}} + 1)$$

You can see that the optimal inflation rate is zero, since that implements the allocation in part (b). But the equilibrium inflation rate is found by using the market clearing condition:

$$c_{1t} + c_{2t} = 3,$$

so

$$\frac{1}{1.9}(2+\frac{p_{t+1}}{p_t})+\frac{0.9}{1.9}(2\frac{p_{t-1}}{p_t}+1)=3$$

which is a difference equation in the inflation rate. Later, you should be able to link this to bubbles and to dynamic inefficiency.

19. This question studies the predictions of growth theory for savings rates over time, using the OLG model with no uncertainty. Agents live for two periods. There is no population growth. Young agents supply labour inelastically, earn wage w_t, and save an amount s_t for retirement. Old agents live off their savings.

(a) Suppose that the utility function is:

$$\frac{c_{1t}^{1-\sigma}}{1-\sigma}+\beta\frac{c_{2t+1}^{1-\sigma}}{1-\sigma},$$

and that $\sigma = 0.5$. Is the period utility function concave?

(b) Solve for optimal savings as a function of factor prices.

(c) The production function is Cobb-Douglas with capital exponent 0.3. The discount factor is 0.9. There is zero depreciation. Find the competitive equilibrium law of motion for the capital stock.

(d) Suppose that, starting from the steady state, an earthquake destroys half the capital stock. Describe the transitional dynamics for the economy using diagrams. Is there a time series correlation between the growth rate and the savings rate in this model?

(e) Empirically, is there a cross-country correlation between savings rates and growth rates? Does this correlation help us to discriminate between exogenous and endogenous theories of economic growth?

Answer (a) The first derivative is $1/c^{1/2}$ which is positive for positive c. The second derivative is $-0.5c^{-3/2}$ which is negative, so the function is increasing and concave.

(b) Combining the Euler equation and budget constraint gives:

$$c_{1t}=w_t\left(\frac{1}{1+\beta^2(1+r_{t+1})}\right),$$

so that

$$s_t=w_t\left(\frac{\beta^2(1+r_{t+1})}{1+\beta^2(1+r_{t+1})}\right),$$

which increases when r increases.

(c)

$$k_{t+1}=s_t=\left(\frac{\beta^2(1+r_{t+1})}{1+\beta^2(1+r_{t+1})}\right)(1-\alpha)k_t^{\alpha}.$$

So with the parameter values given,

$$k_{t+1} = \left(\frac{0.81(1+r_{t+1})}{1+\beta^2(1+r_{t+1})}\right)0.7k_t^{0.3}.$$

then write r as a function of k.

(d) As k rises to k^* the wage rate rises and the interest rate falls. As r falls the savings rate falls. So there is a positive correlation between the growth rate and the savings rate (both are highest initially).

Solow-Swan Growth

1. Consider the Solow-Swan, one-sector growth model with the following production function:

$$Y(t) = K(t)^{\frac{1}{3}}N(t)^{\frac{2}{3}}$$

Suppose that population growth, the savings rate, and the depreciation rate are all constant.

(a) In a steady-state, competitive equilibrium, find the level of the per capita capital stock and the level of per capita output.

(b) Find the real interest rate.

(c) Find per capita consumption and savings.

2. Consider the Cobb-Douglas production function in intensive form:

$$y = k^{\alpha}$$

with $\alpha \in (0,1)$. There is zero depreciation.

(a) Show that if $s = \alpha$ then the competitive equilibrium (in the one-sector growth model with this production function) and the golden-rule optimum coincide.

(b) Now suppose that $s = 2\alpha$. Is the competitive equilibrium dynamically efficient? Explain briefly.

(c) Suppose that $\alpha = 0.5$. In golden-rule equilibrium, find $dc/d\eta$.

3. Suppose that in the Solow-Swan growth model the supply of labour evolves exogenously as follows:

$$N_t = N_{t-1}(1+\eta)$$

In intensive form, the production function is

$$y = log_{10}k.$$

Let $\eta = 0.01$, and let s be the constant savings rate. There is zero depreciation.

(a) If $s = 0.04$, find k^*, the steady-state capital-labour ratio in competitive equilibrium.

(b) If $s = 0.08$, find k^*, the steady-state capital-labour ratio in competitive equilibrium.

(c) Suppose that the savings rate jumps unexpectedly to 0.10. Describe (qualitatively) the growth properties of the economy as it adjusts to this higher savings rate.

4. Consider the Solow-Swan growth model. The production function in intensive form is:

$$f(k) = k^{\frac{2}{3}}$$

Labour inputs grow at rate η. There is depreciation at rate δ. The savings rate is s. Also, the values of s, η, and δ are exogenous.

(a) Find the steady-state value of k in a competitive equilibrium.

(b) In the same equilibrium find the profit and wage rates.

(c) In the same equilibrium find per-capita savings and consumption.

(d) Suppose that the savings rate unexpectedly rises from s to s' where $s' > s$. Find the short-run and long-run effects on the growth rate of output.

(e) If $s = 0.2$ and $\delta = 0.1$ find the values of η for which the steady-state competitive equilibrium is dynamically inefficient.

5. In the Solow-Swan model the production function in intensive form is

$$q(t) = f(k(t)) = k^{\frac{1}{2}}$$

The savings rate is s. The labour force grows exogenously at rate η and there is no technical progress.

(a) Find the capital stock, k^*, in a steady-state competitive equilibrium, in terms of s and η.

(b) In the same competitive equilibrium, find the profit rate π, the wage rate w, and per capita savings and consumption. Find the effects (in a steady state) on all four of these of an increase in η.

(c) Given η, find the savings rates such that the economy is dynamically inefficient.

6. Investigate the fixed proportions production function:

$$Q = F(K, L) = \min[\frac{K}{a}, \frac{L}{b}]$$

with $a > 0$ and $b > 0$ by

(a) calculating marginal and average products and determining the graph of the unit isoquant

(b) verifying constant returns to scale, quasiconcavity, and that both factors are essential.

(c) For this production function, solve Solow's basic differential equation and determine the resulting path of real wage rates, real rentals, and the output/capital ratio. (ref. Solow 1956)

7. This question studies the neoclassical one-sector growth model with constant savings rate. Suppose that

$$f(k(t)) = k(t)^{\frac{1}{3}}$$

in intensive form. Suppose that the depreciation rate is 10%, the savings rate is 20%, and population grows at a constant rate of 5%.

(a) Find output and capital per capita on a steady-state growth path.

(b) Find the wage and profit rates on a competitive steady-state growth path.

(c) Is the economy dynamically efficient?

(d) An extraterrestrial benefactor offers to supply any additional capital necessary to move the economy to the golden rule equilibrium. How much is needed?

Answer

(a) $sf(k)/k = \delta + \eta$; $1/k^{2/3} = 0.75$; $k^{2/3} = 1.333$; so $k = 1.539$ and $y = 1.154$.

(b) $r = f'(k) - \delta = 1/3k{-}2/3 - .1 = .25 - .1 = 0.15$; $w = f(k) - kf'(k) = 1.154 - 1.539(0.25) = 0.769$.

(c) It is dynamically efficient since $r > \eta$.

(d) At the golden rule $r = \eta$: $f'(k_G) = \eta + \delta = 0.15$ gives $k_G = 3.31$ so the gift is 1.771.

8. Consider two countries with the same neoclassical production function in intensive form:

$$y = k^{\frac{2}{3}}.$$

Each country's growth is described by the Solow-Swan growth model in continuous time and with no uncertainty.

(a) Suppose that there is no trade of any kind between the two countries. In country A, the savings rate is $s = 0.20$, the population growth rate is $\eta = 0.10$, and the rate of depreciation of capital is $\delta = 0.10$. Find the steady-state, competitive equilibrium values of the following three measures: the wage rate, the interest rate, and per capita consumption.

(b) Now consider country B. It also faces $s = 0.20$ and $\delta = 0.10$. It has zero population growth, but labour-augmenting technical progress at rate $\lambda = 0.10$. Find the steady-state, competitive equilibrium values of the following two measures: the wage rate, the interest rate. Find the steady-state, competitive equilibrium growth rate of per capita consumption.

(c) Now suppose that there is no movement of capital or labour between the two countries but that ideas can move freely so that labour-augmenting technical progress at rate λ now occurs in country A as well as B. Find the new wage and interest rates and the growth rate of per capita consumption in country A.

(d) Now suppose that movements is both factors, capital and labour, are possible. In which directions, if any, will they tend to move and what will be the effects on the equilibria in the two countries? Do these predictions suggest a test of the standard growth model?

Answer

(a) $sf(k)/k = \delta + \eta$; $f(k) = k^{2/3}$. Thus $k^{-1/3} = 0.20/0.20$ so that $k^* = 1$. Then $f'(k^*) = 2/3 = r + \delta$ so that $r^* = 2/3 - 0.10 = 0.567$ and $w^* = 1 - 2/3 = 1/3$. Total consumption is $C = Q - S = Q(1-s)$. Thus $c^* = q^*(1-s) = 0.8$.

(b) In country B, $sf(k)/k = \delta + \lambda$: here $k^* = 1$, but note that $k = K/L$ not K/N! Then $r^* = 0.567$. The other two variables do not have steady-state levels: c^* grows at rate $\lambda = 0.10$. So does w^* and $w^* = (1/3)k^{2/3}exp(0.10t) = (1/3)exp(0.10t)$.

(c) In country A now $\eta + \lambda = 0.20$. Then $k^{-1/3} = 0.30/0.20 = 3/2$ so $k^* = 8/27$. Thus $r = f'(k) - \delta = (2/3)k^{-1/3} - 0.10$; $r^* = 0.9$. Also, $w = (1/3)k^{2/3}exp(0.10t)$ so $w^* = 0.148 \cdot exp(0.10t)$; and c grows at rate 0.10, as in B.

(d) Country A has lower wages and higher interest rates, so capital tends to move from B to A and humans from A to B. We observe some of this, but we do not observe equalization of growth rates in per capita consumption.

9. One important feature of recent growth in North America is the decrease in productivity growth that apparently occurred in the early 1970s. This question examines the effects of such a slowdown, as predicted by the Solow model of economic growth.

Suppose a constant savings rate s, depreciation rate δ, and population growth rate η. The production function is

$$Q(t) = K(t)^{\frac{1}{3}} L(t)^{\frac{2}{3}}$$

with $L(t) = N(t)exp(\lambda t)$. Q is output, K is the capital stock, L is the effective labour input, N is the population, and λ is the growth rate of labour productivity.

(a) Find the steady-state growth rate of output per capita.

(b) Solve for the real wage and the real interest rate, in the steady state.

(c) Now imagine that λ, the growth rate of labour productivity, declines to a new, steady-state level as it supposedly did in the early 1970s. What are the predicted effects on observable variables?

Answer

(a) The growth rate of output is $\lambda + \eta$. The growth rate of output per capita is λ.

(b) Define k as K/L. Then in steady-state equilibrium:

$$\frac{f(k)}{k} = \frac{\delta + \eta + \lambda}{s}$$

$$k^{\frac{2}{3}} = \frac{s}{\delta + \eta + \lambda}$$

$$k^* = (\frac{s}{\delta + \eta + \lambda})^{\frac{3}{2}}$$

Now, $w = dQ/dN = K^{1/3}(2/3)(Nexp(\lambda t))^{-1/3}exp(\lambda t) = (2/3)k^{1/3}exp(\lambda t)$, so that $w^* = (2/3)[s/(\delta + \eta + \lambda)]^{1/2}exp(\lambda t)$. As you would expect, real wages are growing at rate λ. They decline with higher η or δ and rise with higher s.

Now the interest rate: $\pi = r + \delta = \partial Q/\partial K$ Thus $r = (1/3)k^{-2/3} - \delta$ so $r* = (\frac{1}{3})[\frac{\delta+\eta+\lambda}{s}]^{\frac{1}{2}} - \delta$.

As a check: $Q = wN + \pi K$. Note that w grows at λ, N grows at η, π is constant, K grows at $\lambda + \eta$, Q grows at $\lambda + \eta$.

(c) Lower level of r. Slower growth in real wages, output, consumption.

10. Our growth accounting for North America shows that a slowdown in economic growth over the last three decades largely is accounted for by slower productivity growth. In this question we shall treat productivity growth as exogenous, but study how its rate might affect various, important economic measures. Consider the one-sector Solow-Swan model of growth with standard notation. Suppose that there is zero depreciation. Output is given by

$$Q = K^{\alpha}[N \cdot exp(\lambda t)]^{1-\alpha},$$

where K is capital, N is the labour force, and λ is the rate of growth of labour-augmenting technical progress. Labour is supplied inelastically and we shall assume that there is no population growth. There is a constant savings rate s.

(a) Solve for the steady-state level of the real interest rate.

(b) Solve for steady-state paths of consumption per capita and the real wage.

(c) Suppose that $s = 0.10$ and that $\alpha = 0.30$. Find the steady-state effects on the variables studied in parts (a) and (b) of a decline in productivity growth from $\lambda = 0.02$ to $\lambda = 0.01$. That is roughly the decline in Canada from the 1960s to the 1980s.

Answer

Here $q = k^{\alpha}$, by dividing by effective labour force. Thus $k^{\alpha} = \lambda k/s$, so $k^* = (\lambda/s)^{1/(\alpha-1)}$

(a) The real interest rate is $r = \alpha k^{\alpha-1} = \alpha\lambda/s$.

(b) The real wage is $w = (1-\alpha)k^{\alpha}exp(\lambda t) = (1-\alpha)(\lambda/s)^{\alpha/(\alpha-1)}exp(\lambda t)$. Consumption per capita is growing too. It is $C = Q - S = Q(1-s)$ so $C/N = (1-s)Q/N = (1-s)Lq/N = (1-s)exp(\lambda t)k^{*\alpha} = (1-s)exp(\lambda t)[(\lambda+\eta)/s]^{\alpha/(\alpha-1)}$.

(c) Now $s = 0.10$, $\alpha = 0.30$. And λ falls from 0.02 to 0.01. First the real interest rate: it falls from .06 to .03. Next, real wages from $1.39exp(0.02t)$ to $1.88exp(0.01t)$ Finally, consumption, similar to real wages.

11. This question uses a simple model of economic growth to study a small economy which makes a a transition from autarky to integration in a world economy. Consider the Solow-Swan model of growth with standard notation. Suppose that there is zero depreciation. Output is given by

$$Q = K^{\alpha}[N \cdot exp(\lambda t)]^{1-\alpha},$$

where K is capital, N is the labour force, and λ is the rate of growth of labour-augmenting technical progress. Labour is supplied inelastically, population grows at rate η, and markets are competitive. There is a constant savings rate s.

(a) Suppose that under autarky $\lambda = 0$ so that there is no productivity growth. Solve for the steady-state values of the real wage, the interest rate, and consumption per capita.

(b) Now suppose that ongoing, labour-augmenting technical innovation is available from the world economy at positive rate λ, but that no factor movements are allowed. Describe the steady-state behaviour of the real wage, the interest rate, and consumption per capita.

(c) How do the predicted effects of integration in the world economy change if factor markets are integrated?

(d) Does the answer in part (b) suggest any problems in standard growth accounting?

Answer

(a) If $\lambda = 0$ and $\delta = 0$, then $f(k) = (\eta/s)k$, so

$$k^* = (\frac{\eta}{s})^{\frac{1}{\alpha-1}}$$

Then $r = f'(k) = \alpha k^{\alpha-1} = \alpha(\eta/s)$. Also, $w = (\eta/s)^{\alpha/(\alpha-1)}(1-\alpha)$. And $C/N = (1-s)(\eta/s)^{\alpha/(\alpha-1)}$.

(b) Now the steady state with positive λ.

$$Q = K^{\alpha}[N \cdot exp(\lambda t)]^{1-\alpha} = wN + rK$$

$$r = \frac{\alpha(\eta + \lambda)}{s}$$

$$w = (1-\alpha)k^{\alpha} exp(\lambda t) = (1-\alpha)(\frac{\eta+\lambda}{s})^{\frac{\alpha}{\alpha-1}} exp(\lambda t)$$

$$C/N = (1-s)(\frac{\eta+\lambda}{s})^{\frac{\alpha}{\alpha-1}} exp(\lambda t).$$

So now C/N and w grow and r is larger.

(c) Note that in part (b) output per person grows at the same rate λ in all countries. This equalization does not require capital market integration. If factor markets are open then there will be movements depending on the discrepancies, but we do not have enough information to predict them here.

(d) Explain what growth accounting is. The problem is to speak of country-specific productivity growth or TFP residuals if the shock or exogenous component is really a world component. Notice that Solow residuals need not reflect only neutral technical change.

12. This question uses the Solow-Swan growth model to predict the economic effects of population growth rates that differ across countries. Suppose that there are two, autarkic countries and that they have the same production function, in intensive form,

$$q = f(k) = k^{\frac{1}{3}},$$

where $k = K/N$ is the capital-labour ratio. They also have the same capital depreciation rate $\delta = .05$ and the same savings rate $s = .15$. In country A the population growth rate is $\eta_A = .05$ while in country B it is $\eta_B = .01$.

(a) Solve for wage rates, interest rates, and consumption per capita in the two countries.

(b) In fact, we do not observe that real returns on investment are necessarily higher in countries (such as country A) in which physical capital is relatively scarce relative to labour. Can the model be modified in a sensible way so that it does not make this prediction?

(c) Does the empirical evidence (international cross-section and time series) on the relation between population growth and economic development match up with the predictions of the model?

(d) Suppose that the government in each country can influence the savings rate. Find the golden rule savings rate for each country. What might prevent the savings rate from changing to this value?

Answer

(a)

$$k = \left(\frac{\eta + \delta}{s}\right)^{\frac{1}{\alpha - 1}} = \left(\frac{s}{\eta + \delta}\right)^{\frac{1}{1-\alpha}}$$

$$r = \alpha k^{\alpha - 1} - \delta; \quad w = k^{\alpha}(1 - \alpha) \quad c = k^{\alpha}(1 - s)$$

Thus:

$$\begin{array}{ll} k_A = 1.837 & k_B = 3.95 \\ r_A = 0.17 & r_B = 0.08 \\ w_A = .816 & w_B = 1.05 \\ c_A = 1.04 & c_B = 1.34 \end{array}$$

Thus country B has higher consumption per capita and higher wages but lower real returns.

(b) Simple: differential technical progress. Can show this.

(c) In the model the country with slower population growth has higher income per person. That matches the negative correlation in the cross-sectional data. But

(i) in the data η is endogenous and partly there is feedback from development to η

(ii) in time series countries tend to slow in η if they grow in per capita income,

(iii) also the predictions of the model above change if there is trade between the two countries.

It is important to draw out these general points, rather than cite single observations.

(d) $c = (1-s)q = (1-s)k^{\alpha} = (1-s)[s/(\eta+\delta)]^{\alpha/(\alpha-1)}$ So $c_A = (1-s)(s/.10)^{.5}$ and $c_B = (1-s)(s/.06)^{.5}$. To maximize this is to set $s = .33$ in each country. Then $c_A = 1.22$ and $c_B = 1.57$. We do not observe this because the initial state is dynamically efficient: a sacrifice is demanded of the present generation.

13. One distinction between economic growth in West Germany and that in East Germany from 1950 to 1990 was the faster rate of technical change in the west. This question studies some of the observable consequences of that distinction. Suppose that there is no contact between the two countries and that the population grows at rate $\eta = 0.01$ in both. The depreciation rate is 0.10 in both countries. The savings rate is 0.15 in both countries. But there is labour-augmenting technical progress in West Germany at continuous rate $\lambda = 0.02$ per year, but none in East Germany. Suppose that the production function in intensive form is $y = k^{1/3}$.

(a) If income per capita is the same in the two countries in 1950 then what is the ratio of the two incomes per capita in 1990?

(b) Use the Solow growth model to solve for steady-state wage rates and consumption per capita in both countries.

(c) Is the East German savings rate too high or too low, relative to the golden rule rate which would maximize steady-state consumption per capita?

(d) What factor movements does the model predict to occur at unification?

(e) Does international or historical evidence, more generally, suggest that growth rates or levels of income per capita tend to converge?

Answer

(a) East German output per capita does not change. So the ratio is

$$\frac{y(90)}{y(50)} = exp(0.02 \cdot 40) = exp(.8) = 2.226$$

(b) Now $k = [s/(\delta+\eta+\lambda)]^{3/2}$ so

$$k_E = (\frac{.15}{.10+.01})^{\frac{3}{2}} = 1.59$$

$$k_W = (\frac{.15}{.10 + .01 + .02})^{\frac{3}{2}} = 1.23$$

So in East Germany: $c = q(1-s) = 1.59^{1/3}(.85) = .99$, $w = f(k) - kf' = .778$. In West Germany: $w = (2/3)k^{1/3}exp(.02t) = 0.714 \cdot exp(.02t)$, $C/N = (1-s)q \cdot exp(.02t) = 1.071(.85)exp(.02t) = .9107 \cdot exp(.02t)$.

One also could find levels, rather than these paths, for the West German variables by assuming some initial condition. One possibility would be to assume that West German levels equaled East German ones in 1950. Then in 1990 we would have: $w = 1.73$ and $c = 2.20$.

(c) In East Germany $\eta = 0.01$, while $r = (1/3)k^{-2/3} - \delta = 0.245 - 0.10 = 0.145$. Thus $r > \eta$ so that s is too low relative to golden rule level. That is all we need here, but to see this in terms of consumption, note that:

$$c = q(1-s) = k^{\frac{1}{3}}(1-s) = (\frac{s}{\delta + \eta})^{\frac{1}{2}}(1-s)$$

so to maximize c with respect to s gives $s = .333$.

(d) One can show that $r_W > r_E$ also (because of the ongoing technical progress in the west). Thus the prediction is that both capital and labour would move from East to West, with these factor returns. In fact, at unification we observed labour moving east to west but some capital moving west to east. But, some of that was government capital. And an increase in λ in the east raised r there. Plus, eastern unions tried to maintain high eastern w; the model's assumption of competitive markets may be misleading.

(e) Not really. There are a variety of ways of answering this. See the Baumol and also Jovanovic evidence.

14. Consider the basic Solow-Swan growth model with a constant savings rate s, zero depreciation, constant population growth η, and a Cobb-Douglas production function:

$$Q = K^{\alpha}L^{1-\alpha}.$$

Suppose that there is labour-augmenting technical progress at rate λ.

(a) Consider two countries, denoted 1 and 2, which are identical except that $\eta_1 > \eta_2$. Which will have higher per capita income?

(b) Is this cross-section prediction about the relationship between population growth and per capita income levels supported by the historical evidence?

(c) What other observable differences would there be between the two countries?

(d) What links between economic growth and population growth are omitted from the model?

Answer

(a) Country 2 will have higher per capita income. Its ray will be shallower so higher output level for a given level of population.

(b) Yes. A negative relationship between population growth and the level of per capita income, in cross-sections. That raises the question of the direction of causality in this cross-sectional relationship.

(c) Country 1 will have higher profit rate and lower wage rate than country 2. That also could serve as a test. Country 1 also will have lower capital per person and lower consumption per person. Also there may be observable tendencies for migration from country 1 to country 2 and for direct investment from country 2 in country 1.

(d) The time series evidence is that η declines with development. So there is a feedback; η is endogenous. Plus discussion of immigration, mortality and so on. Plus the composition of the labour force (age, training); plus unemployment and how the labour market functions. Plus urbanization and the alleged role of cities.

15. An economist seeks to explain why Japanese and South Korean firms are locating production in one central American country (called G) but not in another (called H). She uses the Solow-Swan growth model to try to explain this. Suppose that the real interest rate in Japan and South Korea is 5.5%. The technology is assumed to be Cobb-Douglas with parameter α. The investigator estimates the following parameters for countries G and H.

Parameter	Country G	Country H
η	0.01	0.02
λ	0.03	0.00
δ	0.03	0.00
s	0.20	0.20
α	0.30	0.30

(a) Solve for the steady-state real interest rate in country G *and* in country H under autarky.

(b) Are the interest rates implied by the growth model consistent with the observed flows of direct investment mentioned above?

(c) Solve for steady-state real wages in each of the two countries, under autarky.

Answer

(a)

$$r = \alpha k^{\alpha-1} - \delta = \alpha[\frac{\delta+\eta+\lambda}{s}] - \delta$$

so that $r_G = .075$ and $r_H = .03$.

(b) Yes. But autarky may be misleading. The model also predicts that capital will flow from country H to country G.

(c)

$$w = k^{\alpha}(1-\alpha)e^{\lambda t} = [\frac{s}{\delta+\eta+\lambda}]^{\frac{\alpha}{1-\alpha}}(1-\alpha)e^{\lambda t}$$

So

$$w_G = 1.0977exp(.03t)$$

$$w_H = 1.8778$$

16. In the 1930s in the U.S.S.R. the state promoted accumulation and investment in industry. This question uses the simple Solow-Swan growth model to describe optimal savings rates and the transition from one saving rate to another.

Suppose that there is no labour-augmenting technical progress. The initial savings rate is $s = 0.12$. The depreciation rate is $\delta = 0.04$ and the population grows at rate $\eta = .02$. The technology is

$$q = f(k) = k^{\alpha},$$

with $\alpha = 0.4$.

(a) Find the real interest rate r in a steady-state equilibrium.

(b) Is the savings rate above or below the golden rule level? Is the economy dynamically efficient?

(c) Describe (qualitatively and briefly) the effects on output and output per capita of a transition to a higher savings rate.

(d) What proportion of current consumption per capita c would have to be sacrificed to change immediately the capital-labour ratio to the golden rule level?

Answer

(a)

$$r = \alpha k^{\alpha-1} - \delta = \alpha(\frac{\delta+\eta}{s}) - \delta = 0.16$$

(b) Clearly $r > \eta$ so the savings rate is below the golden rule level. The economy is dynamically efficient, and a sacrifice would be required of the current generation (at least) to increase the capital-labour ratio and steady-state consumption per capita.

(c) Output and capital would grow faster than population in the transition. Also, profit rates would fall and wage rates rise as capitalization increased, though these effects might not be detectable in a centrally planned economy. The USSR grew rapidly in the 1930s, 1940s, and 1950s perhaps partly for this reason. A second source of transitional growth was the destruction due to war in the 1920s and especially in the early 1940s. Some western analysts (*e.g.* at Langley, Va.) may not have realized that this growth was transitional rather than steady-state.

17. (a) In data from a cross-section of countries we typically observe a negative relationship between the level of output per capita and the rate of growth of population. Show whether or not the Solow-Swan growth model is consistent with this negative relationship. Does your conclusion depend on any assumptions about labour mobility?

(b) A time series counterpart to this negative relationship is the 'demographic transition'. The observation here, in general, is that growth of per capita output is associated with a decrease in the rate of growth of population. Is the neoclassical growth model consistent with this second fact, whether in transitions or in steady states?

Answer

(a) This is simple to show with the basic diagram for the growth model. At higher η the ray is steeper, so output per capita is lower. The negative correlation depends on limits to labour mobility (or capital mobility!) which seems realistic. If there are no such limits then the model predicts that output per capita will be equalized across countries.

(b) Consider an exogenous decline in the population growth rate. In the transition to a new steady state, output per capita will rise faster than otherwise. That (crudely) reproduces the demographic transition experienced in European countries, for example. But the idea of that transition is that η is endogenous and that causality runs partly from growth in output per capita back to population growth; that channel is not present in the growth model.

18. One of the weaknesses of the basic Solow-Swan model of economic growth is that it predicts that growth should be explained by capital accumulation and population growth. Other factors seems to matter empirically. A second weakness is that it predicts that there should be large capital flows from rich to poor countries. Again, these are not observed very generally. This question studies whether including a third factor, human capital, (in addition to capital and labour) in the production function can resolve these problems. (See section 3.1 of Jones (1998) for more on this modification.)

(a) Suppose that in a developed country the production function is

$$Q = K^{\alpha}(NH)^{1-\alpha},$$

where Q is output, K is the capital stock, N is the population, and H is human capital. Suppose that $\alpha = 0.4$, $\dot{N} = 0.01$, and $\dot{H} = 0.04$. If $\dot{Q} = \dot{K}$, find the growth rate of output per capita.

(b) In the same economy suppose that there is a constant savings rate $s = 0.20$ and a depreciation rate of 5 percent. Find the real interest rate.

(c) Now suppose that the world also contains a less-developed economy, with a production function of the same form and the same depreciation rate. That economy has population growth of $\dot{N} = 0.05$. Assume that capital can move freely internationally. Describe the combinations of savings rates *and* growth rates of human capital in this second economy that make its real interest rate no greater than that in the developed economy. Is this a plausible explanation of limited capital flows?

(d) Is the empirical evidence on convergence consistent with this model?

Answer

(a) The growth rate of output per capita is 0.04 or 4 percent. The model now might be roughly consistent with the panel data study of Mankiw *et al.*

(b) The real interest rate is

$$r = \alpha k^{\alpha-1} - \delta = \alpha(\frac{\lambda + \eta + \delta}{s}) - \delta$$

So $r_D = 0.15$.

(c) We want

$$r_L = 0.4(\frac{\lambda + .10}{s}) - 0.05 \leq 0.15.$$

This means that

$$\lambda + 0.10 \leq 0.5s$$

Because $\lambda > 0$ that means that $s > 0.20$ so that the savings rate in the LDC must be at least as great as that in the DC. That seems implausible. If there is any tendency for human capital to be mobile internationally then λ will certainly be larger than zero in the LDC so that s there will have to be even higher to rationalize the observed non-flow of capital.

(d) Generally the evidence on convergence is not consistent with this.

19. In 1960 South Korea and the Philippines had approximately the same level of output per capita. From 1960 to 1988 output per capita grew at an average rate of 1.8% in the Philippines and 6.2% in South Korea. This question views this evidence from the perspective of some growth theory. Imagine that each country has a constant savings rate.

(a) One possibility is that there has been faster human capital accumulation in Korea than in the Philippines. Suppose that output in country i is given by:

$$Q_i = K_i^{\alpha}(N_i H_i)^{1-\alpha},$$

where Q is output, N is population, K is the capital stock, and H is the stock of human capital. Suppose that human capital accumulates in proportion to time spent in schooling, given by μ, as follows:

$$\frac{dH_i}{dt} = \mu_i H_i.$$

What is the growth rate of output per capita in the steady state of this standard model?

(b) The model in part (a) suggests that output per capita grew faster in Korea because the learning rate there (μ) was large. But suppose that learning also spills over internationally, and specifically that

$$\frac{dH_i}{dt} = \mu_i \frac{\bar{H}}{H_i},$$

where $\bar{H}$ is the average level of human capital across countries. How does this affect the model's ability to mimic sustained differences in growth rates of output per capita?

(c) An alternative view is that capital, labour, and ideas are largely immobile between countries. Suppose that we attempted to explain the faster growth in Korea as arising from capital accumulation. For example, suppose that savings rates initially are equal but that the Korean savings rate increases. Could this explain the facts? (Hint: Can you construct an example with realistic savings and profit rates?)

Answer

(a) This first model is simply the Solow-Swan model with exogenous labour-augmenting technical progress. So output per capita grows at rate μ_i. This would seek to explain the disparities as occurring in the steady state because of different values of μ_i.

(b) With the spillover the Philippines, with lower H_i than average, should grow faster. So output per capita should converge.

(c) Suppose that the two economies are initially identical but then the Korean savings rate rises to a higher level. There will be transitional dynamics and faster growth (growth in output per capita) as a result. But a numerical example shows that the (temporary) change in output growth resulting from a change in the savings rate is approximately the profit rate times the change in the savings rate. That is too small to explain what happened.

20. Measured productivity has grown more slowly recently than it did in the 1950s and 1960s. To evaluate this well-known observation, consider the Solow-Swan growth model with labour-augmenting technical progress. Suppose that $\eta = 0.01$, that $\lambda = 0.03$, and that $s = 0.15$, in standard notation. Suppose that the production function is

$$Q = K^{\alpha}(Nexp(\lambda t))^{1-\alpha}$$

with $\alpha = 0.35$.

(a) Show growth accounting in the steady state; *i.e.* numerically decompose the growth rate of output into components due to capital accumulation, labour force growth, and technical progress.

(b) Next suppose that the savings rate drops immediately to $s = 0.10$. An economist, unaware of this, cannot measure $\dot{K}$ accurately but can measure $\dot{Q}$ and assumes that $\dot{K} = \dot{Q}$, as in the steady state of this model. Show that this economist will find an apparent slowdown in productivity growth.

Answer

$$\dot{Q} = \alpha\dot{K} + (1-\alpha)\dot{N} + (1-\alpha)\lambda$$
$$0.04 = (.35)(.04) + (.65)(.01) + (.65)(.03)$$
$$0.04 = .014 + .0065 + .0195$$

(b) It is still true that $\dot{N} = 0.01$ and $\dot{L} = 0.04$. In the transition K/L and Q/L fall so in the transition $\dot{K} < 0.04$ and $\dot{Q} < 0.04$. Moreover, $\dot{Q}$ is a linear combination of $\dot{K}$ and $\dot{L}$ so $\dot{Q} > \dot{K}$ in the transition. Therefore the investigator will overestimate $\dot{K}$ and underestimate TFP growth.

21. A number of historical descriptions of growth describe the growth rate of output as increasing and then returning to approximately its original level. This question studies the predictions of the Solow-Swan model for the behaviour of the economy on such a path.

Suppose that population growth is $\eta = 0.01$, the depreciation rate is $\delta = 0.05$, and the constant savings rate is $s = 0.20$. The production function in intensive form is

$$\frac{Q}{L} = f(k) = k^{\frac{1}{3}}.$$

There is labour-augmenting technical progress at rate λ.

(a) Suppose that $\lambda = 0.02$. Solve for the real wage rate w *and* the real interest rate r in the steady state.

(b) Now suppose that we attribute an observed increase in growth of output per capita to an increase in λ to 0.04. A burst of growth might be due to a temporarily higher λ; we shall derive implications of that by looking at the steady state with a higher value. Solve for the real wage rate w *and* the real interest rate r in the steady state.

(c) An alternative view is that faster growth which is temporary arises from a temporary increase in the savings rate s, and that we observe the transitional dynamics. Are there any observable implications that distinguish this view from the one in part (b)?

Answer

(a) With $\lambda = 0.02$, then

$$k = 3.95$$
$$r = 0.083$$
$$w = 1.054\, exp(0.02t)$$

(b) With $\lambda = 0.04$. then

$$k = 2.83$$
$$r = 0.1166$$

$$w = 0.9428\, exp(0.04t)$$

(c) From a picture or example, we know that an increase in s which is temporary will tend to raise the value of k. So in the transitional dynamics output growth per capita will be faster than 0.02. One of the main differences though is that the real interest rate will be *lower* during the period of faster growth if this is the cause. Other implications would concern exact dynamics. Note that the definition of k differs between the two cases.

22. Some studies of the international economy suggest that foreign direct investment in less-developed economies is relatively limited. In this question, we see whether the neoclassical one-sector growth model with a constant saving rate can be consistent with that fact.

Consider the Solow-Swan model with labour-augmenting technical progress at constant rate λ and population growth at constant rate η. Suppose that there is no depreciation. There are two countries denoted LDC and DC. In LDC the parameters are $s = .1$, $\lambda = 0$, and $\eta = .03$. In DC the parameters are $s = .12$, $\lambda = .04$, and $\eta = 0$. Both countries have the same production function in capital and effective labour:

$$Q = F(K, L) = K^{0.3} L^{0.7}.$$

(a) Solve for real interest rates (profit rates) in the two countries under autarky, in steady-state competitive equilibrium. Is the result consistent with the evidence that capital does not much tend to flow from DC to LDC?

(b) Other implications of the model must be verified for this explanation to be persuasive. Solve for consumption per capita in each country, under autarky, in steady-state competitive equilibrium.

(c) Find the savings rate s in LDC in which the competitive equilibrium satisfies the golden rule. Is the LDC economy dynamically efficient?

(d) In fact, countries LDC and DC may be integrated in some ways (not autarkic). Does that affect our explanation in part (a) for the absence of capital flows to LDC?

Answer

(a) In the Cobb-Douglas case we have

$$r = \frac{\alpha(\eta + \lambda)}{s}$$

Thus $r(LDC) = 0.09$ and $r(DC) = 0.10$. Thus the profit rate is higher in DC, which is consistent with the evidence.

(b) Consumption per capita is given by:

$$C/N = (1 - s)\Big(\frac{s}{\lambda + \eta}\Big)^{\alpha/(1-\alpha)} exp(\lambda t)$$

Thus in LDC $C/N = .9(.1/.03)^{.3/.7} = 1.507$. In DC $C/N = .88(.12/.04)^{.3/.7} exp(.04t) = 1.41 exp(0.04t)$.

(c) In LDC $r = .09 > .03 = \eta$ so that the economy is dynamically efficient. In the competitive equlibrium:

$$k^* = \frac{s}{\eta}^{1/(1-\alpha)}$$

but also

$$r^* = \alpha \cdot k^{\alpha - 1}.$$

Thus

$$r = \frac{\alpha\eta}{s}$$

So to make $r = \eta$ we need

$$\frac{\alpha\eta}{s} = \eta$$

which for $\alpha = .3$ and $\eta = .03$ implies $s_G = 0.3$.

(d) Yes. Suppose that integration tends to equalize rates of technical progress, even if there are no international movements of labour. Then $r(LDC) > r(DC)$ so a capital flow would be expected. This may cause

23. Consider the simplest Solow-Swan growth model with constant savings rate s, constant population growth rate η and zero depreciation. Begin from a steady-state, competitive equilibrium.

(a) Find the steady-state effects of a permanent increase in s to a new, higher value. Simply give qualitative results, using a diagram or a simple example such as the Cobb-Douglas case.

(b) Describe qualitatively the transitional dynamics for K, L, and Q. Rank the growth rates of these three variables in the transition.

(c) Suppose growth accountants use a Cobb-Douglas production function and know the true α. They thus reckon, correctly, that

$$\dot{Q} = \alpha \, \dot{K} + (1 - \alpha) \, \dot{L} + \dot{z}$$

where $\dot{L} = \eta$. The true productivity growth is $\dot{z} = 0$, but they do not know that. They are unaware that the history includes the transition from one steady-state to another which you have studied in parts (a) and (b). They cannot observe K directly and so they incorrectly use the prediction, from the steady-state of the Solow-Swan model, that $\dot{Q} = \dot{K}$ to estimate $\dot{K}$, because $\dot{Q}$ is observable. Describe the conclusions which they will reach about the Solow residual or productivity growth.

Answer

(a) From the usual diagram, r will be lower, w will be higher, while q and k and c will be higher (below the golden rule level).

(b) We know that for k to rise $\dot{K} > \dot{L}$ in the transition. Thus $\dot{K} > \dot{Q} > \dot{L} = \eta$.

(c) If the accountants impose that $\dot{K} = \dot{Q}$ then clearly they will conclude that $\dot{z} = \alpha(\dot{K} - \dot{Q})$, an overestimate.

24. Consider the Solow-Swan growth model with constant savings rates, labour augmenting technical change at constant rate λ, and population growth at constant rate η. Suppose there are two countries, Canada (CDN) and Mexico (MEX). In CDN $s = 0.12$, $\lambda = 0.04$, $\eta = 0.0$. In MEX $s = 0.1$, $\lambda = 0.0$, and $\eta = 0.03$. Assume that both countries have the same production function in capital and effective labour:

$$Q = F(K, L) = K^{0.3}L^{0.7},$$

there is no depreciation, and all markets are competitive.

(a) Solve for steady-state equilibrium real interest rates in the two countries under autarky. Is the result consistent with the observation that capital (foreign direct investment) does not always flow from more developed to less developed countries?

(b) Is the MEX economy dynamically efficient? Find the MEX savings rate for which the autarky steady-state equilibrium satisfies the golden rule.

(c) Now, in contrast to the autarkic equilibrium discussed above, suppose that NAFTA tends to equalize rates of technical progress even if there are no international movements of labour. What is the steady-state real interest rate for MEX if λ increases from 0 to 0.4? Will this diffusion of technical progress affect the explanation (in part (a)) for the absence of capital flows to the less developed economy? Does this result shed any light on the discussions concerning intellectual property rights during the recent GATT negotiations?

(d) What are the implications of the integration discussed in (c) for real wages?

(e) Suppose that instead of (c) the MEX government successfully encourages an increase in s from 0.1 to 0.3. How does the implied steady-state consumption per capita compare to that in part (c)?

25. An important use for growth models is to help us predict the effects of policy changes. This question uses a simple growth model to predict the effects of changes in the savings rate.

Consider the neoclassical one-sector growth model with constant savings rate s. Suppose that there is no depreciation or technical change. The intensive-form production function is:

$$f(k) = k^{\frac{1}{3}}.$$

(a) An economist proposes a change in tax policy which will increase the savings rate. You do not know the current saving rate or the current population growth rate, η. Nevertheless, you can find the long-run (steady-state) elasticity of the real wage with respect to the savings rate, and hence predict some of the effects. Find $dlog(w)/dlog(s)$.

(b) If in fact $\eta = 0.03$ and s rises from 0.09 to 0.12 find the steady-state effect on per capita consumption.

(c) Imagine two countries, each with $\eta = 0.03$ and each with $s = 0.09$. Capital is mobile between the two countries but labour is not. In one country the savings rate suddenly increases to $s = 0.12$. Describe the transitional dynamics in the world economy.

(d) Now consider a different historical experiment. Again both countries initially have the same savings rates and population growth rates. Again capital is mobile. Now half the capital stock is destroyed in the first country. Describe the transitional dynamics in the world economy.

Answer

(a)

$$k^{\frac{1}{3}} = \frac{\eta}{s}k$$

$$k = \left(\frac{s}{\eta}\right)^{\frac{3}{2}}$$

Thus

$$w = \frac{2}{3}k^{\frac{1}{3}} = \frac{2}{3}\left(\frac{s}{\eta}\right)^{\frac{1}{2}}$$

Thus the elasticity of the wage with respect to the savings rate is 0.5.

(b)

$$c = (1 - s)\left(\frac{s}{\eta}\right)^{\frac{1}{2}}$$

so that the initial value is 1.574 and the final value is 1.76.

(c) In the first country real wages begin to rise and real interest rates to fall. Also capital grows faster than population in transition and so output and consumption grow faster than population. The steady state effect is to have higher real wages and lower interest rates, and higher output and consumption per capita. In the long-run output and consumption again grow at the rate of population growth.

In the case with two countries and immobile labour, though, capital will flow out of the country in which the savings rate has risen and into the other country where profit rates are not falling. This inflow will raise the capital-labour ratio in the capital-importing country so that wages rise there and profit rates fall, and, again, output per capita and consumption per capita rise in the long run. In the steady state capital will flow steadily from the high s country to the low s country and they will have the same wages, interest rates, and per capita consumption and output. [You might think about how this capital flow will be reflected in the external balance conditions and national income accounting of the two countries.]

(d) In the country which loses half its capital stock the profit rate will be high and wages low. Thus capital will flow into that economy; this will raise the capital-labour ratio faster than would be the case under autarky. In fact, in the theoretical model the adjustment can be split evenly between the two countries. The country that has not lost capital will ship capital equal to half the other country's loss, so that profit rates are equalized. Then they will grow in tandem with wages rising and profit rates falling and consumption per capita rising to the initial steady state.

26. The tendency for population growth rates to decline as income rises sometimes is called the 'demographic transition.' In our basic growth model the rate of population growth is exogenous, so we cannot model this transition completely. Nonetheless, we can see what the theory predicts as the economic effects of a decline in η, the growth rate of population.

(a) Consider the simplest Solow-Swan growth model, with no depreciation or technical progress, constant savings rate s, and intensive-form production function $q = k^{0.4}$. Solve for the capital-labour ratio k in a steady state.

(b) Find the elasticity of output per capita with respect to the population growth rate.

(c) If η declines from 0.03 to 0.01 then what is the steady-state percentage change in output per capita?

(d) Would you predict that the rate of profit would rise or fall for an economy which experiences this transition?

Answer

(a)

$$k = (\frac{s}{\eta})^{\frac{1}{1-\alpha}} = (\frac{s}{\eta})^{1.667}$$

(b) The elasticity is 2/3 and is constant.

(c) Population growth declines by 66.7 percent so output per capita rises by 44.4 percent.

(d) Fall, clearly.

27. Some countries in south-east Asia have experienced rapid growth in labour force participation rates. In this question, we shall see how changes in the aggregate participation rate might be predicted to affect economic growth. Suppose that output is produced from the following technology:

$$Q = K^{0.3}L^{0.7},$$

where K is the capital stock and L is the labour force. Suppose that

$$L = \rho N,$$

where N is the population and ρ is the participation rate. Suppose that $\dot{N} = 0.02$, and there is a constant savings rate $s = 0.14$. The depreciation rate is $\delta = 0.05$.

(a) Suppose that ρ is constant, with value 0.5. Solve for the steady-state real interest rate, the real wage rate, and output per capita.

(b) Now suppose that $\rho = 0.75$. Solve for the same three variables in the steady state.

(c) Describe the transitional dynamics, if ρ rapidly changes from 0.5 to 0.75.

(d) Can the behaviour of average labour productivity (output per employed worker) distinguish this explanation for rapid growth from some alternatives (such as an increase in the savings rate)?

Answer

(a) $r = 0.10$, $w = 0.94$. Output per L is 1.345 so output per person is 0.6725.

(b) The real interest rate and wage rate are unchanged, but output per capita is now 1.00.

(c) An increase in the participation rate is like a temporary increase in the population growth rate in the usual Solow-Swan model. So $\dot{L} > 0.02$ temporarily. In the transition we see higher real interest rates and lower real wages (hence also capital inflows if the economy is open); then as capital is accumulated factor prices return to their original levels. To draw this, imagine the economy moved back along the intensive production function, without rotating the ray. Also, remember the intuition that an increase in labour supply will lower real wages, which then rise as capital is accumulated.

(d) Output per employed worker *falls*, in the transition, and then rises. This contrasts with growth explanations based on rapid productivity growth due to sectoral shifts or learning by doing. It seems to fit the experience of Singapore rather well. This can be contrasted with the transitional effects of an increase in the savings rate, say, or with steady-state explanations.

28. Suppose that two countries, denoted D and U, have the same production function:

$$Y = K^{0.4} N^{0.6},$$

where Y is output, K is the capital stock, and N is the population. The depreciation rate is zero. Suppose that population growth rates are $\eta_D = 0.01$ and $\eta_U = 0.04$. Suppose that savings rates are $s_D = 0.10$ and $s_U = 0.06$.

(a) Under autarky, and using the Solow-Swan model, find output per worker and the real interest rate in each country.

(b) Find the capital-labour ratio in country D.

(c) Suppose that capital can move between countries, and imagine that the capital inflow to country U *doubles* the rate of growth of the capital stock that it experienced under autarky. Use growth accounting to see whether this will have a large effect on the output growth rate there.

(d) Now suppose that the production function is

$$Y = K^{0.4} (AN)^{0.6},$$

where $A \geq 1$ is a factor (such as health) which makes workers in country D more productive than workers in country U. In country U $A_U = 1$. If the capital-labour ratio K/N is unchanged from part (b) what value of A_D would make r_D equal to r_U? (The idea is that this might explain the meagre capital flows actually observed between developed and less-developed economies.)

(e) Is the evidence on international capital flows consistent with the evidence on convergence?

Answer

(a) In this case $r = \alpha k^{\alpha-1}$ and $k^{\alpha-1} = \eta/s$. In U the ratio is 2/3 so $r_U = 0.2666$. In D the ratio is 0.1666 so $r_D = 0.04$. Also, $q_U = 1.3162$ and $q_D = 4.641$.

(b) In country D, $k = 6.0^{1.667} = 46.4$.

(c) In autarky: $\dot{Y} = \dot{K} = \dot{N} = 0.04$. Now $\dot{K} = 0.08$. Thus

$$\dot{Y} = 0.4(0.08) + 0.6(0.04) = 0.056.$$

So the growth rate rises only from 4% to 5.6% even when the investment rate doubles.

(d) By differentiating the production function, we see

$$r = 0.4(\frac{K}{AN})^{-0.6} = A^{0.6} \cdot 0.04$$

where 0.04 was the D interest rate we found in part (a). Setting $r = 0.2666$ gives $A = 23.6$. So workers would have to be more than twenty times more productive in country D than in country U.

(e) Most flows are between rich countries not rich to poor. Part (c) shows enormous flows would be necessary to bring about convergence in levels of output. And the evidence on convergence is somewhat mixed; there may be convergence within 'clubs' but not between rich and poor countries.

At a deeper level, the shortage of capital flows in this direction probably does not explain the absence of convergence. We see in part (c) that even an enormous flow would have a modest effect on growth. See the essay by Krugman in Giovannini, ed. (1994).

29. Imagine a developed economy with no population growth. Its production function is:

$$Q = K^\alpha (Ne^{\lambda t})^{1-\alpha},$$

where λ is the rate of labour-augmenting technical progress. The depreciation rate is zero and there is a constant savings rate s.

(a) Find expressions for the real wage and consumption per capita on a steady-state growth path.

(b) Suppose that this country then establishes links with a second country, which has the same savings rate and zero depreciation rate. But the second economy has zero technical progress ($\lambda = 0$) and positive population growth ($\eta > 0$). Describe the transitional dynamics in the world economy.

Answer

(a)

$$w = (\frac{s}{\lambda})^{\alpha/1-\alpha}(1-\alpha)e^{\lambda t}$$

$$\frac{C}{N} = (1-s)(\frac{s}{\lambda})^{\alpha/1-\alpha}e^{\lambda t}$$

(b) This depends on whether λ can be transmitted internationally or not ...

30. This question investigates a two-country model of growth, in continuous time. In country i the production function is:

$$Y_i = K_i^{\alpha}(N_i A_i)^{1-\alpha},$$

where N is the labour force. Assume that $\dot{N} = 0$ and $N = 1$. The technology parameter A_i is given by:

$$A_i = \gamma K_i$$

as a result of learning by doing. This externality is not taken into account by firms, who behave competitively. There is depreciation at rate δ. The savings rate is a constant, s.

(a) Find an expression for the growth rate of output in country i.

(b) Suppose that country j has a smaller capital stock than does country i. Its labour productivity parameter depends on country i's capital stock, because of an international spillover:

$$A_j = \gamma K_i.$$

Find an expression for the growth rate of capital in country j.

(c) Suppose that in both countries $\alpha = 0.30$, $\delta = 0.10$, $\gamma = 0.50$, and $s = 0.20$. Graph the growth rates of output in both countries against time, assuming that initially i has twice as much capital as j.

(d) What is the evidence on conditional and unconditional convergence?

Answer

(a) $\dot{Y} = s(\gamma)^{1-\alpha} - \delta$

(b) $\dot{K}_j = s(\gamma)^{1-\alpha}(\frac{K_i}{K_j})^{1-\alpha} - \delta$

(c) In country i the growth rate is 2.31 percent. In country j the growth rate of output is:

$$\dot{Y}_j = \alpha \dot{K}_j + (1-\alpha)\dot{K}_i.$$

So initially the growth rate in j is $0.3(9.9) + 0.7(2.3) = 4.58$ percent. Then it gradually declines to 2.31 percent.

(d) There is considerable evidence of conditional convergence but very little evidence of unconditional convergence. See Brander and especially Barro and Sala-i-Martin.

31. Suppose that the economy can be modelled with the Solow-Swan growth model in continuous time, with constant savings rate s. The production function is:

$$Y(t) = K(t)^{\alpha} L(t)^{1-\alpha},$$

where L is the effective labour supply. Let λ be the growth rate of labour productivity, N the labour force, and η the growth rate of the labour force. Then:

$$L(t) = exp(\lambda t)N(t) = exp(\lambda t)exp(\eta t).$$

(a) Show how growth accounting could be used to learn the value of λ.

(b) Suppose that λ falls permanently and unexpectedly. What are the predicted effects on real wages and the real interest rate? Comment on the likely size of these effects.

(c) Are there explanations for the productivity growth slowdown other than an exogenous decline in productivity growth?

Answer

(a) Log-differentiate and set $\dot{K} = \dot{Y}$, so $\dot{Y} - \eta = \lambda$.

(b) In equilibrium,

$$k^{\alpha-1} = \frac{\eta + \lambda + \delta}{s}$$

Thus

$$r = \alpha k^{\alpha-1} - \delta = \alpha \frac{\eta + \lambda + \delta}{s} - \delta$$

Notice that $dr/d\lambda = \alpha/s$ so this effect certainly should be noticeable. Also

$$w = (1-\alpha)k^{\alpha} exp(\lambda t) = (1-\alpha)exp(\lambda t)(\frac{s}{\eta + \lambda + \delta})^{\alpha/(1-\alpha)}.$$

(c) Of course. Examples include measurement problems and the oil price shock.

32. In Canada during the past forty years the savings rate has declined and so has the *measured* rate of productivity growth. This question investigates whether the former could be responsible for the latter. Imagine the Solow-Swan growth model with production function

$$Q = K^{0.3}N^{0.7}$$

or in intensive form

$$q = k^{0.3}.$$

The depreciation rate is 0.05, the population growth rate is 0.01, and there is no labour-augmenting technical progress.

(a) If the savings rate is $s = 0.24$, solve for each of the capital-labour ratio, the real interest rate, and the wage rate, in a steady-state equilibrium.

(b) Now suppose s falls immediately and permanently to 0.18. What are the steady-state values for r and w now?

(c) Using a diagram, describe the transitional dynamics between the two steady states. Could this ever be mistaken for a productivity growth slowdown?

Answer

(a) $k = 7.245$; $r = 0.025$; $w = 1.268$

(b) $k = 4.80$; $r = 0.0499$; $w = 1.12$.

(c) In the transition (shown in a diagram) w declines and r rises. A decline in λ in contrast would tend to lower r (the ray from the origin would rotate down instead of up). As for quantities, K would grow more slowly than N in the transition, but if that were correctly measured then there would be no Solow residual since growth accounting does not require a steady state assumption.

33. This question studies the predictions of growth theory for optimal saving rates. Consider the Solow-Swan model with production function

$$q = k^{\alpha}$$

in intensive form, and no technical progress. Denote by s, η, and δ the rates of saving, population growth, and depreciation of capital.

(a) Prove that the golden rule maximizes steady-state consumption per capita.

(b) If $\delta = 0.10$, $\alpha = 0.3$, and $\eta = 0.02$ then what is the golden rule level of s? For what values of s is the economy dynamically efficient?

(c) What is the path of consumption per capita over time in response to an increase in the saving rate, with the parameter values in part (b)? You can answer this question using a diagram, or a numerical example, or a general mathematical example (using the relationship between c and k).

Answer

(a) $c = (1-s)q = q - sq$. But in a steady state $sq = (\eta + \delta)k$, so $c = k^{\alpha} - (\eta + \delta)k$. Differentiating with respect to k gives $\pi = \eta + \delta$ or $r = \eta$ which is the golden rule. Second-order conditions are satisfied from the convexity of the production function.

(b) The golden rule value of s is 0.3. The economy is dynamically efficient for any savings rate less than or equal to that value.

(c) The fundamental equation is

$$\dot{k} = sk^{-.7} - 0.12.$$

Then $c = (1-s)k^{0.3}$, so

$$\dot{c} = \alpha\dot{k}.$$

The main point is that, with an increase in s (starting from $s < 0.3$) c initially drops, then follows a rising path to a higher steady-state level. That initial drop may explain why we are not at the golden rule s. If instead the initial savings rate is 0.3 or greater then consumption per capita falls in both the short run and long run if s rises.

34. This question examines the effects of differential productivity growth across countries. Imagine a country with aggregate production function:

$$Q = K^{0.5}L^{0.5},$$

where $L = e^{\lambda t}N$ and N is the labour force. The depreciation rate is zero. The savings rate s and population growth rate η are constant.

(a) Find expressions for per capita consumption, the real wage, and the real interest rate in this economy.

(b) One of the causes of convergence is alleged to be factor movements, in reponse to international factor price differentials. Imagine a second economy with no technical progress: $\lambda = 0$. Can *both* factor prices be lower in this second economy than in the first one?

(c) Suppose the first country has $s = 0.25$, $\lambda = 0.04$, and $\eta = 0.01$. The second country has $\lambda = 0.00$, and $s = 0.20$. How fast must population grow in the second country for there to be a capital inflow?

(d) Is such an inflow required for convergence?

Answer

(a)

$$r = 0.5(\frac{(\lambda + \eta)}{s})$$

$$w = 0.5(\frac{s}{\lambda + \eta})exp(\lambda t)$$

$$C/N = (1-s) \cdot (\frac{s}{\lambda + \eta})exp(\lambda t)$$

(b) Yes. With the same η and λ for example, r will be lower and w will be lower because it will not be growing. Hence there would be no capital inflow.

(c) If $\eta > 0.04$ then there will be a capital inflow.

(d) Convergence also could occur if λ can be transmitted internationally, or if labour is mobile.

35. This question uses the Solow-Swan growth model to make some predictions about growth in south-east Asia. Suppose that Japan and Malaysia each have the same production function:

$$Q = K^{\frac{1}{3}} N^{\frac{2}{3}}.$$

In both countries the depreciation rate is $\delta = 0.05$ and there is no technical progress. In Japan, $\eta_J = 0.01$ and $s_J = 0.30$, while in Malaysia $\eta_M = 0.03$ and $s_M = 0.15$.

(a) In autarky (with no trade or capital flows), find the wage rates and real interest rates in the two countries.

(b) A growth accountant who is studying Malaysia correctly measures η_M. The accountant measures $\dot{K}$ as

$$\frac{s_M Q_M}{K_M} - \delta.$$

In reality, there are capital flows between the countries (though labour cannot flow from Malaysia to Japan). What will the accountant conclude about technical progress in Malaysia?

(c) Can careful growth accounting shed light on the sources of rapid growth in some south-east Asian countries?

Answer

(a) In Japan $r = 0.0167$ and $w = 1.49$. In Malaysia $r = 0.128$ and $w = 0.913$.

(b) Clearly the accountant would underestimate the growth of the Malaysian capital stock, and so would overestimate technical progress.

(c) [Discuss the examples of Hong Kong and Singapore.]

36. Many models of endogenous growth rely on a positive spillover or externality to keep growth going. An example is 'learning by doing', in which human capital is acquired in proportion to experience working with physical capital. For example, suppose that the production function is:

$$Q = K^{\alpha}(NH)^{1-\alpha},$$

where $N = 1$ so that there is no growth in the labour force. Human capital is proportional to physical capital,

$$H = \theta K \qquad \theta > 0,$$

though firms do not take this into account in their investment decisions. The saving rate is constant at s and there is no depreciation.

(a) Find the wage rate and the growth rate of output.

(b) Does the international evidence support the prediction that higher savings rates lead to higher growth rates?

(c) Describe any other examples of growth based on positive spillovers.

Answer

(a) $\dot{Y} = s\theta^{1-\alpha}$; $w = (1-\alpha)\theta^{1-\alpha}K$ Thus learning raises the wage rate (which is growing) and the growth rate.

(b) Not really. In cross-country growth regressions there is some evidence that equipment investment is significant (see Summers-DeLong and Levine-Renelt) though, which may be related to learning by doing.

(c) Examples might include Big Push externalities (Vishny-Schleifer) and cities (Jacobs).

37. Consider the Solow-Swan growth model with labour-augmenting technical change. The production function is:

$$Q = K^{\frac{1}{3}} L^{\frac{2}{3}},$$

where L is the effective labour force, and

$$L = Nexp(\lambda t),$$

where N is the labour force. The depreciation rate is zero.

Suppose that in Europe $\eta = 0.02$, $s = 0.20$, and $\lambda = 0.02$. Meanwhile, suppose that in North Africa $\eta = 0.04$, $s = 0.15$, and $\lambda = 0.0$.

(a) Solve for real wages and consumption per capita in each region, under autarky.

(b) Does the model predict that capital flows from Europe to North Africa? What might explain the small scale of observed capital flows?

(c) Is there evidence that convergence occurs among regions or countries as a result of some mechanisms other than capital flows?

Answer

(a) In Europe $w = 1.49exp(.02t)$ and $C/N = 1.79exp(.02t)$. In North Africa $w = 1.29$ and $C/N = 1.64$.

(b) Calculate r in each place. See Lucas. There may be a third factor of production, there may be imperfect competition in capital markets, or there may be political risk.

(c) Convergence does not seem to occur as a result of migration (see Barro and Sala-i-Martin on regions within Europe and on Japanese prefectures). Within convergence clubs there may be convergence because of capital flows or sharing technical progress. There is little evidence of convergence between Europe and North Africa, though.

38. Increases in women's access to education seem to lead to decreases in population growth rates, both historically and in a cross-section of countries. This question examines whether this effect might explain the cross-country evidence on the effect of schooling on output per capita.

Consider the Solow-Swan model with production function $q = k^{\alpha}$, constant savings rate s, no depreciation or technical progress, and population growth rate η.

(a) Solve for the logarithm of output per capita.

(b) Suppose that $\eta = yrs^{-\theta}$, where yrs denotes average years of schooling. Find the elasticity of per capita output with respect to years of schooling.

(c) In this model, years spent in school do not add to productivity. Nevertheless, a cross-country regression of log real wages on log years of schooling gives a coefficient (*i.e.* an elasticity) of 4.286. If s is the same for all countries, and $\alpha = .3$ then what must the value of θ be?

(d) Does education matter in cross-country growth regressions?

Answer

(a) $\ln q = \frac{\alpha}{1-\alpha}(\ln s - \ln \eta)$

(b) The elasticity of q with respect to yrs is $\frac{\theta\alpha}{1-\alpha}$

(c) 10; a one percent increase in yrs reduces η by ten percent.

(d) Yes. See Barro *QJE* (1991).

39. The Solow-Swan model can be used to explain differences across countries in output per capita. Suppose that the intensive production function in all countries is $f(k) = k^{\alpha}$. There is no depreciation or technical progress. The population growth rate in country i is η_i, and the savings rate is s_i.

(a) Solve for q^*, the steady-state value of output per capita.

(b) Take logarithms of your answer in part (a) to find a linear regression which could be used to test the model on a cross-section of countries.

(c) Are there any difficulties in testing growth theory with this regression? For example, what happends to the regression if there is labour-augmenting technical progress?

Answer

(a)

$$q^* = (\frac{s}{\eta})^{\frac{\alpha}{1-\alpha}}.$$

(b) The Mankiw-Romer-Weil regression is:

$$\ln q_i = \frac{\alpha}{1-\alpha}s_i - \frac{\alpha}{1-\alpha}\eta_i$$

They measure s by the investment share in GDP.

(c) There may be transitions as well as steady states. There may be flows of capital and labour between countries; though careful measurement of the regressors should reflect that. And in practice MRW find (i) that the two regression coefficients are equal and opposite but both are two large and (ii) that human capital differences across countries (like a third factor of production) also are important statistically.

If labour augmenting technical progress entered, then the residuals in the regression would contain a trend, and the rate of progress should enter as a third regressor, if it differs across countries.

40. This question examines a simple model of endogenous growth, based on a spillover from capital accumulation to labour productivity. Time is continuous. The production function is:

$$Y = K^{0.3}(HN)^{0.7},$$

where N is population, which is constant at $N = 1$. Human capital is given by

$$H = 0.8K$$

though firms do not take this into account in their investment decisions. The savings rate is a constant, $s = 0.25$. The depreciation rate is $\delta = 0.10$.

(a) Find the fundamental equation for the capital-labour ratio.

(b) What is the equilibrium growth rate of output?

(c) If human capital is unaffected by capital accumulation in other countries, can this model characterize international growth better than can the Solow-Swan model?

Answer (a) The fundamental equation is:

$$\dot{K} = s\omega^{1-\alpha} - \delta.$$

(b) In equilibrium $Y = K\omega^{1-\alpha}$ so output grows at rate

$$s\omega^{1-\alpha} - \delta = 0.1138.$$

(c) Yes. See Barro-Mankiw-Sala-i-Martin. Countries with high K need not have predicted low r (and capital outflows) because they also have high H. Countries with high savings rates will grow fastest. But there also are some odd scale effects; for example a country with large N will have low w.

41. It sometimes is argued that the saving rate in North American economies is too low. This question uses the Solow-Swan growth model to predict the outcomes of an increase in the saving rate. The production function in intensive form is:

$$f(k) = k^{0.5},$$

and there is depreciation of $\delta = 0.10$. The population growth rate is $\eta = 0.02$. The saving rate is 0.24. There is no technical progress.

(a) Solve for steady-state consumption per capita *and* the real wage rate.

(b) Suppose that the saving rate rises to 0.27. Solve for the new, steady-state values of consumption per capita *and* the real wage rate.

(c) Using diagrams or differential equations or the numerical example, describe the transition (in a closed economy) in response to this change.

Answer (a) $c^* = 1.52$; $w^* = 1$.

(b) $c^* = 1.6425$; $w^* = 1.125$.

(c) The capital stock per capita rises gradually from 4 to 5.0625, according to the fundamental equation of the growth model. The real wage gradually rises, and the real interest rate gradually falls.

Consumption per capita falls (because of the higher saving rate), then rises gradually as output rises.

42. An economist writes: "Citizens in Japan save more than those in North America, which explains the low interest rates and high wages there." To study this claim, consider the Solow-Swan model. The production function is

$$f(k_t) = k_t^{0.3},$$

in intensive form.

(a) In a steady-state competitive equilibrium, find the elasticity of the wage and interest rates with respect to the savings rate.

(b) If Japan and North America are in autarky, and Japan has a higher savings rate, will it have higher w and lower r? Does your answer depend on the population and productivity growth rates in the two countries?

(c) In fact, Japan is currently in a recession (not a steady-state equilibrium), and is not in autarky. According to the growth model, will the economist's prediction be accurate in either of these circumstances?

Answer

(a)

$$r = 0.3\left(\frac{\eta + \lambda + \delta}{s}\right) - \delta$$

$$w = 0.7\left(\frac{s}{\eta + \lambda + \delta}\right)^{0.428}$$

so the elasticities are -1 and 0.428.

(b) Not necessarily. If Japan had higher s but also higher λ then it need not have higher w and lower r. (We know it does not have higher η so we can dismiss that possibility.)

(c) First, if Japanese k is below the steady state level, then r will be higher and w lower (for a given savings rate) than at k^*. Therefore the economist's predictions may not be supported. Second, with movements of factors and technology, prices may tend to be equalized even if savings rates differ, so again the prediction will not be supported. However, the evidence on convergence is ...

43. This question uses the Solow-Swan growth model to study a developing economy. Consider a small open economy, with a savings rate of $s = 0.12$, a population growth rate of $\eta = 0.03$, and zero depreciation rate. Its production function in intensive form is $q = k^{0.5}$.

(a) In autarky, find the steady-state real wage, real interest rate, and consumption per capita.

(b) Next, the economy's population growth rate falls suddenly and permanently to $\eta = 0.02$. Find a univariate differential equation which describes the evolution of the real wage. What is the new steady-state real wage? What is the new steady-state interest rate?

(c) Suppose labour is immobile internationally. Suppose that if capital markets are opened then capital tends to flow to equalize real returns to investment. The world interest rate is $r_w = 6\%$. A tax applies so that the domestic interest rate is $r = r_w m$ with $m > 1$. Find the quantitative effect of m on the real wage.

Answer

(a) In the steady state: $k = 4^{1/.5} = 16$, $q = 4$, $c = 3.52$, $r = 12.5\%$, and $w = 2$.

(b) With $w = 0.5k^{0.5}$ and log differentiation, we know that $\dot{w} = 0.5\dot{k}$, so translating the fundamental equation from k to w gives

$$\dot{w} = 0.5(\frac{s}{2w} - \eta).$$

Clearly the new steady-state values are $w = 3$ and $r = 0.0833 = 8.33\%$

(c) Clearly higher m lowers w. To find the quantitative effect:

$$r = 0.06m = 0.5k^{-0.5} = \frac{1}{4w},$$

so

$$w = \frac{4.167}{m}$$

which has an elasticity of one.

44. When we add technical progress to the Solow-Swan model we often assume that it occurs at a steady rate. This question instead studies a scenario in which there may be periodic discrete changes in technology. Let time be continuous. Consider the Solow-Swan model with constant rates of saving, depreciation, and population growth, denoted s, δ, and η. The production function is:

$$Y = K^\alpha (AN)^{1-\alpha},$$

where A is the level of labour-augmenting technology.

(a) Solve for steady-state output per capita.

(b) Suppose that $\ln(A)$ rises by one unit, or that A rises by a factor of e. Find the long-run effect on output per worker.

(c) Suppose that you instead performed growth accounting. How much of the increase in output per worker would you find to be due to a change in capital per worker and how much to a change in total factor productivity?

(d) Describe the transitional dynamics of the model economy in response to this step increase in A.

(e) How could you test whether this type of transition explains rapid growth in Singapore and Hong Kong?

Answer

(a)

$$y = A(\frac{s}{\eta + \delta})^{\frac{\alpha}{1-\alpha}}$$

(b) If A is multiplied by 2.7 then so is y, as you can see in part (a).

(c) Do not simply say that the accounting gives roles α and $1 - \alpha$ to capital growth and technology respectively; capital growth is endogenous and known here. The accounting would give measured TFP growth as

$$\dot{y} - \alpha \dot{k}.$$

But in this model y and k grow at the same rate so there would be no evidence of TFP growth.

(d) [A diagram may help here.]

(e) See the Alwyn Young papers. The alternative is simply increases in factors (education, participation, investment) which could be studied directly or via their implications for factor prices.

45. This question uses the Solow-Swan growth model to study the productivity slowdown. Suppose that the technology is:

$$Y = K^\alpha (N exp(\lambda t))^{1-\alpha},$$

with $\alpha = 0.35$. The savings rate is 0.15, the depreciation rate is 0.10, the population growth rate is 0.03 and $\lambda = 0.04$.

(a) Solve for the wage rate, the interest rate, and the path of output per worker.

(b) Suppose that λ drops to 0.02. Find the steady-state effect on the three variables you described in part (a).

(c) Describe how growth accounting could be used to detect the slowdown. How can α be estimated?

(d) K is difficult to measure. Suppose a macroeconomist measures output per worker. Would that series accurately represent the slowdown?

Answer

(a) Here $k = 0.82476$ and so $r = 0.297$ $w = 0.6076exp(0.04t)$ and
$Y/N = 0.9348exp(0.04t)$.

(b) Here $k = 1$ and so $r = 0.25$ $w = 0.65exp(0.02t)$ and $Y/N = exp(0.02t)$.

(c) Write in log differences; estimate α from factor shares; subtract.

(d) This gives an accurate long-run answer. What about the transition?

46. This question uses the growth model to predict the effects of a change in the savings rate. The environment is the Solow-Swan model, with production function in intensive form:

$$y = k^{\frac{1}{3}},$$

and $\delta = 0$, $\eta = 0.01$, and $\lambda = 0$. The savings rate is $s = 0.10$.

(a) Find the steady-state values of k, y, and r.

(b) A natural disaster reduces k by 10 percent. At the same time, s rises to a value of 0.12. Find the new steady-state values of k, y, and r.

(c) What is the effect on steady-state consumption of each of the events (the fall in k and the rise in s) separately? Is the economy dynamically efficient?

(d) Financial economists often describe the economy using a differential equation for the interest rate. In this model, what is the first-order differential equation describing r in the transition?

Answer (a) $k = 31.62$, $y = 3.162$, $r = 3.33\%$

(b) $k = 41.46$, $y = 3.46$, $r = 2.77\%$

(c) The fall in k has no steady-state effect. The rise in s changes consumption to 3.0448 from 2.8458, a rise of 7 percent. It is dynamically efficient since $s < \alpha$ and since steady-state consumption rose in this example.

(d)

$$r = \frac{1}{3}K^{\frac{-2}{3}}.$$

Use

$$\frac{dr}{dt} = \frac{dr}{dk}\frac{dk}{dt} = \frac{dr}{dk}\dot{k}k$$

then divide by r. This gives

$$\dot{r} = 0.00666 - 0.24r.$$

which has the steady-state value found in part (b). We also could easily find the starting value for r: 3.576 percent.

47. Suppose that Japan has this production function:

$$Y_j = AK_j^{\alpha}(N_jH_j)^{1-\alpha},$$

where there is no population growth and H measures human capital. There is no depreciation, and the saving rate is s_j, a constant. Also, human capital is proportional to actual capital:

$$H_j = K_j,$$

an externality resulting from learning-by-doing.

(a) Solve for the competitive equilibrium.

(b) Hong Kong has the same production function and saving rate as Japan. But suppose that initially Hong Kong has half the capital per worker that Japan does. And there is an international spillover, with

$$H_h = K_j.$$

Describe the behaviour of the macroeconomic variables in Hong Kong.

(c) Does this model fit the evidence on convergence and on the sources of productivity growth in Hong Kong?

Answer (a) $\dot{K}_j = \dot{Y}_j = sA$ and $w = (1-\alpha)AK$ and $r = \alpha A$ if $N = 1$ in each country.

(b) $\dot{K} = sA\left(\frac{K_j}{K_h}\right)^{1-\alpha}$, so the growth rate is initially $sA2^{1-\alpha}$. Also, w will be lower and r will be higher than in Japan. Hong Kong will converge steadily on Japan.

(c) It does fit with convergence, since there is conditional convergence in the model and in the data (both Hong Kong and Japan may be in a convergence club). It does not fit the growth accounting evidence, since it implies there is rapid measured TFP in Hong Kong. In fact, much of the rapid growth in Hong Kong was due to a high savings rate and to increases in labour-force participation.

Also notice that (i) convergence would be even faster with factor movements (such as Japanese FDI in Hong Kong), and (ii) there can be a scale effect of N.

48. An economist writes: "Economic theory predicts that capital (in the form of foreign direct investment) flows from high-wage to low-wage countries." To study this claim, suppose that two countries have the same production function:

$$Y = K^{0.3}(HN)^{0.7},$$

where H is an indicator of education and N is the labour force. The depreciation rate is zero.

(a) Suppose that the capital-labour ratio in a rich country is three times the capital-labour ratio in a poor country. Find the relative education level that equalizes their real interest rates.

(b) Given your answer to part (a), what are the relative wages and relative outputs-per-worker in the two countries?

(c) Is this a plausible explanation for cross-country differences in output per capita?

(d) In financial economics, prices of assets are often calculated starting from a first-order differential equation which describes interest rates. Using this same production function, a depreciation rate of zero, and the Solow-Swan model with constant savings rate s and population growth rate η, find a univariate (*i.e.* involving only r and parameters) differential equation which describes the real interest rate. Assume that H is constant.

Answer (a) The H-ratio also must be 3 to equalize real interest rates.

(b) The wage ratio also is 3, as is the ratio of output-per-worker.

(c) Sources: Jones; Mankiw-Romer-Weil.

(d) From the Solow-Swan model, we know that

$$\dot{k} = sk^{-0.7}H^{0.7} - \eta.$$

But

$$r = .3k^{-.7}H^{.7},$$

so

$$\dot{k} = s\frac{r}{.3} - \eta.$$

Finally,

$$\dot{r} = \frac{dr}{dk}\frac{k}{r}\dot{k} = .7\dot{k} = .7\left(\frac{sr}{.3} - \eta\right).$$