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Abstract

Dammon and Green (1987) in their two-period model, showed that there are well-known difficulties in dealing with taxes in a general-equilibrium setting. When tax rates differ across investors, there will be tax-arbitrage and therefore, an equilibrium will fail to exist unless short-sale restrictions are imposed that prevent investors from exploiting such arbitrage opportunities. In the present paper, we introduce two frictions: limited tax rebates on capital losses and ordinary income taxes on long and short asset positions; and we assume transaction costs on asset transactions. With reasonable conditions on taxes and transaction technologies, we show that a general equilibrium exists. Consequently, these frictions are suggested as one possible explanation for why an apparent arbitrage may in fact not exist and a general equilibrium does exist.

1 Introduction

There is an extensive literature addressing the role of taxation and transaction costs in competitive financial markets. In an earlier paper, Jin and Milne (1999), we proposed a general competitive asset economy with very general transaction technologies and provided sufficient conditions for the existence of a competitive equilibrium. This model allowed us to consider economies with brokers, dealers and a wide range of market constraints on trade (eg. short sale constraints). We considered economies with convex and non-convex transaction technologies. In a complimentary paper, Milne and Neave (2002) provide characterisations

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of agent and competitive equilibria in such an economy and demonstrate that most known models can be accommodated as special cases. In addition they provide some new results on asset pricing and arbitrage.

In this paper we address another major imperfection in asset markets: taxation. As pointed out by Schaefer (1982) and Dammon and Green (1987), there are well-known difficulties in dealing with taxes in a general equilibrium setting. To clear markets, when there are no restrictions on asset trading, relative prices must reflect the after tax marginal rates of substitution of all agents simultaneously. When tax rates differ across investors, however, this condition can be impossible to achieve. Schaefer (1982), Dammon and Green (1987) give examples showing that when tax rates between investors are sufficiently different, there may exist arbitrage opportunities for at least one investor: this is inconsistent with the existence of general competitive equilibrium. The main reason that investors can exploit tax arbitrage opportunities is that they can infinitely short sell assets, and exploit unlimited tax rebates on capital losses. An alternative and more realistic view, which can accommodate asset short selling and individuals in different tax brackets, is where potential tax arbitrage opportunities exist, but there are realistic restrictions on their exploitation.

Unlike Schaefer (1982), we will investigate a model where tax rebates are limited (ultimately) by the government budget constraint. Long before government revenues and wealth are exhausted, sections of the tax code limit the ability of private agents or organisations to exploit unbounded loopholes in the tax law. For example, our modelling captures the essential features of the current US tax code. The Tax Reform Act of 1986 has eliminated the net capital gain deduction. Prior to this date, individuals could deduct 60 percent of net long-term capital gains from income. A $3000 loss limitation was applied to net long-term capital losses. Under the new rules, the capital gain deduction is eliminated, but the $3000 loss limitation is retained. More generally, the government can always make contingent provisions in the application of tax laws to avoid large revenue losses from implausible (unbounded) claims. More realistically its revenues from wages and salaries dwarf any revenue or drains from financial taxes: in other words, the government cannot/will not promise infeasible tax rebates to consumers and firms. We call this a No Ponzi Game (NPG) condition. We stress that this constraint is a weak bound and that more restrictive conditions could be introduced by appealing to detailed tax laws or financial regulations.

Conceptually, the forms of tax on capital gains and losses are very complicated because the investment strategies depend on the whole history of the investment (For example see Dammon and Spatt (1996) and Dammon, Spatt and Zhang (1999)). Instead of giving concrete examples of capital gains and tax rebate rules, we will impose an upper bound on rebates for each investor. The bound can be regarded as being exogenously determined by government and legal considerations. Realistically it will be far less than the bound that would exhaust government resources.

Rather than considering bounds on short positions, the upper bound on rebates is the first friction in the present paper and is an asymmetric treatment of tax on long and short positions. If the asymmetry is sufficiently pronounced to eliminate arbitrage opportunities, the only motivation for individuals to take short positions is to construct a profile of portfolio cash flows which is not feasible with entirely nonnegative portfolio weights. However, when there is a large number of securities, and tax asymmetries have eliminated arbitrage opportunities, individuals may have little motivation to take short positions. Consequently
prices may be similar to those which would result if short sales were explicitly disallowed.

The second friction considered in this paper is transaction costs. Effectively we have extended our earlier paper (Jin and Milne(1999)) on transaction costs to include taxation. This means short-sales have additional costs over and above taxation assymetries and is consistent with the discussion of Allen and Gale(1994). In this paper, we do not impose any constraints on short-selling, and short-sales are determined endogenously. Limited tax rebates on capital losses and transaction costs are suggested as possible explanations for the lack of apparent arbitrage and why a general competitive equilibrium can exist.

Another difficult issue is to construct a model which is sufficiently general to deal with the complexities of the tax law, and yet remain tractable. There are many papers that consider the implications for asset prices or asset allocations in specialised models (for a small sample see: Constantinedes (1983), Dammon and Green (1987), Dammon and Spatt (1996), Dammon, Spatt and Zhang (1999), Dybvig and Ross (1986), Ross (1987), Green (1993), Zechner (1990)). Before discussing the properties of an equilibrium, it is important that there are imposed sufficient restrictions on the economy to imply the existence of an equilibrium. As we indicated above, if agents face agent specific tax functions that allow tax arbitrage possibilities, then an equilibrium may not exist. This issue has been addressed in a two period exchange economy by Dammon and Green (1987) and Jones and Milne (1992). Dammon and Green (1987) consider restrictions on tax functions to eliminate arbitrage and define a set of arbitrage-free asset prices. Their paper considered tax functions of considerable generality and exploited the theory of recession (asymptotic) cones to determine arbitrage free prices. Jones and Milne (1992) argued that Dammon and Green (1987) did not include the government sector explicitly, ignoring the feasibility constraints implicit in the government budget constraint. Once these constraints were introduced and recognised by the agents, there were natural bounds on the possible asset trades consistent with tax arbitrage. These restrictions allowed more general and complex tax functions to be consistent with equilibrium. Of course the differences in the two models could be resolved by assuming that Dammon and Green’s tax functions included the implicit tax rules that come into play as soon as large tax arbitrages were claimed from the government by consumers. In that sense the two models could be made consistent.

The models in Dammon and Green (1987) and Jones and Milne (1992) are two-period exchange economies. In this paper we extend the two-period model to a more general multi-period economy: We assume an explicit government sector, productive firms, brokers/dealers, and spot commodity markets. The aim of the paper is to show how the basic ideas can be extended in a number of realistic directions. In particular the introduction of firms allows us to accomodate models (eg. Zechner (1990)) that discuss the interaction of personal and corporate taxation and its impact on corporate financial structure. The introduction of government is more general than the modelling in Jones and Milne (1992) in that we allow the government to choose commodity trades optimally given its net tax revenues. We show that the introduction of spot commodity markets is easily accomodated. (For simplicity we assume only one commodity is traded in the last period, but this is an expositional restriction that can be relaxed). We allow general tax functions that can accomodate most properties of tax codes. For example: the tax functions can be non-linear, convex or piece-wise linear; they can depend upon dynamic asset strategies so that capital gains or other complex tax systems can be incoroporated; the tax functions can include financial
subsidies as well as taxes; and the state contingent taxation functions can be interpreted to include (random) legal interpretations of the tax code where a dynamic asset position is deemed to violate the code and subsequent income or capital gains are taxed at a higher rate with possible penalties. In the body of the paper we rule out non-convex tax functions due to subsidy/tax thresholds- in the conclusion we discuss how this (and other extensions) could be incorporated in a more extensive model.

The remainder of the paper is organized as follows: in Section 2 we set out the model and present fundamental assumptions; Section 3 is devoted to the proof of the main theorem; in Section 4 we discuss extensions or variations of the model and how they could be incorporated in an extended version of the model; proofs of some lemmas are provided in the Appendix.

2 The Economic Setting

Consider an economy with uncertainty characterized by a event tree such as that depicted in Fig.1 of Duffie(1987). This tree consists of a finite set of nodes $E$ and directed arcs $A \subset E \times E$, such that $(E, A)$ forms a tree with a distinguished root $e_0$. The number of immediate successor nodes of any $e \in E$ is denoted $\#e$. A node $e \in E$ is terminal if it has no successor node. Let $T$ denote the set of all terminal nodes. The immediate successor nodes of any non-terminal node $e \in E$ are labeled $e^{+1}, \ldots , e^{+K}$, where $K = \#e$. The sub-tree with root $e$ is denoted $E(e)$. Particularly, $E = E(e_0)$. Suppose there are $\# E$ nodes(including $e_0$)in the tree, $N$ securities and $M$ commodities at any node $e \in E' = E - T$. This assumption can be relaxed by assuming that the numbers of securities can vary across nodes because we will not require every asset to be held by agents at node $e_0$. This relaxation covers the case where some securities are issued and some mature in interim periods. For the sake of simplicity, we assume that there is only one commodity at each terminal node (this is largely for expositional convenience and avoids some minor technical issues). At each node, all securities first distribute dividends, and then are available for trading: that is, all security prices are ex-dividend.

Let $p^c(e) = (p^c_1(e), \ldots , p^c_M(e))$ denote the spot price of commodities at node $e \in E'$. At each node $e \in E'$, asset $n(n = 1, \ldots , N)$ has a buying price $p^b_n(e)$ and a selling price $p^s_n(e)$ and a dividend $p^d(e)D_n(e)$, where we take the first commodity as numeraire. Moreover, let $D_n(e)$ be the dividend of asset $n$ at the terminal node $e' \in T$. We denote $p^b(e) = (p^b_1(e), \ldots , p^b_N(e))$, $p^s(e) = (p^s_1(e), \ldots , p^s_N(e))$ and $p(e) = (p^b(e), p^s(e))$ and $D(e) = (D_1(e), \ldots , D_N(e))$. And it is assumed that dividends are always non-negative.

2.1 Intermediaries

Suppose that there are $H$ brokers or intermediaries(indexed by $h$) with objective (utility) function $U^B_h(\cdot)$ and commodity endowment vector $\omega^B_h(e)$ at each node $e$. They are intermediaries specializing in the transaction technology that transforms bought and sold assets. Let $\phi^{B,h}_n(e)/(\phi^{S,h}_n(e))$ be the number of bought(sold) asset $n$ supplied by intermediary $h$ at node $e \in E'$ (denote $\phi^{B,h}_n(e) = (\phi^{B,h}_1(e), \ldots , \phi^{B,h}_N(e))^T$ and $\phi^{S,h}_n(e) = (\phi^{S,h}_1(e), \ldots , \phi^{S,h}_N(e))^T$, where $T$ is the transpose transformation) and $z_h(e) = (z_{h,1}(e), \ldots , z_{h,M}(e))^T$ be the vector of contingent commodities used up in the activity of intermediation at node $e \in E'$. Then the broker pays
tax on capital gains via a general tax function \( T_h^{BC}(e) = T_h^{BC}(e)((p^B(e'), p^S(e'), \phi_h(e'))_{e' \in \text{PA}(e)}) \),
where \( \text{PA}(e) \) is a path from \( e_0 \) to \( e \); and pays tax on dividends via a general tax function
\[
T_h^{BD}(e) = T_h^{BD}(e)(D(e)[\sum_{e' \in \text{PA}(e) \setminus \{e\}} (\phi_h^B(e') - \phi_h^S(e'))])
\]
At any terminal node \( e \in \mathbf{T} \), the broker receives dividends
\[
D(e)[\sum_{e' \in \text{PA}(e) \setminus \{e\}} (\phi_h^B(e') - \phi_h^S(e'))]
\]
and pays tax via a general tax function
\[
T_h^{BD}(e) = T_h^{BD}(e)(D(e)[\sum_{e' \in \text{PA}(e) \setminus \{e\}} (\phi_h^B(e') - \phi_h^S(e'))])
\]
Denote \( z_h = (\ldots, z_h(e), \ldots)_{e \in \mathbf{E}'} \in R_+^{E' \times M} \) and portfolio plan by
\[
\phi_h = (\phi_h(e))_{e \in \mathbf{E}'} = (\phi_h^B(e), \phi_h^S(e))_{e \in \mathbf{E}'} \in R_+^{2(|\mathbf{E}'| \times N)}
\]
And set
\[
\gamma_h = (p^B(e_0)\phi_h^S(e_0) - p^S(e_0)\phi_h^B(e_0) - T_h^{BC}(e_0)
+ p(e_0)(\omega_h^B(e_0) - z_h(e_0)),
(\phi_h^B(e) - \phi_h^S(e)) \sum_{e' \in \text{PA}(e) \setminus \{e\}} (p^B(e') - p^S(e'))
+ p_c(e)D(e)[\sum_{e' \in \text{PA}(e) \setminus \{e\}} (\phi_h^B(e') - \phi_h^S(e'))]
- T_h^{BC}(e) - T_h^{BD}(e) + p(e)(\omega_h^B(e) - z_h(e)))_{e \in \mathbf{E}'} - \{e_0\},
(D(e)[\sum_{e' \in \text{PA}(e) \setminus \{e\}} (\phi_h^B(e') - \phi_h^S(e'))] - T_h^{BD}(e))_{e \in \mathbf{T}}
\]
For intermediary \( h \), let \( \mathbf{T}(h, e) \subseteq R_+^N \times R_+^N \times R^M \) denote its technology at node \( e \).
The maximization problem of broker \( h \) can be stated as:
\[
\sup_{(\phi_h, z_h) \in \Gamma_h^B(\tilde{p})} U_h^B(\gamma_h)
\]
where \( \Gamma_h^B(\tilde{p}) \) is the space of feasible trade-production plans \( (\phi_h, z_h) = (\phi_h^B, \phi_h^S, z_h) \) given \( \tilde{p} = (p^B, p^S, p^C) \), which satisfies:
\[
(2.1) (\phi_h^B(e), \phi_h^S(e), z_h(e)) \in \mathbf{T}(h, e) \text{ and } z_h(e) \geq 0;
(2.2) p^B(e_0)\phi_h^S(e_0) - p^S(e_0)\phi_h^B(e_0) + p(e_0)(\omega_h^B(e_0) - z_h(e_0)) - T_h^{BC}(e_0) \geq 0, \text{ and }
\]
\[
p^B\phi_h^S(e) - p^S\phi_h^B(e) + p_c(e)(\omega_h^B(e) - z_h(e)))
+ p_c(e)D(e)[\sum_{e' \in \text{PA}(e) \setminus \{e\}} (\phi_h^B(e') - \phi_h^S(e')) - T_h^{BC}(e) - T_h^{BD}(e) \geq 0, \forall e \in \mathbf{E}' - \{e_0\};
(2.3) D(e)[\sum_{e' \in \text{PA}(e) \setminus \{e\}} (\phi_h^B(e') - \phi_h^S(e'))] - T_h^{BD}(e) \geq 0, \forall e \in \mathbf{T}.
\]
Comments: The intermediary formulation allows the agent to trade on its own account; or by interpreting the transaction technology more narrowly, it can be restricted to a pure broker with direct pass through of assets bought and sold(see Milne and Neave(2002) for further discussion). Note: our tax functions are sufficiently general in formulation that our capital gains tax function could incorporate dividend taxes as well (apart from the final date).
2.2 Firms

Suppose there are $J$ firms. At node $e_0$, the firm $j \in J = \{1, \ldots, J\}$ has an initial endowment $\omega_j^F(e_0)$, which gives the firm a positive cash flow. The firm chooses an input plan $y_j^-(e_0) \in R_+^M$ and a trading strategy $\beta_j(e_0) = (\beta_j^B(e_0), \beta_j^S(e_0))^T = (\beta_{j,1}^B(e_0), \ldots, \beta_{j,n}^B(e_0), \beta_{j,1}^S(e_0), \ldots, \beta_{j,n}^S(e_0))^T \in R_+^{2n}$, where $\beta_{j,n}^B(e_0)(\beta_{j,n}^S(e_0))$ represents the purchase(sale) of asset $n$ by firm $j$ at node $e_0$. Then, the firm pays tax according to the tax function $T_j^F(e_0)$. At every node $e (\in E')$ other than node $e_0$, the firm produce an output $y_j^+(e) \in R_+^M$ and receives a net dividend

$$p_1^C(e)D(e)[ \sum_{e' \in PA(e)-\{e\}} (\beta_j^B(e') - \beta_j^S(e'))],$$

then, chooses an input plan $y_j^-(e) \in R_+^M$ and a trading strategy $\beta_j(e) = (\beta_j^B(e), \beta_j^S(e))^T = (\beta_{j,1}^B(e), \ldots, \beta_{j,n}^B(e), \beta_{j,1}^S(e), \ldots, \beta_{j,n}^S(e))^T \in R_+^{2n}$, and pays tax $T_j^{FC}(e)$ and $T_j^{FD}(e)$ on capital gains and ordinary income. It is assumed $(y_j^+(e) - y_j^-(e)) \in Y_j(e) \subseteq R_+^M$, where $Y_j(e)$ is a production set. At each terminal node $e$, the firm $j$ produces $y_j^+(e) \in R_+^M$, get its dividend and then pays tax $T_j^{FD}(e)$. Set

$$\delta(e_0) = p^C(e_0)(\omega_j^F(e_0) - y_j^-(e_0)) + p^S(e_0)\beta_j^S(e_0) - p^B(e_0)\beta_j^B(e_0) - T_j^{FC}(e_0),$$

$$\delta(e) = p^C(e)(y_j^+(e) - y_j^-(e)) + p^S(e)\beta_j^S(e) - p^B(e)\beta_j^B(e)$$

$$+ p_1^C(e)D(e)[ \sum_{e' \in PA(e)-\{e\}} (\beta_j^B(e') - \beta_j^S(e'))] - T_j^{FC}(e) - T_j^{FD}(e), e \in E' - \{e_0\},$$

and

$$\delta(e) = p^C(e)y_j^+(e) + p_1^C(e)D(e)[ \sum_{e' \in PA(e)-\{e\}} (\beta_j^B(e') - \beta_j^S(e'))] - T_j^{FD}(e), e \in T.$$

Therefore, the problem of the firm $j$ can be stated as:

$$\max_{(y_j, \beta_j) \in \Gamma_j^F(\bar{p})} U_j^F((\delta(e))_{e \in E}),$$

where $\Gamma_j^F(\bar{p})$ is the feasible set of production-portfolio plan of firm $j$ given $\bar{p}$, which satisfies:

$$\delta(e) \geq 0, e \in E.$$

Remark: The modeling of the objective of the firm avoids the failure of profit to be well-defined, and the non-applicibility of the Fisher Separation Theorem. We can think of the firm being either a sole proprietorship or that the utility function is derived from some group preference arrangement (for a more detailed discussion of these issues see Kelsey and Milne(1996)). The utility function can incorporate a number of interpretations: for example, it could include job perquisite consumption by a manager, where her optimal perquisite consumption is endogenously determined in equilibrium.
2.3 Consumers

In addition to broker/intermediaries and firms, there are $I$ consumers, indexed by $i \in I = \{1, \ldots, I\}$. The consumer $i$ has endowment $\omega_i(e)$ of commodity at any node $e \in E$. At any node $e \in E'$, the consumer $i$ has a consumption set $x_i(e) = (x_{i,1}(e), \ldots, x_{i,m}(e))^T \in \mathbb{R}_+^m$ and chooses an asset portfolio $\alpha_i(e) = (\alpha_i^B(e), \alpha_i^S(e))^T = (\alpha_{i,1}(e), \ldots, \alpha_{i,n}(e))^T \in \mathbb{R}_+^n$, where $\alpha_{i,n}(e)(\alpha_{i,n}(e))$ represents the purchase(sale) of asset $n$ by consumer $i$ at node $e$. The consumer pays capital gains tax via a tax function

$$T_i^{CC}(e)((pB(e'), pS(e'), \alpha_i(e'))_{e' \in \text{PA}(e)})$$

and dividend taxes via a function defined similarly to the broker tax function, $T_i^{CD}(e)$. At any terminal node $e \in T$, the consumer receives dividends

$$D(e)[\sum_{e' \in \text{PA}(e) - \{e\}} (\alpha_i^B(e') - \alpha_i^S(e'))]$$

and pays tax via a general tax function $T_i^{CD}(e)$, and then consumes. We denote the consumption plan by $x_i = (\ldots, x_i(e), \ldots)_{e \in E'} \in \mathbb{R}^{E'|M}$ (where $|E'|$ denote the number of nodes in the set $E'$) and portfolio plan by $\alpha_i = (\alpha_i(e))_{e \in E'} = (\alpha_i^B(e), \alpha_i^S(e))_{e \in E'} \in \mathbb{R}^{2(|E'| \times N)}$.

The consumer $i$ has a consumption set $X_i \subseteq \mathbb{R}^{E'|M}$ and a utility function $U_i^C(\cdot)$ over the consumption plan $x_i$ and the consumption

$$D(e)[\sum_{e' \in \text{PA}(e) - \{e\}} (\alpha_i^B(e') - \alpha_i^S(e'))] - T_i^{CD}(e)$$

at terminal nodes. Denote

$$\tilde{x}_i = (x_i, (D(e)[\sum_{e' \in \text{PA}(e)} (\alpha_i^B(e') - \alpha_i^S(e'))] - T_i^{CD}(e)))_{e \in T}.$$

So the problem of consumer $i$ can be expressed as:

$$\max_{(x_i, \alpha_i) \in \Gamma_i^C(\tilde{p}, \gamma, \delta)} U_i^C(\tilde{x}_i),$$

$\Gamma_i^C(\tilde{p}, \gamma, \delta)$ is the feasible set of portfolio-consumption plan of consumer $i$ given $(\tilde{p}, \gamma, \delta)$, which satisfies:

$$p^C(x_i(e)) + p^B(e)\alpha_i^B(e) - p^S(e)\alpha_i^S(e) + T_i^{CC}(e) + T_i^{CD}(e)$$

$$\leq p^C(e)\omega_i(e) + p^C_i(e)D(e)[\sum_{e' \in \text{PA}(e) - \{e\}} (\alpha_i^B(e') - \alpha_i^S(e'))]$$

$$+ \sum_j \eta_{i,j} \delta_j(e) + \sum_h \theta_{i,h} \gamma_h(e)$$
\[ \forall e \in E' \text{ and} \]
\[ D(e) \left[ \sum_{e' \in \text{pa}(e)} (\alpha_i^{R}(e') - \alpha_i^{S}(e')) \right] - T_{i}^{CD}(e) \geq 0. \forall e \in T \]

where \( \eta_{i,j} \) is the share of the profit of \( j \)th producer owned by the \( i \)th consumer. The numbers \( \eta_{i,j} \) are positive or zero, and for every \( j \), \( \sum_{i=1}^{l} \eta_{i,j} = 1 \). The numbers \( \theta_{i,h} \) have the same explanation.

### 2.4 Government

The government is also included as part of the economy. Rather than providing a detailed analysis of the operations of the government, we will place weak restrictions on government activity. For simplicity assume that the government has net resources \( \omega_{G}(e) \in \mathbb{R}_{+}^{M} \) at node \( e \), sets tax rates ex ante, and then consumes \( x_{G}(e) \) at node \( e \). The government can always propose precommitted tax laws that avoid government bankruptcy due to errors or subtle legal interpretations; and its revenues from taxes on wages and salaries dwarfs any revenue or drains from financial taxes. In other words, the government cannot promise infeasible tax rebates to consumers and firms. We call this a No Ponzi Game (NPG) condition\(^1\). We stress that this constraint is a weak bound and that more restrictive conditions could be introduced by appealing to detailed tax laws or financial regulations.

Suppose that the government has preferences over its consumption set \( X_{G} = \mathbb{R}_{+}^{E'| \times M} \), represented by a utility function \( U_{G} \) (this is simplistic but avoids more complex issues of government decision-making). At node \( e \in E' \), it spends its income, which comes from its endowment \( \omega_{G}(e) \) and tax revenue

\[ T(e) = \sum_{h} (T_{h}^{BC}(e) + T_{h}^{BD}(e)) + \sum_{j} (T_{j}^{FC}(e) + T_{j}^{FD}(e)) + \sum_{i} (T_{i}^{CC}(e) + T_{i}^{CD}(e)). \]

The government’s problem can be stated as:

\[ \max_{x_{G} \in \Gamma_{G}} U^{G}(x_{G}), \]

where \( \Gamma_{G} \) denote the set of government’s consumption \( x_{G} \) which satisfies

\[ p^{C}(e)x_{G}(e) \leq p^{C}(e)\omega_{G}(e) + T(e) \]

### 2.5 Competitive Equilibrium and Assumptions

To conclude the section, we give the definition of a competitive equilibrium and assumptions made throughout the remainder of this paper.

\(^{1}\)The No Ponzi Game condition has been introduced in macroeconomic models to eliminate unbounded borrowing positions by consumers and/or governments(see Blanchard and Fisher(1989)).
**Definition 2.1.** A competitive equilibrium with asset taxation and transaction costs is a non-negative vector of prices \( p^* = (p^{C\cdot}, p^{B\cdot}, p^{S\cdot}) \) and allocations \( \{(x_i^*, \alpha_i^*) \text{ for all } i \in I; (z_h^*, \phi_h^*) \text{ for all } h \in H; (y_j^*, \beta_j^*) \text{ for all } j \in J; x_G^* \text{ such that:} \)

(i) \( (z_h^*, \phi_h^*) \) solves the broker problem for each \( h \in H; \)
(ii) \( (y_j^*, \beta_j^*) \) solves firm’s problem for each \( j \in J; \)
(iii) \( (x_i^*, \alpha_i^*) \) the consumer problem for each \( i \in J; \)
(iv) \( x_G^* \) solves government’s problem; 

(v) \( \sum_i x_i^* + \sum_h z_h^* + \sum_j y_j^* + x_G^* = \omega_G + \sum_j y_j^+ + \sum_i \omega_i^C + \sum_j \omega_j^B + \sum_h \omega_h^B; \)

(vi) \( \sum_i \alpha_i^B + \sum_j \beta_j^B = \sum_h \phi_h^S + \sum_i \alpha_i^S + \sum_j \beta_j^S \) if \( p_n^B > p_n^S; \) \( \sum_i \alpha_i^B + \sum_j \beta_j^B = \sum_h \phi_h^S + \sum_i \alpha_i^S + \sum_j \beta_j^S \) if \( p_n^B = p_n^S. \)

Notice that condition (vi) allows for the extreme case where the transaction technology is costless.

The following assumptions are made in the remainder of this paper.

For consumer \( i: \)

(A1) \( X_i = R_{+}^{E_i \times M}; \)

(A2) \( U_i^C \) is a continuous, concave and strictly increasing function;

(A3) \( T_{i\cdot}^{CC} \) and \( T_{i\cdot}^{CD} \) are continuous, convex functions and there exist positive constants \( c_i^{CC}(e) < 1, c_i^{CD}(e) < 1 \) such that \( T_{i\cdot}^{CC}(e) [x] \leq c_i^{CC}(e) x, T_{i\cdot}^{CD}(e) [x] \leq c_i^{CD}(e) x \) for any \( x \geq 0, \)

that is, the taxes can never be larger than revenues;

(A4) We assume that the tax rebates on capital losses and ordinary income on long and short positions have lower bounds: that is, \( T_{i\cdot}^{CC}(e) + T_{i\cdot}^{CD}(e) > \alpha_i^C(e) \), where \( \alpha_i^C \) is determined by the government;

(A5) \( \omega_i^C(e) > 0, \) \( \forall e \in E_i; \)

For broker \( h: \)

(A6) For each \( e, T(h, e) \) is a closed and convex set with \( 0 \in T(h, e); \)

(A7) For any \( e \) and given \( x = (x_1, \ldots, x_{2N}, z_1, \ldots, z_M) \in T(h, e), \) if \( y = \sum_{n=1}^{2N} x_n \longrightarrow \infty, \) then \( \{(z_1, \ldots, z_M) \} = \sum_{n=1}^M z_n \longrightarrow \infty; \)

(A8) For each \( e, (\psi, z) \in T(h, e) \) and \( z' \geq z, \) then \( (\psi, z') \in T(h, e) \) (free disposal).

(A9) \( U_i^B(\cdot) \) is a continuous, concave and strictly increasing function;

(A10) \( \omega_i^B(e) > 0, \) \( \forall e \in E_i. \)

(A11) \( T_h^{BC} \) and \( T_h^{BD} \) satisfy assumption (A3) and (A4) for some \( c_h^{BC}(e), c_h^{BD}(e) \) and \( \alpha_h^B(e) \).

For firm \( j: \)

(A12) For each \( e \in E_j, Y_j(e) \) is a closed and convex set; 

(A13) For each \( e \in E_j, Y_j(e) \cap R_{+}^{2M} = \{0\}; \)

(A14) For each \( e \in E_j, \sum_{j} Y_j(e) \cap (- \sum_{j} Y_j(e)) = \{0\}; \)

(A15) \( U_j^F(\cdot) \) is a continuous, concave and strictly increasing function;

(A16) \( \omega_j^F(e_0) > 0; \)

\footnote{For } For \( e = (x_1, \ldots, x_n) \in R^n, x \gg 0 \text{ means } x_i > 0, i = 1, \ldots, n; x > 0 \text{ means } x_i \geq 0, i = 1, \ldots, n \text{ but } x_{i_0} > 0 \text{ for at least one } i_0. \)
(A17) $T_j^{FC}$ and $T_j^{FD}$ satisfies assumption (A3) and (A4) for some $c_j^{FC}(e), c_j^{FD}(e)$ and $\alpha_j^{F}(e)$.

For government;
(A18) $U^G$ is a continuous, concave and strictly increasing function;
(A19) $\omega_G(e) > 0$;
(A20) For each security and any $e \in E'$ there exists a terminal node $e' \in E(e)$ such that the dividend of this security is positive at this node;
(A21) $\sum_i \alpha_i^C(e) + \sum_j \alpha_j^F(e) + \sum_h \alpha_h^B(e) \geq 0, \forall e \in E'$.

All assumptions except (A7), (A15), (A4), (A11), (A17) and (A21) are standard. (A7) says that transactions must consume resources. Assumptions (A4), (A11), (A17) and (A21) say that the government has designed into the tax code, in the application of tax laws, measures that avoid large revenue losses from implausible (unbounded) claims. In other words, the government cannot/will not promise infeasible tax rebates to consumers and firms, preventing consumers from exploiting unlimited arbitrage opportunities through tax rebates on capital losses. According to the Federal Tax Code, the government often recognizes realized capital gains and does not recognize realized capital losses. Observe that the tax functions and trading restrictions are sufficiently general to allow the government to levy taxes on capital gains, interest and dividends, and delay paying tax rebates on capital losses. Also the tax functions are state contingent allowing for random audits or (random from the point of view of agents and the government) legal interpretations. Strictly speaking, a fully rational system would model the intricacies of the legal tax system - here we simply assume that is an exogenous mechanism.

3 The Main Theorem

The main theorem of this paper is as follows:

**Theorem 3.1:** Suppose that the Assumptions 1-21 hold. Then there exists an equilibrium $(\xi^*, \tilde{p}^*)$: that is, $(\xi^*, \tilde{p}^*)$ satisfies $(i) - (vi)$ of Definition 2.1, where $\xi^* = ((x_i^*, \alpha_i^*)_{i \in I}, (z_h^*, \phi_h^*)_{h \in H}, (y_j^*, \beta_j^*)_{j \in J}, x_G^*)$.

3.1 The Modified Economy

To show the existence of equilibrium, as in Arrow and Debreu(1954), we will construct a bounded commodity spot market. Unfortunately this method does not apply directly because of the possible emptiness of the budget correspondence of agents in a multi-period assets market. To overcome this problem we will approximate the original economy with a sequence of new economies with positive commodity prices which has the property that the limit of the equilibria of the new economy is an equilibrium of the original economy. In the following, we will first truncate the commodity spot market as in Arrow and Debreu(1954) and then truncate the asset space.

First of all, we truncate the the commodity space. For broker $h$, define
\[ Z_h = \{ z_h : \text{there exists } (\phi_h^B, \phi_h^S) \gg 0 \text{ such that } (\phi_h^B(e), \phi_h^S(e), z_h(e)) \in T(h, e), \forall e \in E' \} \]

\[ \hat{Z}_h = \{ z_h \in Z_h : \text{there exist } x_i \in X_i, i = 1, ..., I, y_j \in Y_j, j = 1, ..., J, z_h \in Z_h, h = 1, ..., H \text{ and } x_G \in X_G \text{ such that } \sum_i x_i(e) + \sum_j y_j^-(e) + \sum_h z_h(e) + x_G(e) - \omega_G(e) - \sum_j y_j^+(e) - \sum_i \omega_i^C(e) - \sum_j \omega_j^F(e) - \sum_h \omega_h^B(e) < 0, \forall e \in E' \}. \]

For consumer \( i \), define

\[ \hat{X}_i = \{ x_i \in X_i : \text{there exist } x_{i'} \in X_{i'}, i' \neq i, \text{ and } y_j \in Y_j, j = 1, ..., J, z_h \in Z_h, h = 1, ..., H \text{ and } x_G \in X_G \text{ such that } \sum_i x_i(e) + \sum_j y_j^-(e) + \sum_h z_h(e) + x_G(e) - \omega_G(e) - \sum_j y_j^+(e) - \sum_i \omega_i^C(e) - \sum_j \omega_j^F(e) - \sum_h \omega_h^B(e) < 0, \forall e \in E' \}. \]

For firm \( j \), set

\[ \hat{Y}_j = \{ y_j \in Y_j : \text{there exist } x_i \in X_i, i = 1, ..., I, y_j' \in Y_j', j' \neq j, \text{ and } z_h \in Z_h, h = 1, ..., H \text{ and } x_G \in X_G \text{ such that } \sum_i x_i(e) + \sum_j y_j^-(e) + \sum_h z_h(e) + x_G(e) - \omega_G(e) - \sum_j y_j^+(e) - \sum_i \omega_i^C(e) - \sum_j \omega_j^F(e) - \sum_h \omega_h^B(e) < 0, \forall e \in E' \}. \]

For the government, set

\[ \hat{X}_G = \{ x_G \in X_G : \text{there exist } x_i \in X_i, i = 1, ..., I, y_j \in Y_j, j = 1, ..., J, z_h \in Z_h, h = 1, ..., H \text{ such that } \sum_i x_i(e) + \sum_j y_j^-(e) + \sum_h z_h(e) + x_G(e) - \omega_G(e) - \sum_j y_j^+(e) - \sum_i \omega_i^C(e) - \sum_j \omega_j^F(e) - \sum_h \omega_h^B(e) < 0, \forall e \in E' \}. \]

As in Arrow and Debreu (1954), we have the boundedness of sets \( \hat{X}_i, \hat{Y}_j, \hat{Z}_h \) and \( \hat{X}_G \), which is stated in the following lemma. The proof is omitted because it is standard.

**Lemma 3.1.** The sets \( \hat{X}_i, \hat{Y}_j, \hat{Z}_h \) and \( \hat{X}_G \) are all convex and compact.

Now we turn to truncating trading sets of asset. For broker \( h \), define

\[ \Phi_h = \{ \phi_h = (\phi_h^B, \phi_h^S) : \text{there exists } z_h \in \hat{Z}_h \text{ such that } (\phi_h(e), z_h(e)) \in T(h, e), \forall e \in E' \}. \]

In the following lemma, \( \Phi_h \) will be shown to be bounded, which will play an important role in proving the existence of the general equilibrium.

**Lemma 3.2.** The set \( \Phi_h \) is a compact and convex subset of \( R^{E' \times N} \), \( h = 1, ..., H \).

**Proof:** See the Appendix. \( \square \)

By Lemma 3.2, there exists a positive constant \( M_0 \) such that for any \( (\phi_h^B, \phi_h^S) \in \Phi_h \),

\[ \phi_{h,n}^B(e) \leq M_0 \text{ and } \phi_{h,n}^S(e) \leq M_0, n = 1, ..., N, h = 1, ..., H, \forall e \in E'. \]

For consumer \( i \), define

\[ \Psi_i = \{ \alpha_i = (\alpha_i^B, \alpha_i^S) : \alpha_{i,n}^B(e) \leq HM_0, n = 1, ..., N, \forall e \in E' \}. \]

For firm \( j \), define

\[ \Theta_j = \{ \beta_j = (\beta_j^B, \beta_j^S) : \beta_{j,n}^B(e) \leq HM_0, n = 1, ..., N, \forall e \in E' \}. \]

Define

\[ C^1 = \{(c_1, ..., c_{2(|E'| \times N)}) : |c_i| \leq M_1, i = 1, ..., 2(|E'| \times N) \} \subseteq R_{\geq}^{2(|E'| \times N)} \]

where \( M_1 > HM_0 \).

Set \( \bar{X}_i \) be the set of \( x_i \in X_i \) with property that

\[ \sum_i x_i(e) + \sum_j y_j^+(e) + \sum_h z_h(e) + x_G(e) - \omega_G(e) - \sum_j y_j^+(e) - \sum_i \omega_i^C(e) - \sum_j \omega_j^F(e) - \sum_h \omega_h^B(e) \ll D(e)[\sum_{e' \in PA(e) - \{e\}} M_1(e')], \forall e \in E'. \]
for some $x_i' \in X_i', i' \neq i, y_j \in Y_j, j = 1, \ldots, J, z_h \in Z_h, h = 1, \ldots, H$ and $x_G \in X_G$. Here

$$M_1(e') = (3M_1, \ldots, 3M_1).$$

Likewise, we can define $\widetilde{Y}_j, \tilde{Z}_h$ and $\widetilde{X}_G$. Like Lemma 3.1, it can be shown that $\widehat{X}_i'$, $\widehat{Y}_j'$ and $\widehat{X}_G'$ are compact and convex and therefore, there exists a cube $C^2 = \{ (c_1, \ldots, c_{|E'| \times M}) : |c_i| \leq M, i = 1, \ldots, |E'| \times M \} \subseteq \mathbb{R}^{E' \times M}$ such that $C^2$ contains in its interior all sets $\widehat{X}_i'$, $\widehat{Y}_j'$, $\tilde{Z}_h$ and $\tilde{X}_G$. Define $\widehat{X}_i = C^2 \cap X_i, \widehat{Y}_j = C^2 \cap Y_j, \tilde{Z}_h = C^2 \cap Z_h$ and $\tilde{X}_G = \mathcal{I}_i = \Theta = C^1$. And let $\tilde{\Gamma}_{\cdot}^C(\tilde{p}, \gamma, \delta), \tilde{\Gamma}_{\cdot}^F(\tilde{p}), \tilde{\Gamma}_{\cdot}^B(\tilde{p})$ and $\Gamma_G$ be the resultant modification of $\Gamma_{i}^C(\tilde{p}, \gamma, \delta), \Gamma_{j}^F(\tilde{p}), \Gamma_{h}^B(\tilde{p})$ and $\Gamma_G$ respectively. Define $3$

$$\Delta = \{ \tilde{p} = (p^C, p^B, p^S) : \sum_{n=1}^{N} (p_{n}^B(e) + p_{n}^S(e)) + \sum_{m=1}^{M} p_{m}^C(e) = 1, \quad p_{n}^B(e) \geq p_{m}^B(e) \geq 0, p_{m}^C(e) \geq 0, m = 1, \ldots, M, n = 1, \ldots, N, \forall e \in E' \}$$

and

$$\Delta_k = \{ \tilde{p} = (p^C, p^B, p^S) \in \Delta : p_{m}^C(e) \geq 1/k, \quad m = 1, \cdots, M, \forall e \in E' \},$$

where $k \geq M$.

In the next section, we will prove the existence of the modified economy and then prove the existence of equilibrium of the original economy.

### 3.2 The Proof of Existence of General Equilibrium

We will adopt the Arrow-Debreu technique to prove existence in the modified economy.

Given $\tilde{p} \in \Delta_k$, let

$$\mu_{i}^{C} = \mu_{i}^{C}(\tilde{p}, \gamma, \delta) = \{ (\psi_i, x_i) : U_{i}^{C}(x_i) = \sup_{(\psi_i, x_i) \in \Gamma_{i}^{C}(\tilde{p}, \gamma, \delta)} U_{i}^{C}(\tilde{x}_i) \};$$

$$\mu_{j}^{F} = \mu_{j}^{F}(\tilde{p}) = \{ (\beta_j, y_j) : U_{j}^{F}(\delta_j) = \sup_{(\beta_j, y_j) \in \Gamma_{j}^{F}(\tilde{p})} U_{j}^{F}(\delta_j) \};$$

$$\mu_{h}^{B} = \mu_{h}^{B}(\tilde{p}) = \{ (\phi_h, z_h) : U_{h}^{B}(\gamma_h) = \sup_{(\phi_h, z_h) \in \Gamma_{h}^{B}(\gamma_h)} U_{h}^{B}(\tilde{z}_h) \};$$

$$v_{G} = v_{G}(\tilde{p}) = \{ x_{G} : U_{G}(x_{G}) = \sup_{x_{G} \in \Gamma_{G}} U_{G}(\tilde{x}_G) \};$$

$3$ In this model excess demand is not necessarily homogeneous of degree zero in prices: thus the equilibrium prices may depend on the normalization chosen. For example, this will be the case for specific taxes.
\[ \mu^M(e) = \{ p \in \Delta_k | p^B(e)(\sum_i \alpha_i^B(e) + \sum_j \beta_j^B(e) - \sum_h \phi_h^S(e)) \]
\[ - p^S(e)(\sum_i \alpha_i^S(e) + \sum_j \beta_j^S(e) - \sum_h \phi_h^B(e)) \]
\[ + \sum_{m=2}^M p_m^C(e)(\sum_i x_{i,m}(e) + \sum_j y_{j,m}(e) + \sum_h z_{h,m}(e) + x_{G,m}(e)) \]
\[ - (\omega_{G,m}(e) + \sum_j y_{j,m}^+(e) + \sum_i \omega_{i,m}^C(e) + \sum_j \omega_{j,m}^F(e) + \sum_h \omega_{h,m}^B(e)) \]
\[ + p_i^C(e)[D(e) \left( \sum_{e' \in \mathbb{P}(e) - \{ e \}} (\alpha_i^B(e') - \alpha_i^S(e')) \right) \]
\[ + D(e) \left( \sum_{e' \in \mathbb{P}(e) - \{ e \}} (\beta_j^B(e') - \beta_j^S(e')) \right) \]
\[ + D(e) \left( \sum_{e' \in \mathbb{P}(e) - \{ e \}} (\phi_h^B(e') - \phi_h^S(e')) \right) \} \text{ is maximum} \]

Now we make the last assumption, the Boundary Condition for government.

(A22) If \( \tilde{p}(k) = (p^C(k), p^B(k), p^S(k)) \), \( p_m^C(k) \to 0 \) as \( k \) goes to infinity, then \( x_{G,m}(\tilde{p}(k)) \to \infty \), where \( x_{G,m}(\tilde{p}(k)) \) is the government’s optimal consumption of commodity \( m \) corresponding to price \( \tilde{p}(k) \).

This assumption makes the plausible condition that the government can increase its consumption of a commodity to infinity when the price of that commodity declines to zero. A sufficient condition to imply this, would be to assume that the government utility function has the property that the marginal utility of its consumption goes to infinity as consumption goes to zero. Here we merely assume the condition directly.

Now we turn to the proof of the lower hemi-continuity of \( \tilde{\Gamma}_i^C(p, \gamma, \delta), \tilde{\Gamma}_j^F(p), \tilde{\Gamma}_h^B(\gamma_h) \) and \( \tilde{\Gamma}^G \).

Lemma 3.3. \( \tilde{\Gamma}_i^C(p, \gamma, \delta), \tilde{\Gamma}_j^F(p), \tilde{\Gamma}_h^B(\gamma_h) \) and \( \tilde{\Gamma}^G \) are lower hemi-continuous on \( \Delta_k \) for each \( k \) and therefore, \( \mu_i^C(p, \gamma, \delta), \mu_j^F(\tilde{p}), \mu_h^B(\tilde{p}) \) and \( v_G(\tilde{p}) \) are upper hemi-continuous on \( \Delta_k \) for each \( k \).

Proof. It suffices to show that the correspondences \( \tilde{\Gamma}_i^C(p, \gamma, \delta), \tilde{\Gamma}_j^F(p), \tilde{\Gamma}_h^B(\gamma_h) \) and \( \tilde{\Gamma}^G(\tau) \) all have interior points for the given price \( \tilde{p} \in \Delta_k \). From (A5), \( \tilde{\Gamma}_i^C(p, \gamma, \delta) \) has an interior point; From (A10), \( \tilde{\Gamma}_j^F(p) \) has an interior point; from (A16), \( \tilde{\Gamma}_j^F(p) \) has an interior point since the firm can buy some security at the initial node and sell it gradually afterwards; and from (A19) and (A20), \( \tilde{\Gamma}^G \) has an interior point. By Berge’s Maximum Theorem and standard methods, we can prove that the correspondences \( \mu_i, \mu_j, \mu_h \) and \( v_G \) are upper hemi-continuous and convex valued. \( \square \)

Define

\[ \Psi(\xi, \tilde{p}) = \prod_{i=1}^I \mu_i^C \prod_{j=1}^J \mu_j^F \prod_{h=1}^H \mu_h^B \otimes v_G \otimes \prod_{e \in E'} \mu_M \]

13
Ψ is also upper hemi-continuous and convex valued. Under these conditions, Ψ satisfies all the conditions of the Kakutani fixed point theorem. Thus there exists $(x^*_i(k), \alpha^*_i(k))$ for all $i \in I$; $(z^*_h(k), \phi^*_h(k))$ for all $h \in H$; $(y^*_j(k), \beta^*_j(k))$ for all $j \in J$; $x^*_G(k)$ and $\bar{p}^*(k)$ such that $(\xi^*(k), \bar{p}^*(k)) \in \Psi(\xi^*(k), \bar{p}^*(k))$, where $\xi^*(k) = ((x^*_i(k), \alpha^*_i(k))_{i \in I}, (z^*_h(k), \phi^*_h(k))_{h \in H}, (y^*_j(k), \beta^*_j(k))_{j \in J}, x^*_G(k))$. Particularly, for all $\bar{p} \in \Delta_k$

\[
\begin{align*}
p^B(e) & (\sum_i \alpha^*_i(k,e) + \sum_j \beta^*_j(k,e) - \sum_h \phi^*_h(k,e)) \\
- p^S(e) & (\sum_i \alpha^*_i(k,e) + \sum_j \beta^*_j(k,e) - \sum_h \phi^*_h(k,e)) \\
+ M \sum_{m=2}^M & p^C_m(e) \sum_i x^*_{i,m}(k,e) + \sum_j y^*_{j,m}(k,e) + \sum_h z^*_{h,m}(k,e) + x^*_G(k,e) \\
- \omega_{G,m}(e) & + \sum_j y^*_{j,m}(k,e) + \sum_i \omega^C_{i,m}(e) + \sum_j \omega^F_{j,m}(e) + \sum_h \omega^B_{h,m}(e) \\
+ p^C_{e'}(e) & [D(e) \left( \sum_{e' \in \mathcal{PA}(e)-\{e\}} (\alpha^*_i(k,e') - \alpha^*_i(k,e')) \right)] \\
+ D(e) & \left[ \sum_{e' \in \mathcal{PA}(e)-\{e\}} (\beta^*_i(k,e') - \beta^*_i(k,e')) \right] \\
+ D(e) & \left[ \sum_{e' \in \mathcal{PA}(e)-\{e\}} (\phi^*_h(k,e') - \phi^*_h(k,e')) \right]
\end{align*}
\]

where $(\alpha^*_i(k,e))_{e \in \mathcal{E}'} = \alpha^*_i(k,e)$ and the other variables are defined in exactly the same manner.

Since $(\xi^*(k), \bar{p}^*(k))$ are bounded, we may assume that the sequence $(\xi^*(k), \bar{p}^*(k))$ converges, say to $(\xi^*, \bar{p}^*)$. By Lemma 3.3, in order to prove that $(\xi^*, \bar{p}^*)$ is an equilibrium of the modified economy, we should show that $p^*_m(e) > 0, m = 1, ..., M, e \in \mathcal{E}'$ and $\xi^*$ satisfies market clearance.

**Lemma 3.4.** $p^C(e) \gg 0, p^B(e) \gg 0, e \in \mathcal{E}'$.

**Proof:** See the Appendix.

**Lemma 3.5.** $\xi^*$ satisfies market clearance.

**Proof:** See the Appendix.

Combining above three lemmas, we have proved the existence of general equilibrium in the modified economy. Consequently, in exactly the same manner as in Arrow and Debreu (1954), it can be shown that $(\xi^*, \bar{p}^*)$ is an equilibrium of the original economy, finishing proof of Theorem 3.1.

4 Conclusion

We have provided sufficient conditions for the existence of a competitive equilibrium for a multiperiod asset economy with taxes and transaction costs. The point of the paper is to show how common assumptions on taxes and transaction costs can be incorporated into a competitive asset economy in a consistent manner. For expositional reasons we have
introduced a number of assumptions to simplify the analysis. We will indicate how one could relax these assumptions in a more general model and how one could modify the strategy of proof to deal with this added complexity.

First, one could introduce multiple physical commodities in the final period. We assumed a single commodity in the last period to simplify the technical analysis. Apart from some additional technical restrictions, the addition of multiple commodities in the last period is a straightforward generalisation.

Second, the brokers and firms have utility functions. We assumed this type of objective because profit maximisation is no longer the unanimous objective for share holders with incomplete markets or transaction costs. Similar problems arise in general with differing taxation across firms and shareholders in multiperiod economies. Rather than attempt to address this difficult issue here we have avoided it by assuming utility functions for brokers and firms. There are attempts to discuss more abstract firm objectives - see Kelsey and Milne (1996) and their references, but we will not pursue that line of argument here. Similar arguments can be applied to the government utility function. Clearly, assuming a utility function is a gross simplification of government decision making, but it suffices to provide sufficiently regular responses in the government demands for goods and services and closes our economy. It would be possible to introduce less regular preferences for government - see the techniques on voting outcomes for firm decision-making in Kelsey and Milne (1996) for some possible ideas that could applied to firms and governments. If the economy has a single physical commodity and the government cannot trade securities then government consumption is specified by the budget constraint and we can dispense with the government utility function.

Third, we have assumed that the transaction technology is convex. It is well known that transactions involve set-up costs, or non-convex technologies. In Jin and Milne (1999) we address that issue by considering approximate equilibria in economies with non-convex transaction technologies (for details see Jin and Milne). Clearly these ideas could be introduced here for our transaction technologies. Similar arguments could be applied to non-convex tax functions that arise from tax/subsidy thresholds that induce sharp nonconvexities in tax schedules. As long as the non-convexities in these functions are not large in comparison to the overall economy, an approximate equilibrium could be obtained.

Fourth, we assumed that taxation over dividends and capital gains are additively separable. This is assumed for expositional convenience. We could have assumed that the capital gains tax function included dividend taxation and suppressed the dividend tax function. Notice that agent’s asset demands and supplies will depend upon buying and selling prices, their utility functions, endowments, technology and their tax functions. This allows considerable generality in taxation law. For example our model allows for tax timing in buying or selling assets (for more discussion and examples of this type of behavior see Dammon, Spatt and Zhang (1999)).

Fifth, although we consider the simple case of broker/intermediaries, the model can be adapted quite easily to accommodate restrictions on consumer and/or firm asset positions that are imposed by regulation (see Milne and Neave (2002) for either a reinterpretation of our model, or a more straightforward modification of the consumer model). Thus our model could be adapted to include discrete versions of Detemple and Murthy (1997) and Cuoco and Liu (1998) on trading restrictions.
Six, one can consider the government precommitting to a general taxation system of laws that specify contingent rates and rules depending upon the behaviour of the agents. Given that the government could compute the equilibrium given the competitive actions of all other agents, then the government could simulate different tax codes and choose among the equilibria. The equilibria we have discussed here is merely one of that set.

To conclude: we have constructed a model that is sufficiently general to include most known models of competitive asset economies with "frictions" in asset markets. We have omitted two important classes of frictions: the first is price making behavior where agents’ asset trades impact on prices. Clearly this would violate our competitive assumption. The second friction involves asymmetric information between agents. Nearly all of the latter literature is restricted to partial equilibrium frameworks, although there is some recent work modifying the competitive equilibrium framework.

5 Appendix

Proof of Lemma 3.2: If the assumption (A6) holds, then the set \( \Phi_h \) is a closed convex set. The convexity of \( \Phi_h \) is obvious. It remains to show its closedness. To this end, suppose that for each \( k \), there exists \( \phi_h^{(k)} \in \Phi_h \) and \( z_h^k \in \mathbf{Z}_h \) such that \( (\phi_h^{(k)} , z_h^k) \in T(h,e), \forall e \in \mathbf{E}' \), and, in particular, by (A8), \( (\phi_h^{(k)} , z_h^k) \in T(h,e), \forall e \in \mathbf{E}' \), where \( z_h^k = (\max_k z_{h1}^k, \ldots , \max_k z_{hM}^k) \).

If \( \{\phi_h^{(k)}\} \) is unbounded, we may suppose \( (\phi_h^{(k)})_{h1} \rightarrow \infty \) without loss of generality. But, by assumption (A7),

\[
\lim_{k \rightarrow \infty} |z_h^k| = \infty
\]

which provides a contradiction since \( \mathbf{Z}_h \) is bounded by Lemma 3.1 and proves the boundedness of \( \{\phi_h^{(k)}\} \). Hence, we can choose a subsequence \( \{\phi_h^{(k_n)}\} \) from \( \{\phi_h^{(k)}\} \) such that

\[
\lim_{n \rightarrow \infty} \phi_h^{(k_n)} = \phi_h,
\]

this implies, by closedness of \( T(h,e) \), that \( \Phi_h \) is compact. □

Proof of Lemma 3.4: First of all, we prove that \( p^{C*} \gg 0 \). It is obvious that for all \( \bar{p} \in \triangle \), \( \xi^* \) satisfies (1) and particularly,

\[
\sum_i \alpha_i^{B*}(e) + \sum_j \beta_j^{B*}(e) - \sum_h \phi_h^{S*}(e) - (\sum_i \alpha_i^{S*}(e) + \sum_j \beta_j^{S*}(e) - \sum_h \phi_h^{B*}(e)) \ll 0; \quad (2)
\]

\[
\sum_i \alpha_i^{B*}(e) + \sum_j \beta_j^{B*}(e) - \sum_h \phi_h^{S*}(e) \ll 0; \quad (3)
\]

\[
\sum_i x_{i,m}^*(e) + \sum_j y_{j,m}^*(e) + \sum_h z_{h,m}^*(e) + x_{G,m}^*(e)
\]

\[
- (\omega_{G,m}(e) + \sum_j y_{j,m}^*(e) + \sum_i \omega_i^{C,m}(e) + \sum_j \omega_j^{F,m}(e) + \sum_h \omega_h^{B,m}(e)) \leq 0, \quad m = 2, \ldots , M,
\]

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and

\[
\sum_i x_{i,1}^e + \sum_j y_{j,1}^e + \sum_h z_{h,1}^e + x_{G,1}^e \\
- (\omega_{G,1}^e + \sum_j y_{j,1}^e + \sum_i \omega_i^e + \sum_j \omega_j^e + \sum_h \omega_{h,1}^e)
\]

\[
\leq D(e)[ \sum_{e' \in \mathcal{PA}(e)-\{e\}} (\alpha_i^{B*}(e') - \alpha_i^{S*}(e'))] \\
+ D(e)[ \sum_{e' \in \mathcal{PA}(e)-\{e\}} (\beta_i^{B*}(e') - \beta_i^{S*}(e'))] \\
+ D(e)[ \sum_{e' \in \mathcal{PA}(e)-\{e\}} (\phi_h^{B*}(e') - \phi_h^{S*}(e'))]
\]

\[
\leq 0.
\]

The last inequality follows from (2). This means that \(x_i^* \in \hat{X}_i, y_j^* \in \hat{Y}_j, z_h^* \in \hat{Z}_h \) and \(x_G^* \in \hat{X}_G \) and therefore, \((\phi_h^{B*}, \phi_h^{S*}) \in \Phi_h, h = 1, ..., H, (\alpha_i^{B*}, \alpha_i^{S*}) \in \Phi_i \) and \((\beta_i^{B*}, \beta_i^{S*}) \in \Theta_j \).

Thus, by assumption (A22), \(p_m^{C*}(e) > 0, m = 1, ..., M, \forall e \in E' \), otherwize, \(x_G^* \notin \hat{X}_G \).

Now we turn to proving \(p_{B*}(e) \geq 0, \forall e \in E' \). To the contrary, suppose \(p_n^{B*}(e_0) = 0 \) for some \(n \) and some \( e_0 \in E' \). By considering the consumer \(i\) and noticing that \(p_{C*}(e) \geq 0, \forall e \in E' \), \((\alpha_i^{B*}, \alpha_i^{S*}) \neq \Phi_i \), since, by assumption (A20), the consumer can unlimittedly increase her wealth at one of terminal node by buying asset \(n\) at the node \(e_0\). Consequently, \(p_{B*}(e) \geq 0, \forall e \in E' \).

**Proof of Lemma 3.5**: Note that from the proof of Lemma 3.4, \(x_i^* \in \hat{X}_i, y_j^* \in \hat{Y}_j, z_h^* \in \hat{Z}_h, x_G^* \in \hat{X}_G \). Hence,

\[
p_{B*}(e)[ \sum_i \alpha_i(x_i^*, e) + \sum_j \beta_j(x_j^*, e) - \sum_h \phi_h(x_h^*, e)] \\
- p_{S*}(e)[ \sum_i \alpha_i(x_i^*, e) + \sum_j \beta_j(x_j^*, e) - \sum_h \phi_h(x_h^*, e)]
\]

\[
+ \sum_{m=2}^M \sum_i x_{i,m}^e + \sum_j y_{j,m}^e + \sum_h z_{h,m}^e + x_{G,m}^e \\
- (\omega_{G,m}^e + \sum_j y_{j,m}^e + \sum_i \omega_i^e + \sum_j \omega_j^e + \sum_h \omega_{h,m}^e)
\]

\[
+ p_{C*}(e)[ D(e)[ \sum_{e' \in \mathcal{PA}(e)-\{e\}} (\alpha_i^{B*}(e') - \alpha_i^{S*}(e'))] \\
+ D(e)[ \sum_{e' \in \mathcal{PA}(e)-\{e\}} (\beta_i^{B*}(e') - \beta_i^{S*}(e'))] \\
+ D(e)[ \sum_{e' \in \mathcal{PA}(e)-\{e\}} (\phi_h^{B*}(e') - \phi_h^{S*}(e'))]
\]

\[
= 0, e \in E'
\]

Suppose \(p_n^{B*}(e) > p_n^{S*}(e)\). Define \(p_n^{S*}(e) = p_n^{S*}(e) + \varepsilon, p_M^{C*}(e) = p_M^{C*}(e) - \varepsilon\) for \(\varepsilon\) sufficiently small, \(p_l^{S*}(e) = p_l^{S*}(e), l \neq n, p_l^{B*}(e) = p_l^{B*}(e), l = 1, ..., N, p_m^{C*}(e) = p_m^{C*}(e), m = 1, ..., M - 1\).
Plugging this price in to (1) and by above equality and (5),
\[\varepsilon(\sum_h \phi_{h,n}^*(e) - \sum_i \alpha_{i,n}^*(e) - \sum_j \beta_{j,n}^*(e))\]
\[\leq \varepsilon(\sum_i x_{i,M}^*(e) + \sum_j y_{j,M}^*(e) + \sum_h z_{h,M}^*(e) + x_{G,M}^*(e)\]
\[-(\omega_{G,M}(e) + \sum_j y_{j,M}^*(e) + \sum_i \omega_{i,M}^C(e) + \sum_j \omega_{j,M}^F(e) + \sum_h \omega_{h,M}^B(e)]\]
\[\leq 0, \ e \in E'.\]

implying
\[\sum_h \phi_{h,n}^*(e) - \sum_i \alpha_{i,n}^*(e) - \sum_j \beta_{j,n}^*(e) \leq 0, \ e \in E'. \quad (6)\]

On the other hand, as in the proof of \(p_{B^*}(e) \gg 0\), we can show that if \(p_{m}^{S^*}(e^0) = 0\) for some \(m\) and some \(e^0 \in E'\), then, by assumption (A20), \(\alpha_{i,n}^{S^*}(e^0) = \beta_{j,n}^{S^*}(e^0) = 0, \forall i, j\), and therefore, by (6), \(\phi_{h,n}^*(e^0) = 0\). Consequently, by noticing that \(p_{C^*}(e) \gg 0, p_{B^*}(e) \gg 0, e \in E'\), and combining (2), (3), (4), (5) and (6),
\[\sum_i x_{i}^*(e) + \sum_j y_{j}^*(e) + \sum_h z_{h}^*(e) + x_{G}^*(e)\]
\[-(\omega_{G}(e) + \sum_j y_{j}^*(e) + \sum_i \omega_{i}^C(e) + \sum_j \omega_{j}^F(e) + \sum_h \omega_{h}^B(e)) = 0;\]
\[\sum_i \alpha_{i,n}^{B^*}(e) + \sum_j \beta_{j,n}^{B^*}(e) - \sum_h \phi_{h,n}^{S^*}(e) - (\sum_i \alpha_{i,n}^{S^*}(e) + \sum_j \beta_{j,n}^{S^*}(e) - \sum_h \phi_{h,n}^{B^*}(e)) = 0\]

if \(p_{B^*}(e) = p_{B^*}^{S}(e)\); if \(p_{B^*}(e) > p_{B^*}^{S}(e)\),
\[\sum_i \alpha_{i,n}^{B^*}(e) + \sum_j \beta_{j,n}^{B^*}(e) - \sum_h \phi_{h,n}^{S^*}(e) = 0\]
and
\[\sum_i \alpha_{i,n}^{S^*}(e) + \sum_j \beta_{j,n}^{S^*}(e) - \sum_h \phi_{h,n}^{B^*}(e) = 0\]

finishing the proof of Lemma 3.5. \(\square\)

References


