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- An Overview -

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Market Liquidity Risk
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Market liquidity is the ease of trading an asset. Its risk is the potential loss, because a security can only be traded at high or prohibitive costs. While the omnipresence and importance of market liquidity is widely acknowledged, it has long remained a more or less elusive concept. Treatment of liquidity risk is still under development.

This paper provides an overview on important aspects of market liquidity and its risk. We also survey existing models to integrate market liquidity risk into risk frameworks. We place special emphasis on practical usability and discuss relevant strengths, weaknesses and their implications.

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1 Introduction

Many recent crises have been liquidity crises. The two large hedge fund breakdowns of LTCM in 1998 and Amaranth Advisors in 2006 were mainly caused, because they took positions that were too large to be liquidated without substantial price impact.[1] In the recent sub-prime crises of 2007/08 banks around the world were troubled by liquidity shortages and had to liquidate assets to reduce risk exposure. Stock prices slumped because many funds were forced to sell-off positions due to margin calls and fund outflows.

The regulators are alert and the Basel II committee has already published several reports and guidelines on liquidity in recent months. Banks are requested to "use appropriately conservative assumptions about the marketability of assets" and "incorporate liquidity costs, benefits and risks in the internal pricing, performance measurement and new product approval process for all significant business activities"[2]. Still, the BIS survey among banks revealed, that market liquidity remains the single risk factor across all asset classes, that is not easily captured.[3]

In this paper, summarize the current state of research on liquidity definition and its relevant aspects. We also provide an up-to-date overview on the treatment of liquidity risk. We describe existing liquidity risk models and clarify when and under which assumptions they can be applied. We also analyze strengths and weaknesses from a practical point of view. Finally, we try to sketch open research questions, which we believe to be most relevant for the proper treatment of liquidity risk in practice.

In contrast to existing overviews[4] we take a more critical and more practical point of view. We also outline implied, less transparent assumptions and limitations of liquidity risk models, but also characterize their specific range of applications.

The paper is structured as follows. Section 2 defines liquidity and outlines its characteristics. Section 3 provides an overview on existing liquidity risk models and describes their assumptions, strengths and weaknesses. Section 4 summarizes and sketches possible venues for future research.

[1] Cp. Jorion (2007)
[2] Cp. Basel committee (2008), p. 6 and p. 9.
[3] Cp. Basel committee (2005), p. 10.
[4] Cp. Mahadevan (2001); Erzegovesi (2002); Loebnitz (2006); Bervas (2006); Jorion (2007).

2 Definition of liquidity

In this section we will clarify relevant terms and concepts as well as clearly delimit the topic. The term 'liquidity' is used in three different settings.[5] First, liquidity can designate the liquidity of a firm, also called solvency. From the corporate perspective, this is the net liquidity of assets and liabilities. Liquidity of the liability side is also called 'funding liquidity'. Second, liquidity is a characteristic of an asset, also called 'asset liquidity' or 'market liquidity' depending on whether the balance sheet or the market is in focus. From an investor's perspective it describes the marketability or "ease of trading an asset"[6]. Third, liquidity is also used from a monetary perspective and addresses the liquidity of the whole economy.

While solvency is quite well understood, this paper addresses issues of market and asset liquidity, which have more recently been brought into focus.

2.1 Definition of market liquidity

Market liquidity can be defined as the cost of trading an asset relative to fair value.[7] Fair value is set at the middle of the bid-ask-spread, the mid-price. This has the advantage that it is most objective, but the disadvantage, that the fair, fundamental value fluctuates heavily, which is slightly less intuitive.

We distinguish three components of liquidity cost $L_t(q)$ in percent of the mid-price for an order quantity q at time t

$$L_t(q) := T(q) + PI_t(q) + D_t(q) \tag{1}$$

where $T(q)$ are direct trading costs, $PI_t(q)$ is the price impact vs. mid-price due to the size of the position, $D_t(q)$ are delay costs if a position cannot be traded immediately.[8]

Direct trading costs comprise exchange fees, brokerage commissions and transaction taxes. They are also called explicit transaction costs, because they are known beforehand and time invariant, i.e. deterministic.[9] The *price impact* is the difference between the achieved transaction price and the mid-price.[10] They result from

[5]Extended from Jorion (2007), p. 334.
[6]Cp. Longstaff (1995).
[7]Cp. Dowd (2001), p. 187 ff. and Buhl (2004); Amihud and Mendelson (2006).
[8]This closely follows Amihud and Mendelson (2006), but additionally differentiates by the size of the position. Compare also similar in Aitken and Comerton-Forde (2003); Torre (1997).
[9]Also cp. Loebnitz (2006), p.18 f.
[10]Similarly Demsetz (1968) defines transaction cost as the price concession needed for an immediate exchange of an asset into money (p.35). This is also called market impact.

imperfectly elastic demand and supply curves for an asset at a specific point in time, which makes the price impact increase with the size traded.

Liquidity costs increase with order size for two reasons. First, investors have heterogeneous expectations with respect to the fair value of an asset and are subject to capital restrictions. They are therefore willing to trade only a limited quantity at their own prespecified price. When trading a small position, a trader is likely to find a counterparty which is willing to exchange the full position at or close to the trader's fair value expectation. The larger the position to be traded, the more counterparties have to be found. The achievable transaction price falls. Compared to the trader's fair value expectation, the liquidation cost rises with the size of the position. Second, liquidity costs are also a price for immediacy. An immediate transaction at a certain price is essentially an American option paired with an exchange.[11] The option component comprises the right to receive a certain amount of shares at order execution with the current market price as strike. This optionality has an immanent value, which depends on price volatility and the order size relative to expected transaction volume, because this determines the future liquidity of the position for the buyer. Due to these two components, price impact cost can be expected to rise with the size of the position.

Delay costs comprise the costs for searching a counter-party and the cost imposed on the investor due to bearing risk, because prices and price impact cost might change during the delay.[12] For many assets, like most stocks and bonds on an exchange, search costs are negligibly small, but costs of additional risk during delay can remain substantial.

Because liquidity costs increase with size, a trader faces a possible trade-off between cost and delay. He can save on price impact cost by deliberately delaying parts of the transaction. But then he has to face delay risk for the remaining portion of the position. This deliberate delay is optimal if the savings on price impact costs exceed the additional delay cost. These strategies are analyzed in the literature on optimal trading strategies.[13] As a consequence, there are two types of delay, forced and deliberate.

Relation to other liquidity definitions Above cost definition takes a practical, concrete investor's perspective and can integrate other definitions in the literature.

[11] Cp. Chacko et al. (2008).
[12] Almgren (2003) calls price impact risk "trading enhanced risk".
[13] Cp. for example Bertsimas and Lo (1998); Almgren and Chriss (1999, 2000); Almgren et al. (2005); Almgren (2003); Subramanian and Jarrow (2001) and section 3.4.1.

In our view, it also provides a suitable framework to integrate the multitude of perspectives and makes liquidity a less elusive concept.

In the cost framework, liquidity is the effect a transaction has on an investor. The importance of other, more indirect liquidity measures like transaction volume, zero trading days, depth, etc.[14] can be much better understood from a cost perspective. If a liquidity aspect results in high liquidity costs in economic downturns, it will have a large effect on asset prices. The cost perspective provides the economic explanation for the validity of many liquidity measures.[15]

The most often cited dimensions of liquidity are tightness, depth, resiliency and immediacy.[16] They can be easily understood in above cost framework. *Tightness*, “the cost of turning a position around in a short time”, corresponds to the sum of direct trading costs T and price impact costs PI. *Depth*, “the size of an order flow innovation required to change prices a given amount”, is the quantity q transactable at a specific price impact PI, i.e. $PI^{-1}(q)$. *Resiliency*, “the speed with which prices recover from a random, uninformative shock”, is the mean reversion speed of liquidity cost, i.e. the time dimension of liquidity cost. *Immediacy*, the time between order submission and settlement, directly corresponds to the delay time of the delay cost component D. Thus, all four dimensions can be analyzed in the cost framework introduced above.

Kempf (1999) defines liquidity in more abstract terms and cites the dimensions price and time. Price directly corresponds to cost, but time is - in above view - also converted into a cost component. While time is a more direct aspect of liquidity, its conversion into cost make it more concrete from an investor’s perspective.

2.2 Important aspects of market liquidity

Degrees of market liquidity Liquidity is a continuous characteristic. Hence, assets can have different degrees of liquidity.[17] The liquidity degree is determined by the type of the asset, the size of the position and the liquidation horizon. It is useful to distinguish at least four categories of liquidity degrees as illustrated in figure 1 on the facing page. They are closely related to the magnitude of liquidity costs and require substantially different treatment.

If an asset is ’fully liquid’ any position in the asset can be immediately traded without a cost. Cash is the primary example. For practical purposes, liquidity

[14] Cp. Datar et al. (1998); Liu (2006); Bekaert et al. (2007); Goyenko et al. (2008) and others.
[15] Cp. Stange and Kaserer (2008a), p.4.
[16] Cp. Kyle (1985), p. 1361 for the first three dimensions and the citations and Black (1971), p.30 for the latter. Tightness is also sometimes called ’width’ or ’breadth’.
[17] Cp. also discussion in Stange and Kaserer (2008a), p. 4f.

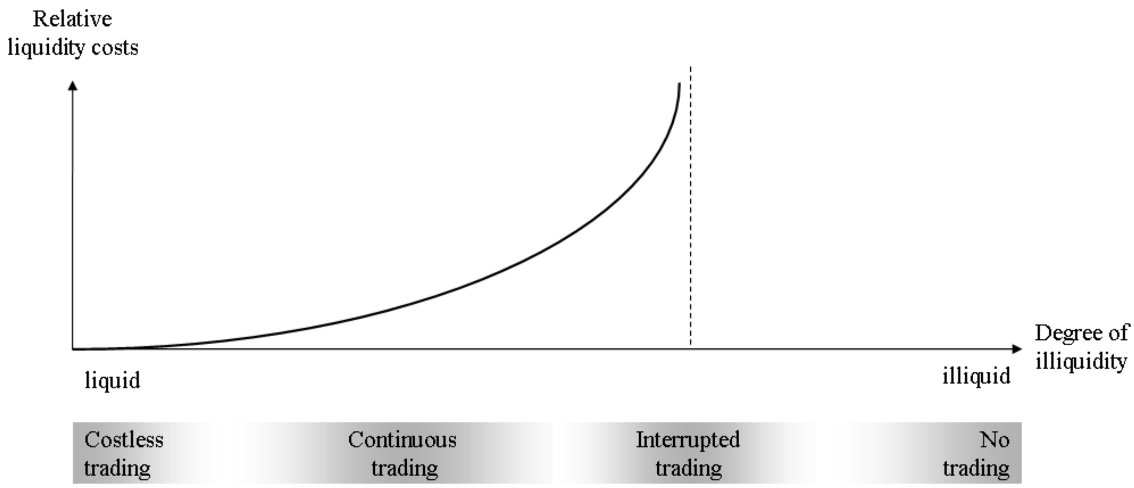

Figure 1: Degrees of market liquidity
Figure 1 illustrates the different degrees of market liquidity and the resulting important liquidity categories.

adjustments to its value are not necessary. An asset can be called 'continuously tradable' when most positions can be traded albeit with a cost. A good example are limit order books of developed stock markets. The determination of the costs of trading is the main issue from a liquidity perspective. If liquidity deteriorates further, the asset becomes 'disruptively tradable', i.e., it can be traded from time to time. While market price provide an indicator for the fair value of the asset, delay and its incorporation into liquidity measures is a major issue - in addition to trading costs. A good example are over-the-counter markets of exotic bonds. Finally, an asset is 'illiquid' if no position size can be traded. Market prices are thus non-observable and value has to be determined by intrinsic methods. Rare art or currently collateralized debt obligations (CDOs) can be considered illiquid.

Not only the type of the asset, but also the size of the position determines the degree of liquidity. In most cases, it is the position size relative to the prevailing trading volume, that determines the degree of liquidity, which also shows the relation between asset and market liquidity. Is the position size much larger than traded volume, we can expect significant trading delay. The asset position is only interruptedly tradable. If it is too large, it might even be illiquid in the short term due to the lack of counter-parties.

The liquidation horizon is another determinant of a position's liquidity degree. A security might be illiquid in the short term because of a lack of counter-parties, but interruptedly tradable at longer liquidation horizons. If an asset is held to maturity, then, obviously, liquidity costs are zero and irrelevant.

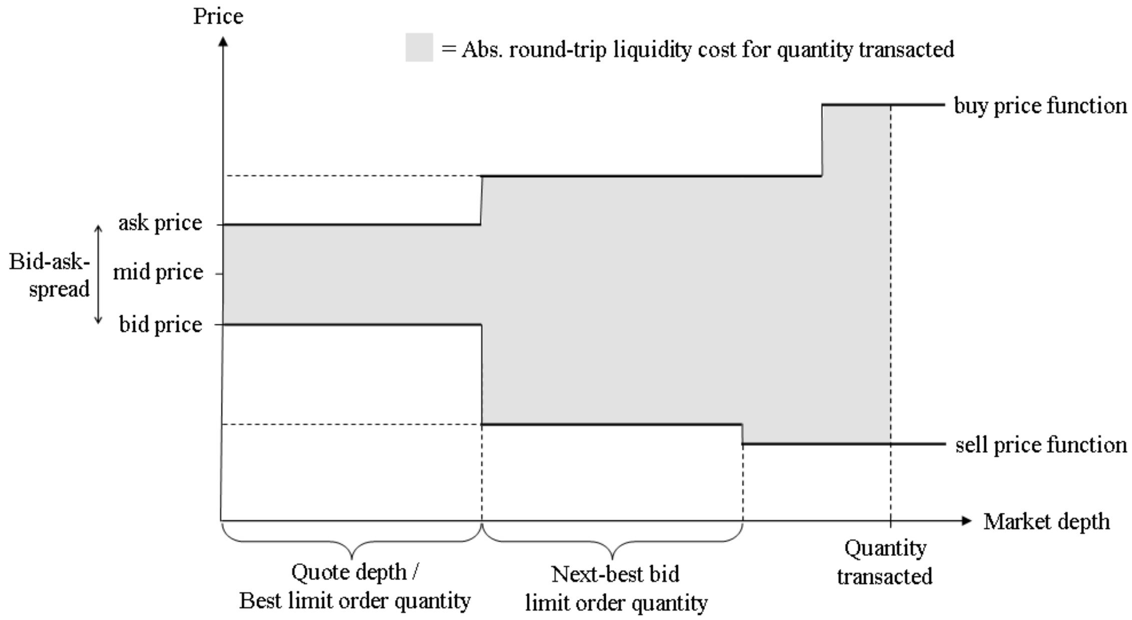

Figure 2: Price quantity function

Figure 2 illustrates the price impact function as difference between the buy-price and the sell-price function as well as important term in this context.

Characteristics of market liquidity When measuring market liquidity, ex-ante, *committed* liquidity and possible *hidden* liquidity have to be distinguished.[18] The advantage of market organization on the basis of order books lies in the fact, that more liquidity is ex-ante committed and transparent to market participants.

The price impact component of asset liquidity can be described in a price-quantity diagram, which collects all potential counterparty orders with their order size and their willingness to pay. In case of committed liquidity in a limit order book, these are limit orders. These counterparty orders, if sorted by best price construct the buy- or sell-price function. The cost of liquidity of a round-trip[19] can be then described by a price-quantity function, which is the difference between the buy- or sell-price function and the mid-price as displayed in figure 2. The trader buys at the buy price function and sells at the sell price function. The difference between the two is the liquidity cost from the transaction.

For small orders, not larger than the quote depth, this cost of a round-trip corresponds to the bid-ask-spread. For larger orders the liquidity cost of a round-trip is the weighted spread between the buy- and sell-side functions up to the traded quantity. The spread of the individual limit orders are weighted with their respective limit order quantity. In general, this weighted spread is called 'price impact'.

[18] Cp. Irvine et al. (2000).

[19] I.e. buying and immediately selling a position.

Because the limit order book only measures committed liquidity, due to hidden liquidity, transactions can and do occur inside the bid-ask-spread. Therefore the commonly used quoted spread measures ex-ante committed liquidity.

Up to the quote depth, liquidity costs are sometimes called *exogenous* and beyond *endogenous*.[20] It is argued, that bid-ask-spread up to the quote depth is exogenous, because it is common to all market participants while the weighted spread is endogenous depending on the individual trader's position. We believe that this argument is imprecise with respect to the structure of liquidity costs. The whole price impact curve is exogenously given, because it is determined by the market. This is also true beyond the quote depth. The size of the trade (endogenously) determines the point on the curve valid for a specific trade. In this way, the bid-ask-spread is also endogenous - determined by a very small specific trade position. As a consequence, the cost itself is neither exogenous nor endogenous - at any size - but can be decomposed into an exogenous price-quantity curve and an endogenous point on this curve.

A possible cause for this misleading distinction is the usual graphical representation, which shows a flat price impact curve similar to the display above, but a continuous increases of liquidity cost beyond the spread. This falsely implies that liquidity costs would be structurally different beyond the spread.

Above graphical display necessarily neglects the temporal dynamics of liquidity. Important is the distinction between temporary and permanent price impact.[21] *Temporary price impact* is the portion of the price impact, that will dissipate over time and is closely related to the notion of resiliency.[22] It is driven by order imbalances when trades are purely motivated by liquidity needs. Temporary price impact might also occur under information asymmetries, if the market reacts on perceived informational content, i.e. it occurs due to adverse effects. *Permanent price impact* is the portion of the price impact that will permanently move mid-prices. In an efficient market, the permanent part is directly related to the real informational content of the trade. Measurement of temporary and permanent price impact separately is still difficult.[23]

[20] Cp. Bangia et al. (1999), p.68 f., also in Jorion (2007), p. 336 or Bervas (2006), 3.
[21] Holthausen et al. (1987) first introduced this setup.
[22] Cp. section 2.1..
[23] Cp. Amihud (2002); Pastor and Stambaugh (2003), who try to extract temporary price impact from prices..

2.3 General definition of liquidity risk

Traditional risk measurement assumes that liquidity costs can be neglected if the liquidation horizon is long enough.[24] Therefore, there is no adjustment for liquidity cost in many practical market valuation models: Liquidity cost is assumed to be zero and positions to be liquidated at mid-prices.

Liquidity risk can generally be defined as the potential loss due to time-varying liquidity costs. Several empirical papers have already shown that liquidity risk is a substantial risk component, already when only cost at the bid-ask-spread level is accounted for. Bangia et al. (1999) find underestimation of total risk by 25-30% in emerging market currencies in daily Value-at-Risk. Le Saout (2002) estimates that the bid-ask liquidity component can represent over 50% of total risk for illiquid stocks. Lei and Lai (2007) reveal a 30% total intraday risk contribution by liquidity in small-price stocks.

Also, the adjustment for the full price impact cost - beyond the spread - is significant. Francois-Heude and Van Wynendaele (2001) find a 2-21 % contribution of price impact in one stock. Giot and Grammig (2005) show that 30-minute liquidity-adjusted VaR is 11-30 % for three stocks. Angelidis and Benos (2006) estimate that liquidity risk constitutes 11 % of total VaR in low capitalization stocks. Stange and Kaserer (2008b) show in a large stock sample that liquidity risk amounts to over 25 % of price risk in a 10-day, 99 % VaR when trading large positions. A detailed discussion of risk measurement methods and more concrete liquidity risk definitions will be provided in section 3.

Furthermore, there is an important conceptual distinction to be made when defining 'horizons' in the liquidity risk management framework. The *reaction horizon* is the time until management takes a decision vis-a-vis the liquidation of an asset, while the *liquidation horizon* is the period during which the position is liquidated. Although this distinction is usually neglected, it has important consequences. Usually, the horizon is used as a forecast period. Based on this information a decision is taken now, i.e. the reaction horizon is zero and the liquidation horizon is equal to the forecast period. Although the position is said to be orderly liquidated during the liquidation horizon, its worst value is calculated for the end of the liquidation horizon, which is logically inconsistent but conservative.

When directly adjusting for liquidity risk, it is possible to be more precise and logically consistent. However, 'horizon' then has to be distinguished into above aspects.

[24]Cp. Jorion (2007), p. 333.

3 Models including market liquidity risk

The choice of liquidity risk model strongly depends on the purpose as well as the type of asset position in question. In the following, we will look at models for regular risk measurement, which are not necessarily suitable for stress testing. If intraday forecasts are not aimed for or the integration of intraday data is too computational intensive, several models based only on intraday data are ruled out.

Assets on the balance sheet have to be categorized according to the following three criteria: General degree of market liquidity, typical size of a position and data availability.

What is the general liquidity degree of the asset? If the asset is continuously traded, liquidity cost models are in focus, if it is only traded with large interruptions, models incorporating execution delay have to be applied. If the asset is illiquid, i.e. generally not traded, value has to be determined with internal models. The same is true, if data is hardly available or of limited quality, e.g. in some over-the-counter markets. Internal value models and possible liquidity adjustments therein are outside the focus of this paper.

How large is the typical position size relative to traded volume? If sizes are relatively small, models which neglect the price impact of position size can be applied. If sizes get larger, these models are naturally imprecise. If positions are especially large, like block holdings, even models which incorporate price impact will loose precision.

What type of data is available? The precision of the price impact measurement depends directly on the amount of data available. On the basis of spread data, price impact is generally neglected. On the basis of transaction data, price impact approximations are possible.[25] With limit order book data, price impact can be quite precisely estimated. The type of data determines the liquidity measure than can be used.

In the following, we will introduce relevant liquidity risk models and indicate, which assumptions are made and when they can be applied. We want to emphasize at this point, that our discussion is based on our very own interpretation of the liquidity risk models, because many aspects we point out are only implicit in the model structure and not explicitly discussed by the original authors. We also used our own consistent notation to allow for better comparisons between the different models.

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[25] Cp. also Erzegovesi (2002), p. 9 ; Torre (1997) argues that large costs cannot be observed because trades at such cost are not executed and transaction data is most sparse in illiquid assets where expected price impact are largest.

3.1 Models based on bid-ask-spread data

3.1.1 Add-on model based on bid-ask-spread: Bangia et al. (1999)

Bangia, Diebold, Schuermann and Stroughair (1998, 1999) include time-varying, empirical bid-ask-spreads into a parametric Value-at-Risk (VaR). Transaction price is modeled as mid-price with an add-on for the bid-ask-spread,

$$P_{mid,t+1} = P_{mid,t} exp(r_{t+1}) - \frac{1}{2} P_t S_{t+1} \tag{2}$$

where P_{mid} is the middle of the bid-ask-spread, r is the continuous mid-price return between t and t+1 and S is the time-varying bid-ask-spread. Relative liquidity-adjusted total risk (L-VaR) is then the sum of the mean-variance-estimated price-risk percentile and the empirically-estimated spread percentile.

$$L - VaR = 1 - exp(z_\alpha \sigma_r) + \frac{1}{2} P_{mid} \left(\mu_S + \hat{z}_\alpha \sigma_S\right) \tag{3}$$

where σ_r is the variance of the continuous mid-price return and μ_S and σ_S are the mean and standard deviation of the bid-ask-spread. z_α is the percentile of the normal distribution, $\hat{z}_\alpha$ is the empirical percentile of the spread distribution.[26] As spread is not normally distributed, it is not possible to take percentiles from theoretical distribution tables. Therefore, Bangia et al. take the percentile of the empirical spread distribution, which ranges - in their 99% case - between 2.0 and 4.5, which is partially far away from 2.33, the 99% cut-off of the normal distribution.

Bangia et al. (1999) also address the problem of moving from single asset to portfolio VaR. They argue, that aggregating single asset L-VaR's by using the spread covariance matrix is of dubious value, because spreads are non-normally distributed. Instead, they suggest to aggregate single asset's price risk in a more traditional way and then deduct a weighted average spread from the portfolio VaR. Single currency and portfolio L-VaRs are calculated as illustration in their paper. Other empirical applications of this model include Mahadevan (2001), Lei and Lai (2007) and Roy (2005).

The great advantage of the methodology of Bangia et al. is the low data requirement. Spread data is available at all frequencies for most assets, often also in over-the-counter (OTC) markets. It is also quickly implementable, because the liquidity-adjustment can be simply added to existing price risk measures.

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[26] The empirical percentile is calculated as $\hat{\alpha}_S = (\hat{S}_\alpha - \mu_S)/\sigma_S$, where $\hat{S}_\alpha$ is the percentile spread of historical distribution.

The greatest drawback is the neglect of price impact, the fact that only small order sizes can be traded at the spread and liquidity costs quickly increase with order size. As consequence, liquidity risk will be heavily underestimated for large positions.

Further, their add-on approach is logically inconsistent, because spread is calculated on the current mid-price and not on the crises mid-price, which is however easily correctable.[27] Bangia et al. also make the assumption of perfect tail correlation between spread and price, i.e. they assume that worst liquidity costs and lowest prices occur simultaneously. Because tail correlations are much lower in reality, this technical assumption overestimates liquidity risk.[28]

Another problem is the estimation of the spread distribution. As stated in their paper, spreads are often far from normal, because regime-switching leads to multimodality and because trending creates skewness and fat tails.[29] Accounting for non-normality by using empirical percentiles remains difficult, because this requires longer time series as a basis for estimation, which might themselves exhibit structural breaks with several modi. Structural breaks might especially occur in crises. These distributional properties make further underestimation of liquidity risk highly likely.

Although, the Bangia, Diebold, Schuermann and Stroughair (1999)-model suffers from several imprecisions, it is one of the few models of choice, when data is scarce, especially on transaction volumes or transactions. We recommend to keep the add-on approach under the assumption of perfect correlation, because this (partially) compensates the tendency to underestimate due to the neglect of position size and regime-switching.

3.1.2 Modified add-on model with bid-ask-spread: Ernst et al. (2008)

Ernst, Stange and Kaserer (2008) apply a Cornish-Fisher approximation to determine percentiles instead of taking them from the historical empirical distribution. The Cornish-Fisher approximation adjusts percentiles from the normal distribution to account skewness and kurtosis.[30] The approximate adjusted percentile $\tilde{z}_\alpha$is calculated as

$$\widetilde{z}_\alpha \approx z_\alpha + \frac{1}{6}(z_\alpha^2 - 1) * \gamma + \frac{1}{24}(z_\alpha^3 - 3z_\alpha) * \kappa - \frac{1}{36}(2z_\alpha^3 - 5z_\alpha) * \gamma^2 \tag{4}$$

[27] Critique noted and corrected by $L - VaR = 1 - exp(\alpha\sigma_r) \times 1/2(\mu_S + \tilde{\alpha}\sigma_S)$. in Loebnitz (2006), p.71 f.

[28] Cp. critique in Francois-Heude and Van Wynendaele (2001); Angelidis and Benos (2006); Jorion (2007) and empirical results of Stange and Kaserer (2008b).

[29] Cp. discussion of the distributional characteristics of spread in Stange and Kaserer (2008a).

[30] Integration of higher moments is also possible.

where z_α is the α- percentile of a $N(0,1)$ distribution, γ denotes the skewness and κ the excess-kurtosis estimate of the random variable. Modified, relative, liquidity-adjusted total risk can then be calculated as

$$L - VaR = 1 - exp(\mu_r + \tilde{z}_\alpha(r) \times \sigma_r) \times \left(1 - \frac{1}{2}\left(\mu_S + \tilde{z}_\alpha(S) \times \sigma_S\right)\right) \quad (5)$$

where $\tilde{z}_\alpha(r)$ is the Cornish-Fisher-approximated percentile of the return distribution and $\tilde{z}_\alpha(S)$ and the approximated spread distribution percentile.

The procedure is shown to yield empirically more precise results than the specification of Bangia et al. (1999). However, the critique similarly applies. It assumes, that positions can be traded at the bid-ask-spread (although the approach can also be used on other liquidity approaches, see section 3.3.3). Perfect correlation between mid-price return and liquidity costs are similarly a problem of this add-on approach.

Overall, Ernst et al. (2008) provides an alternative and more precise approach for bid-ask-spread data than Bangia et al. (1999).

3.2 Models based on volume or transaction data

Several papers have used different price impact measures with increasing preciseness to address the shortcomings of Bangia et al. (1999).

3.2.1 Transaction regression model: Berkowitz (2000)

Berkowitz (2000a,b) estimates the liquidity price impact from past trades. While controlling for the influence of other risk factors, price impact is measured from the time-series of trades in a linear regression.

$$P_{TA,t+1} = P_{mid,t} + C + \theta N_t + x_{t+1} + \epsilon_t \quad (6)$$

where $P_{TA,t+1}$ is the transaction price at time $t+1$, N_t is the number of shares sold, θ is the regression coefficient, x_{t+1} is the effect of risk factor changes on the mid-price, C is a constant and ϵ_t the error term of the regression. The regression coefficient θ acts as liquidity measure and can be seen as the absolute return due to changes in volume, i.e. the absolute liquidity cost per share traded.

To construct a liquidity-adjusted risk measure in a convenient way, Berkowitz assumes that liquidity and other risk factors are independent from each other, which is equivalent to zero liquidity-return correlation. They also build on Bertsimas and Lo (1998), who show that under linear price impact an optimal execution strategy within a horizon of h days is to liquidate $\frac{1}{h}$th of the portfolio each day during the liquidation period. Similar to equation (6), price then follows

$$P_{TA,t+1} = P_{mid,t} + x_{t+1} - \theta \frac{N_t}{h} \tag{7}$$

Risk can then be derived from the general probability distribution. The choice of concrete risk measurement (numerical, simulation, parametric) is left to the reader.

The advantage of the Berkowitz-approach is the integration of price impact of order size beyond the bid-ask-spread. While being more computationally extensive through the regression methodology, it only uses transaction data for the liquidity measurement, which is available in many markets. However, intraday data are required to calculate the price impact cost from single trades. Otherwise, the estimation can get very approximate.[31]

The liquidity measure used in their approach, however, is quite imprecise. In general, it closely resembles the liquidity measure of Amihud (2002). Berkowitz additionally controls for risk factor changes in his empirical regression. One problem is, that θ can become positive or negative, which is counter-intuitive as size should always lead to a price discount. Further research should empirically verify in how far this measure proxies for real liquidity cost.

Also the liquidity concept as such has to be criticized. Berkowitz assumes linear, non-time-varying price impact, which is clearly not the case and most likely underestimates liquidity risk impact. The assumption of zero liquidity-return correlation in his risk estimates leads to further underestimation, because, empirically, positive correlations can be observed. Further, as will be discussed at the beginning of section 3.4.1, we doubt that an optimal trading strategy applied above is as such a suitable approach in crises situation. A correction is however simple, because traded volume does not have to be divided by the liquidation horizon.

Overall, Berkowitz (2000a,b) provides an approach to integrate price impact of order size into a risk framework, but liquidity measurement remains highly approximate.

3.2.2 Crises transactions regression model: Jarrow and Protter (2005)

Jarrow and Protter (2005) use a framework which is very similar to Berkowitz (2000a). Price impact is also measured in a regression from transaction data. How-

[31] Cp. relatively poor results of implementation in daily data by Ernst et al. (2009).

ever, they do not explicitly control for other risk factors and only take a sample of crises transactions to derive a crises price impact coefficient.

$$log\left(\frac{P_{TA,t+1}}{P_{TA,t}}\right) = \left(\mu_{r_t} - \frac{1}{2}\sigma^2_{r_t}\right) + \theta_c(N_{t+1} - N_t) + \epsilon_t \tag{8}$$

where μ_{r_t} and $\sigma^2_{r_t}$ are continuous mean and variance of the mid-price return, θ_c is the crises price impact coefficient and N_t is the number of shares traded at time t.[32] The restriction to crises introduces time-variation into the price impact which is neglected by Berkowitz. The additional, relative liquidity component in a VaR when selling a position immediately in crises can then be calculated as

$$VaR_L = (1 - \theta_c N) \tag{9}$$

where N is now the trader's quantity to be traded.[33]

The advantage of Jarrow and Protter (2005) is the integration of time-varying price impact. The crises specific coefficient also implicitly accounts - at least in approximation - for the liquidity-return correlation in crises. Similar to the Berkowitz critique, this type of empirical liquidity measure remains generally highly approximate. Running the regression in crises periods only might, however, severely shrink the sample, which further reduces the validity of the liquidity estimate θ. Therefore, their approach is overall of similar value than Berkowitz (2000b).

3.2.3 Volume-based price impact: Cosandey (2001)

Cosandey (2001) proposes a simple framework to estimate price impact from volume data. The price is a function of the number of shares traded, $P = Q/N$, where Q is the (constant) quantity of money traded and N is the number of shares traded. Under the assumption, that traded amount of money Q is independent of a single trade, price including the impact of trading ΔN shares can be simply estimated as

$$P_{mid,t}(\Delta N) = \frac{Q}{N + \Delta N} = P_{mid,t} \times \frac{N}{N + \Delta N} \tag{10}$$

where the number of traded shares N is assumed to be constant over time. The trade fully increases the number of shares traded in the market. The price impact is thus

[32] To keep notation consistent, we used the Greek letters from Berkowitz (2000b), which carry different meaning than the original Greeks in Jarrow and Protter (2005).

[33] To simplify, we neglect that in the original paper the position is only partially liquidated.

assumed to be linearly related to relative traded volume. Relative liquidity-adjusted total risk can then be calculated as

$$L - VaR(\Delta N) = perc\left(r_{t+1} \times \frac{N}{N + \Delta N}\right) \tag{11}$$

where *perc* determines the percentile from simulated distributions. The effect of mid-price change and order size is jointly modeled.

Cosandey (2001) already addresses his shortcoming of the linearity of the price impact function in (10) and proposes to model it as

$$P_{mid,t}(\Delta N) = P_{mid,t} \times \left(\frac{N}{N + \Delta N}\right)^{\frac{1}{a}} \tag{12}$$

where a is the - possibly time-varying - curvature parameter, but leaves its measurement to future research.

The approach of Cosandey offers a major improvement over Bangia et al. (1999), because the price impact of order size is accounted for. While the important determinant of order size is integrated, the integration of price impact remains simple and has very few data and computational requirements. Volume data are available for many markets and a large range of frequencies. However, not only single transaction data, as in Berkowitz (2000a) or Jarrow and Protter (2005) are required, but the overall market volume. The linear implementation is simple and straight forward.

At the same time, the linearity of the price impact in the standard specification is one main source of imprecision. Empirically, price impact is shown to be concave, which makes a linear functional form overestimate liquidity risk for large order sizes.[34] Curvature parameters in this functional specification are difficult to measure, which makes this problem hard to solve in this setup.

The second reason for imprecision is the assumption, that the amount of trading in the market, N, does not vary over time. This is equivalent to assuming zero volume elasticity. The dynamics of trading volume in crises might significantly alter the picture. The much cited 'flight-to-liquidity' effect can introduce complicated mechanics, because liquid assets improve in liquidity while illiquid assets deteriorate.[35] If this is consistently the case, the liquidity risk of more illiquid positions will be underestimated, which should be a major concern. As a conservative solution, trading volume can be assumed to dry up in crises, e.g. by assuming that trading volume falls to the lowest percentile of the volume distribution. But if this sugges-

[34]Cp. Stange and Kaserer; Stange and Kaserer.
[35]Cp. Longstaff (2004).

tion more precisely captures liquidity effects in reality is unclear. Overall, neglect of time variation is a problem difficult to solve.

Further, liquidity is assumed constant between stocks apart from differences in trading volume. However, Stange and Kaserer (2008a) show, that liquidity cost also greatly vary with market capitalization. Integration of this fact might possibly capture flight-to-liquidity effects but requires further research.

In summary, Cosandey offers a framework, which can integrate price impact in a simple way, especially in markets where data availability is limited.

3.2.4 Structurally implied spread: Angelidis and Benos (2006)

Angelidis and Benos (2005, 2006) develop an implied liquidity cost model from structural considerations, i.e. liquidity is traced to its underlying drivers. They combine an inventory model of a market maker with a fundamental model of information asymmetry. This yields an implied spread, where the impact of traded volume depends on the degree of information asymmetry and the price elasticity with respect to volume and a volume-independent minimum cost component.

$$L = \sqrt{N_t}(\theta + \kappa) + \Phi \tag{13}$$

where N_t is the absolute number of total shares traded, θ is the degree of information asymmetry, κ is price elasticity with respect to volume and Φ is the size-independent cost per share. The Greek letters are estimated from intraday data with a Generalized Method of Moments (GMM).

This liquidity measure is then integrated into relative VaR as add-on similar to the quoted spread in Bangia et al. (1999).

$$L - VaR = VaR + \left[(\theta + \kappa)\sqrt{N_t^{\alpha'}} + \Phi\right] \tag{14}$$

where VaR is mid-price risk and $N_t^{\alpha'}$ is the top α' percentile of traded volume.

Angelidis and Benos assume, that the individual position size of a trader dissipates in the volume of the market and does not increase total traded volume as long as the position size is smaller than traded volume. This is the opposite extreme to Cosandey (2001), who assumed, that the trader's volume fully increases traded volume. Angelidis and Benos take a less conservative approach. On the other hand, the assumption that liquidity cost is calculated for the top percentile of traded volume, probably captures the volume increase in the case of liquidation implicitly.

Angelidis and Benos (2006) provide a new approach of liquidity modeling by tracing liquidity cost to its underlying determinants. This allows to estimate liquidity even in markets, where other liquidity cost estimations are not available. However, their approach requires intraday data and heavy computations to get estimates for the structural coefficients.

For practical purposes the main question is, if the structural model is correct. If main liquidity effects are not captured, liquidity estimates will be strongly biased. We would hypothesize for example, that volume elasticity strongly varies over time, which is not captured. This might substantially influence results if these effects are of large magnitude. Also, the degree of information asymmetry can be expected to change over longer periods. Therefore, this model is probably most useful when calculating intraday risk.

The second critique addresses the mechanics of integrating liquidity into the VaR-approach. As discussed above, adding liquidity risk to price risk assumes perfect price-liquidity correlation, which might overestimate risk. Since the dynamics of volume are not fully researched yet, we do not know if the assumption of increased volume in crises is really valid and if we can then safely assume, that the trader's position disappears in the generally increased market volume without additional impact.

Overall, Angelidis and Benos (2006) provide an interesting intraday model of liquidity risk, but relies on a large amount of intraday data as well as some strong structural assumptions. Testing the validity of the structural approach or empirically verifying the real dynamics of traded volume in crises could take this line of research to the next level.

3.3 Models based on limit order book data

3.3.1 Price impact from limit orders: Francois-Heude and Van Wynendaele (2001)

Francois-Heude and Van Wynendaele (2001) estimate price impact of order size by using more information from the limit order book. They suggest to estimate the price impact for a certain order size by interpolating the price impact function from the best five limit order quotes made available by the Paris Stock Exchange. This estimation of the spread $S(q)$ for a specific positions size q makes their approach quite precise, at least for smaller order sizes.

Relative liquidity-adjusted total risk is then calculated in the following intraday model

$$L - VaR(q) = 1 - exp(-z_\alpha \sigma_r)\left(1 - \frac{\bar{S}(q)}{2}\right) + \frac{1}{2}\left(S(q) - \bar{S}(q)\right) \tag{15}$$

where z_α is the normal percentile and σ_r the standard deviation of the mid-price return distribution. $\bar{S}_t(q)$ is the average spread in the market for order quantity q and $S_t(q)$ is the spread of the asset. Market spreads are subtracted from worst mid-prices. However, as market average spread and individual asset spread might differ, the second term tries to correct for this difference.

Because it seems logically inconsistent to us that the correction term is multiplied with current and not with worst mid-prices, we suggest to modify the risk term into

$$L - VaR(q) = P_{mid,t} \times \left[1 - exp(-\alpha\sigma_r)\left(1 - \frac{S_t(q)}{2}\right)\right] \tag{16}$$

which is simpler, more consistent and does not require average market spread data.

Still, time variation of liquidity is not accounted for in the Francois-Heude and Van Wynendaele (2001)-model, but could be similarly implemented as in Bangia et al. (1999) using mean and variance of the spread distribution. This would, however, require the estimation of liquidity cost distributions for all order sizes.

This approach generally requires intraday data to estimate the price impact function, which restricts its application to risk estimation at intraday frequencies. Also, the type of data described above needs to be available. A suitable degree of precision is restricted to order sizes that are not too large, because extrapolation much beyond the fifth limit order quote is approximate.

Overall, it is difficult to judge whether the increased preciseness through integration of price impact or the lacking time-variation dominate in a specific situation. If the approach of Francois-Heude and Van Wynendaele (2001) is used, we would suggest to integrate time-variation in a suitable way.

3.3.2 Price impact from weighted spread: Giot and Gramming (2005)

In order to address price impact, Giot and Grammig (2005) extend the idea of Bangia et al. (1999) by using spread data beyond the spread depth. They assume, that the position is immediately liquidated as market order against limit orders in the limit order book. Liquidity costs can then be calculated as the average weighted spread of those limit orders necessary to liquidate a certain position size. In this way, the liquidity costs of different order sizes can be extracted from the limit order book.

In detail, price impact is calculated as

$$WS_t(q) = \frac{a_t(n) - b_t(n)}{P_{mid,t}} \tag{17}$$

where WS is weighted spread in percent and q is the size of the position in mid-price value. $a_t(n)$ is the weighted ask price of trading n shares calculated as $a_t(n) = \sum_i a_{i,t}n_{i,t}/n$ with $a_{i,t}$ being the ask-price and $n_{i,t}$ being the ask-volume of individual limit orders. Individual limit orders add-up to the size of the position, i.e. $\sum_i n_i = n = q/P_{mid}$. $a_t(n)$ is defined analogously.

The liquidity measure defined above can be used to calculate the net return, return net of liquidity cost at time t over horizon h as

$$rnet_t(h,q) = r_t(h) \times (1 - \frac{WS_t(q)}{2}) \tag{18}$$

where $r_t(h)$ is the h-period mid-price return at time t. Net return including price impact is then integrated in a parametric, intraday VaR-framework. Relative liquidity-adjusted total risk over horizon h is estimated by using tails of the student distribution as

$$L - VaR(h,q) = 1 - exp\left(\mu_{rnet(h,q)} + z_{t,\alpha}\sigma_{rnet(h,q)}\right) \tag{19}$$

where $\mu_{rnet,t}$ is the mean and $\sigma_{rnet,t}$ is the variance of net returns, while allowing for diurnal variation of spreads and time-varying clustering of return volatility by modeling conditional heteroskedasticity.[36] $z_{t,\alpha}$ is the α-percent percentile of the student distribution.

The main advantage of using weighted spreads is the precise modeling of the price impact of positions size. As discussed in Stange and Kaserer (2008b), weighted spread is a precise liquidity measure in a large range of situations, despite the assumption of immediate liquidation. It is accurate in markets, where asset positions are generally continuously traded.

Time variation and non-normality is accounted for by using the parametric specification. While it is possible, that the assumption of the t-distribution is a source of imprecision, this would need empirical testing. A further advantage is the modeling of net-return instead of separating mid-price return and liquidity cost, because the correlation between return and liquidity cost does not have to be explicitly modeled.

[36] For details please refer to the original paper.

Total risk is measured when the combination of mid-price return and liquidity cost are lowest.

Unfortunately, this method requires a transparent limit order book market such as the London Stock Exchange, the NASDAQ, the Deutsche Börse Xetra or the Euronext. If weighted spread data have to be manually calculated from the full intraday limit order book, the method is highly computationally intensive due to the large amount of data. However, some exchanges, like the German Xetra, provide weighted spread data, which can be integrated into a risk framework with limited computational requirements.[37]

Overall, the weighted spread approach allows for highly precise integration of liquidity risk including price impact of order size - if limit order book data is available.

3.3.3 Alternative weighted spread models

Stange and Kaserer (2008b) employ empirical percentiles instead of the t-distribution approach and define relative, liquidity-adjusted total risk as

$$L - VaR(q) = 1 - exp\left(\mu_{rnet(q)} + \hat{z}_\alpha(q) \times \sigma_{rnet(q)}\right) \tag{20}$$

where $\hat{z}_\alpha$ denotes the empirical percentile of the net return distribution. This accounts for non-normality in the net return distribution in a less restrictive way than the t-distribution approach of Giot and Grammig (2005).

An analogous application of the Cornish Fisher approximation according to Ernst et al. (2008) is also possible. Risk is then defined as

$$L - VaR(q) = 1 - exp\left(\mu_{rnet(q)} + \tilde{z}_\alpha(q) \times \sigma_{rnet(q)}\right) \tag{21}$$

where $\tilde{z}_\alpha$ is the percentile estimated with the Cornish-Fisher approximation (4). This approach more precisely accounts for non-normality than the t-distribution approach, but remains still parametric.

3.4 Theoretical models

3.4.1 Models based on optimal trading strategies

General remarks In addition to the models analyzed so far, a different class of models has been suggested by academia in the context of liquidity risk measurement. As discussed in section 2.1, optimal trading strategies try to find an optimal balance

[37] Available as Xetra Liquidity Measure (XLM).

between price impact costs and delay cost by delaying parts of a transaction. They are very helpful in determining a valid liquidity cost estimate when liquidating a large stock position in normal situations.

We only provide a short overview, because we believe that in risk management the usefulness of these strategies is limited for three reasons.[38] First, we doubt that optimal trading strategies are suitable approach from a risk perspective in general. They assume, that there is enough time to delay portions of a trade, which is rather unrealistic in a crises situation. Calls on margin accounts and strong expected momentum enforce a fast liquidation, leaving little room for patient optimal delay. If we assume a 10-day forecast horizon and a crises occurs on day one, does a trader really wait the nine remaining days to liquidate the position? Second, even if there is enough time, optimization parameters must be stable enough to yield an optimized result. Otherwise, it might be that the optimized trading strategy yields worse results than by trading as quick as possible. This is especially the case, if a position is to be liquidated due to informational advantage with respect to the further development of a crises.[39] Third, optimal trading strategies are usually based on a large amount of parameters that are difficult or impossible to estimate in practice. The more aspects are mathematically integrated, the more difficult and possibly unstable is the implementation. All of the model suggestions have yet failed to demonstrate that they can be empirically applied in real crises data.[40] To prove the validity of optimal trading strategies, empirical estimation procedures need to be developed and it needs to be shown, that the analytical optimal strategies are stable in crises situations. We believe that optimal trading strategies have their greatest validity when trying to liquidate block holdings in normal market situations, but have limited applicability in risk management.

Nevertheless, for sake of completeness, we provide a brief overview. Papers with optimal trading strategies usually assume some form of price impact function and a particular structure of the temporal dynamics. We will highlight those two main characteristics for each model to clarify the differences.

Model overview Lawrence and Robinson (1995) include liquidation costs, delay costs, which are measured as risk exposure during liquidation, and hedging costs into a net sales value. Risk is then measured as the maximum net sales price when setting the liquidation horizon in an optimal way. Unfortunately, the problem of

[38] More detailed discussions of these theoretical models can be found in Erzegovesi (2002), Loebnitz (2006) and Jorion (2007).
[39] This translates into high permanent vs. temporary price impact.
[40] Cp. also critique in Bangia et al. (1999), p. 69.

liquidity cost measurement is left to be specified by the reader. It seems, that liquidity costs are measured as constant bid-ask-spread only, i.e. price impact and time variations are neglected. The general critique on optimal trading strategies applies as discussed above. In addition, it can be also doubted that maximizing expected proceeds and neglecting potential shortfall due to proceed variance is a suitable way from a risk perspective. Also, using an unbounded liquidation horizon is a questionable procedure in crises. Therefore, their approach can only serve as a very general framework for analyzing the problem.

Jarrow and Subramanian (1997)/Subramanian and Jarrow (2001) include liquidity cost and execution delay in an optimized framework maximizing liquidation proceeds within a given horizon. They assume that liquidity costs are non-decreasing with order size and that trading has economies of scale, i.e. that liquidating the full position at once is always optimal. Liquidity-price correlation is assumed to be zero. The trader is treated as risk neutral. Under these assumptions, an analytically optimal solution is derived. Unfortunately the framework must place heavy restrictions on reality to find an analytical solution. If the liquidation strategy is optimal in real data remains to be seen. The critique on optimal trading strategies in general and on the neglect of proceed variance analogously applies. How the parameters used in the optimization are to be empirically estimated will have to be developed.

Almgren and Chriss (2000) construct an optimal trading strategy within a given liquidation horizon. They decompose liquidation cost into a temporary and a permanent component and construct a liquidity-adjusted VaR by minimizing VaR itself. This approach is extended in Almgren (2003) by including non-linearity in the price impact. However, the question of measuring these parameters remains unsolved in both papers. This especially concerns the magnitude and functional form of permanent and temporary price impact as well as the duration of the temporary price impact. If time-variation of liquidity is incorporated, distributional estimations are also necessary.[41] Concerns with respect to the validity of optimal trading strategies in crises as such apply.

Hisata and Yamai (2000) also construct an optimal trading strategy by minimizing the cost of liquidation, also including normally-distributed permanent and temporary price impact. They determine the optimal holding period at constant sales speed by maximizing expected sales proceeds with a penalty for proceed variance. Liquidity risk then is the price impact variance under the condition, that the sales strategy is optimized. Several variations as well as portfolio considerations are

[41] Almgren et al. (2005) present a calibration procedure based on internal trade data. This is, however, less helpful when trades are sparse for certain assets in general or the specific institution.

discussed. Unfortunately, the paper also fails to specify how to empirically estimate the parameters used in the framework.[42] Several assumptions, that are required to technically find an analytical solution, might not be robust in reality. Also using an unbounded liquidation horizon is questionable as discussed above.

Dubil (2003) analyzes the optimal execution strategy between delaying parts of a position and the price impact. Liquidation costs are also decomposed into a permanent and a temporary component. He optimizes the liquidation horizon by maximizing the total VaR of the transaction when assuming a constant liquidation speed, i.e. when price impact is linear. Above critique on optimization strategies, unbounded horizon optimization in particular as well as empirical parameter estimation applies.

Engle and Ferstenberg (2007) optimize the sales trajectory within a given horizon to maximize expected proceeds with a penalty for proceed variance. Similar to Almgren and Chriss (2000), they assume that permanent and temporary price impact can be measured and solve this theoretical problem, but fail to address how these parameters can be estimated.

This line of research will proceed quickest to practical implication, if two questions are addressed. It needs to demonstrate the empirical estimation technique for the multitude of parameters and prove if or under which circumstances optimal trading strategies yield superior results in crises situations compared with instant liquidation. In the end, integration of many aspects might not be the best way because implementation and result stability are relevant aspects as well.

3.5 Synopsis

Liquidity risk measurement has to take two problematic steps: Measurement of liquidity and integration of the measure into a risk framework. The measurement technique is closely connected to the data available. The preciseness should increase the more information is used in determining the price impact curve. The correct risk integration technique is generally a balance between simplicity and applying suitable, non-distorting assumption. In table 1, we summarized the traceable models based on these criteria. While this provides a theoretical indication, which models should be most suitable, the ultimate test must be empirical. An empirical analysis of their precision will provide further impetus on which models to apply in practice.

[42] The numerical illustration takes important parameters such as temporary price impact recovery and permanent price impact coefficient as given or sets them to zero.

Model	Short description	Relevant assumptions	Strengths	Weaknesses
Bangia et al. (1999)	• L. measured with bid-ask-spread • Parametric worst liquidity cost added to price risk • Non-normality accounted for through empirical percentiles	• Position size without significant influence • Liquidity cost and price perfectly correlated	• Only spread data required • Simple add-on to existing risk measures	• Underestimation for larger order sizes • Overestimation because correlation less than perfect and spread calculated based on current price • Regime switching (multi-modality) of spread is neglected
Ernst, Stange, Kaserer (2009)	• Improvements on Bangia et al. (1999) • Risk modeled based on net return • Non-normality accounted for by Cornish-Fisher approximation	• Position size without significant influence	• Only spread data required	• Underestimation for larger order sizes • Regime switching (multi-modality) of spread is neglected
Cosandey (2001)	• L. measured as position size relative to traded shares • L. adjustment from worst market price	• No time variation of l. • Price impact linear to relative traded shares • Further l. differences neglected	• Market volume data required and available at all frequencies • Accounts for price impact of order size	• Price impact approximation • Underestimation because deterioration of liquidity in crises neglected
Berkowitz (2000) Jarrow and Protter (2005)	• L. measured from transaction prices, Berkowitz controlling for other risk factor changes	• Cost precisely extractable from transaction data • Linear price impact • Time variation and correlation issues solved by Jarrow, Protter (2005)	• Accounts for price impact of order size	• Intraday data on single transaction prices and volumes required • Price impact only very approximate
Francois-Heude and Van Wynendaele (2001)	• Quantity adjusted L. measured from best five limit-orders • Current market avg. liquidity costs added to price risk with ad-hoc add-on of difference between market and individual liquidity	• No time variation of l. • Liquidity cost and price perfectly correlated	• Accounts for price impact of order size • More precise price impact measurement than when extracted from transaction prices	• Intraday data of best limit orders required • Price impact approximation most precise for small sizes only • Underestimation because deterioration of liquidity in crises neglected • Somewhat arbitrary spread-adjustment leads to overestimation
Angelidis and Benos (2006)	• L. measured by estimating a model of structural liquidity determinants • Worst liquidity cost added to price risk	• Specific structural model • Volume increase in crises • Liquidity cost and price perfectly correlated	• Partially accounts for price impact of order size if larger than worst market volume	• Intraday data required • Assumptions empirically not verified • Overestimation because correlation less than perfect • Complex, time-consuming estimation of parameters
Giot and Gramming (2005) Stange and Kaserer (2008)	• L. measured by weighted spread in limit order book for a specific position size • Risk modeled based on net return	• Worst case perspective (because immediate liquidation) OR Efficient market for liquidity	• Accounts for price impact of order size • Precise liquidity measure • Correlation correctly accounted for	• Only applicable in limit order book markets • If weighted spread data not provided by exchange, intraday data of full limit order book required • Possible overestimation if instant liquidation highly suboptimal

Table 1: Overview of traceable models integrating liquidity risk

4 Conclusion and outlook

4.1 Summary

In this paper we provided an overview on the current status quo of research on market liquidity from a risk perspective. We defined liquidity from a cost perspective as the cost of liquidation. The main components of liquidity are direct trading costs, price impact of order size and delay costs. We argued that it provides a useful framework which integrates other existing liquidity definitions.

Liquidity can have different degrees determined by the type of the asset, the size of the position and the liquidation horizon. If an asset is continuously traded, the precise determination of price impact function is the main issue. If the asset displays trading interruptions, delay costs become an additional problem. If an asset is not traded, value has to be determined by intrinsic methods.

We also provided a survey on existing models to integrate market liquidity risk into a risk framework, which we structured by the type of data available. Models exist when only bid-ask-spread data are available, when transaction data or when limit order book data are accessible. We also uncovered the relevant assumptions implicit in the modeling approaches, their implications and proposed several alternative specifications. In the theoretical part, less traceable approaches were discussed.

Overall, some of the most important problems have not been addressed yet. In recent years, scientific research has mainly worked on developing optimal trading strategies and integrating them into risk frameworks. However, their empirical implementation and their effectiveness have yet to be proven. Delay risk, a major factor in many markets, has been rather neglected.[43] An integrated model of price, price impact and delay risk would be an important improvement for the practical measurement of liquidity risk.

4.2 Management of market liquidity risk

While we have summarized which liquidity aspects are important and how liquidity risk can be measured, we have neglected the important questions of when these methods should be applied and how liquidity can actually be managed.

As argued before, the liquidity of an asset depends on the liquidation horizon. Therefore, an assert held to maturity, where profits come from cash-inflows such as dividends or interest coupons, carries no liquidity risk.[44] Therefore, the main question is, which assets would possibly have to be liquidated and how fast?

[43] We sketch an approach in section (5.1), which still has to be refined and empirically tested.
[44] Cp. Berkowitz (2000a), p. 105.

Some assets are intended to be liquidated. The trading book is the typical example, which will most certainly be liquidated. But also non-strategic stock market investments of banks, insurances or investment funds will be liquidated at some point in time.

From a risk perspective, funding risk is the major determinant of the probability of liquidation. If a financial institution has unexpected cash outflows or cash requirements, it will be forced to liquidate some of its assets. A bank facing margin calls on its trading book or generally increased risk will be forced to close positions in order to bring total risk in line with available regulatory capital. A financial institutions having cash shortages, because (short-term) refinancing is not available will also be required to convert some of the asset base into cash. Mutual or hedge funds might have to pay out investors. All these examples demonstrate sources of funding risk, which have immediate consequences on the probability which assets have to be liquidated. Therefore, a forecast of funding requirements and risks is the first major step in solvency risk management.

From an internal point of view, it is prudent and recommended for management to know the liquidation value in crises situations for the whole asset base, i.e. the full risk including liquidity risks. Knowing the liquidation value has two components, marking the asset to market (mid-prices) in crises and also incorporating the liquidity cost of the asset. This allows for a full picture of the liquidity situation of the firm and allows to actively manage and control liquidity risk. This can have important consequences for internal pricing, performance measurement or the evaluation of new products or investment and trading strategies. Transparency should be as full as possible, not only for the trading book.

From an accounting point of view, those assets should be marked to market, that are expected to be liquidated including liquidity costs. Risk is not incorporated. For assets held to maturity value can also be determined by internal methods if market prices are lacking or inefficient.

The regulatory perspective takes a balance between these two extremes, the internal and the accounting view. A "necessary" amount of assets should be valued at crises liquidation prices. Positions might be included which would have to be sold in crises situations only.

An important aspect should be kept in mind when regulating liquidity risk. If regulatory capital requirements are strictly tied to worst possible market sales prices, a feedback mechanism might create a downward spiral. Depressed market prices decrease available regulatory capital. Increased volatility and liquidity costs and therefore higher risk increase the need for additional regulatory capital. Trading strategies face margin calls when is especially difficult to get additional capital. Crises news lead to fund outflows. Cramped from all sides, financial institutions are forced to sell assets. If this happens for the financial system as a whole, the concerted sale will further depress prices, which will start the vicious cycle. A downward momentum might destabilize markets.

One main cause for this downward spiral is the continuous adjustment of risk measures and regulatory capital to current market conditions. It also seems logically inconsistent that regulatory capital is supposed to cover worst losses but cannot be consumed if worst losses really occur. A possible solution could be smoothed regulatory capital, more prudent in normal times and more lax in times of market turmoil. This would leave room to take losses without increasing bankruptcy risks. It can also be argued, that this procedure does not even decrease shareholder value as it increases cost of capital but also decreases earnings volatility. However, this regulatory capital smoothing requires more discussion, before it can be used as regulatory mechanism.

4.3 Questions for future research

Market liquidity risk still provides a large realm of topics that require future research. We believe, that answers to the following questions would be especially interesting. A better understanding of certain aspects of market liquidity would be helpful and liquidity risk management also shows some loose ends.

First, although we hypothesized that optimal trading strategies do not possibly provide benefits from a risk perspective, they are certainly valid in normal market conditions and for block sales. The pressing question is how to estimate the parameters required for the optimal trading algorithms. What is then the empirical benefit of different optimal trading strategies? In which situations are they (most) valid?

This issue can get tackled from a different perspective as well: When are liquidity prices efficient? If they are, then any optimal trading strategy will have to fail. It also only makes sense to add liquidity cost risk to price risk if price not yet suitably reflects liquidity. If mid-prices already reflect overall liquidity, must any further adjustment be restricted to the individual trader's situation, must common liquidity effects be neglected?

Second, asset pricing questions based on more precisely estimated price impact curves would clarify the importance of liquidity costs to investors. Combining the weighted spread measure of the price impact curve with the distribution of trading volume yields the total cost paid by investors per stock. Is this total cost reflected in prices? It might also be possible to describe the whole price impact curve with theoretical, calibrated liquidity processes - similar to theoretical descriptions of the interest rate curve. This might help in situations, where the price impact curve is non-observable or where forecasting is very difficult.

Third, the most important issue for liquidity risk measurement is, in our view, the under-researched treatment of delay risk. The dynamics of delay (in crises) and its relation to the price dynamics is still unclear. When and for which assets does trading break down in crises? Further insight into empirical delay properties would help to choose an appropriate approach to integrate delay risk into liquidity risk measurement. This research topic would also have to tackle the question of how to measure and forecast delay, especially in markets where delay is important and market data is quite perforated. A subsequent empirical comparison of methods and magnitudes of liquidity risk in different asset classes would be interesting.

Fourth, the specification of size has been handled differently by different authors. When analyzing liquidity cost and risk, which specification is most suitable? Size can be defined as number of shares, volume in value or in volume relative to the traded volume in the market. From the theoretical as well as the empirical perspective an analysis could be fruitful, which determines liquidity in a more precise and stable way.

Fifth, the literature on market liquidity has been enriched by approaches that have not yet been used in liquidity risk management. Chacko et al. (2008) calculate liquidity cost in an option pricing framework, which is possible because liquidity can be interpreted as marketability option as discussed in section (2.1). Because it is implementable based on transaction data, it provides a traceable approach that is theoretically rigorous at the same time. It might be an interesting venue to explore from a liquidity risk perspective.

Sixth, liquidity risk management could still need some refinement. Duffie and Ziegler (2003) describe optimal liquidation strategies of portfolios in crises. We believe, that liquidity risk treatment of portfolios still has neglected potential for further insight. It might also be interesting to understand if it is possible to construct specific liquidity options, that could be used to hedge away the liquidity cost risk. Not long ago, volatility options became a traded contract in financial markets. Is there similar potential for liquidity options?

While it is possible, that some of above questions have been answered, which we are not aware of, and some lead to dead ends, we hope part of those questions help to spur further research and lead to a better understanding of the reoccurring and important topic of market liquidity risk.

5 Appendix

5.1 A model proposition with forced delay risk for seldomly traded assets

Models usually neglect the additional risk of delay in assets characterized by interrupted trading. We propose a simple incorporation of delay in liquidity risk measures. It is based on the idea that the expected delay period $E(d)$ prolongs the liquidation horizon T.

$$T = h + E(d) \tag{22}$$

where h is the initially chosen liquidation horizon and $E(d)$ is the expected delay during crises. We thereby assume, that the liquidation period T concerns the decision period and not the period of liquidation as such (cp. discussion in section (2.3)). Thereby, delay increases the total liquidation period also if it is smaller than the period required for liquidation. If the original horizon h comprises decision and liquidation period, delay increases the total horizon only if it is larger, which can be modeled with a simple maximum rule.[45]

The major problem is the exact specification of the delay process conditional on the price development. Using expected delay is valid, if we assume d to be independent of the return process, i.e. zero price-delay correlation. More complicated dynamics could be integrated, but if delay worsens or improves in market turmoil is yet unclear. If there is a flight-to-liquidity effect and liquid asset positions get more tradable in crises while less liquid asset positions get less tradable, this asymmetry needs to be accounted for. For now, we take expected delay.

We suggest to empirically estimate expected delay as average delay length in the past estimation period.

$$E(d) = \frac{number\,of\,zero\,trading\,days}{number\,of\,delays} \tag{23}$$

As further simplification, we assume full non-tradability of the stock and do not account for delay effects by order size. Because the market of a traded stock has a limited depth, any position larger than this depth cannot be traded as such. It can be partially transacted, the remaining portion has to be delayed. If the trade clears the market by consuming the full market depth, this might significantly alter prices for a longer period of time. We leave this complication to future research.

[45] $T = max(h, E(d))$.

No-trade periods have already received some attention in the liquidity literature, where it has been shown, that zero trading days are a risk factor that prices assets.[46] In the following framework, we provide the economic rational for this fact.

We define net return at time t over the period T as the T-period mid-price return net of liquidity costs

$$rnet_t(T, q) = r_t(T) \times (1 - L(q)) \tag{24}$$

where r is the mid-price return and $L(q)$ is the liquidity cost of transacting a position of quantity q.

Relative liquidity-adjusted total risk in a VaR-framework can then be defined as the worst loss over the horizon T with confidence α. We employ a parametric specification based on the assumption that net returns are student-distributed as follows.

$$L - VaR(T, q) = \mu_{rnet(T,q)} + t_\alpha \times \sigma_{rnet(T,q)} \tag{25}$$

where μ_{rnet} is the mean and σ_{rnet} is the variance of net return, t_α is the α-percent percentile of the student distribution.[47] This specification describes the liquidation value of an asset including delay by non-trade periods. Delay amplifies total risk by increasing mean and volatility, because it increases the uncertainty of the liquidation value.[48]

The VaR for a given horizon h can also be calculated by discounting above VaR with a suitable crises, short-term interest rate i over the delay period.

$$L - VaR(h, q) = \frac{L - VaR(T, q)}{(1 + i)^{E(d)}} \tag{26}$$

The discounting might be negligible if the delay is short and the crises interest rate is low. However, we basically see the VaR-position of the asset at time T as collateral for the credit. Risk-adjusted discounts can get quite large, even when already accounting for future price drops. This assumes that short-term financing is still possible. Otherwise, the liquidation value at any time $h < T$ will be zero. If an asset cannot be sold or collateralized it is not worth anything.

Delay induces a measurement problem of return mean and variance, which has to be accounted for. Daily data returns of assets with delay occurrences must be falsely more fat-tailed if all returns are sampled, because 1-day returns are mixed with d-day returns, with the later being in the order of d-times larger. To get unbiased

[46] Cp. for example Liu (2004) and others.

[47] We suggest to take the number of estimation period observations less one (for the mean calculation) as degrees of freedom.

[48] This is similar to the volatility scaling in Jarrow and Subramanian (1997); Subramanian and Jarrow (2001).

moment estimates for mean and variance, either returns with delay are deleted from the sample, the sampling frequency is set as to get equal period returns or delay returns are appropriately scaled with the delay period. The latter methodology eliminates the least observations.

The advantage of our approach is its simplicity and empirical traceability. However, the simplification has its natural drawbacks. If we assume that the flight-to-liquidity effect holds, assuming constant delay overestimates risk from delay for liquid and underestimates it for illiquid assets.

Also, the general delay perspective has its difficulties. Price risk could be hedged away until the position is sold. Unfortunately, derivatives are often not available in illiquid markets. Therefore, traders apply a proxy hedge technique to cancel the price risk while liquidating the position. When, for example, holding a currently unsellable position, they short a highly-correlated, but more liquid asset. This cancels price risk to a certain degree, but multiplies liquidation costs (for entering and selling the hedge) and leaves the trader with hedging risk. This technique replaces the delay risk problem with a hedging problem. In contrast to optimal delay strategies (see section (2.1) and (3.4.1)), it can be quickly implemented. But what is a suitable liquid proxy asset? What is the optimal balance between a liquidity and correlation of the proxy hedge? The solution to this question will be just another optimal trading strategy, albeit one that has not been on the scientific radar, and one that still has to prove its benefit. The issue of hedging during delay is valid when looking at single asset liquidity. On the portfolio level this complication can be neglected, because the hedge itself is integrated as part of the trader's portfolio.

The discussion provides only a brief sketch of a possible treatment of delay risk, which neglects any optimization strategies. It estimates an upper bound on delay risk where hedging or optimization are not possible. Although we described a possible empirical specification, we leave the empirical test to future research.

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