Chapter Five

The Pricing of Credit Derivatives

Patience is a minor form of despair, disguised as a virtue. (Ambrose Bierce)

The pricing of most credit derivatives is more complex than that of equity, commodity, interest rate, or foreign exchange derivatives. One reason for the higher complexity is that the market price for the underlying variable (i.e. the bond or loan) is often not easily observable. This is especially true for loans, which typically do not trade in a secondary market. Even if substantial research is conducted, measuring the credit quality of a debtor can be difficult, since credit quality criteria such as quality of the management or intangible assets are difficult to quantify.

However, if the underlying company is rated by an agency, traders can use the rating of the agency as a proxy for the value of the debt. Nevertheless, this might be problematic, since different rating agencies sometimes derive different ratings for the same debt. In addition, published ratings are often outdated, since agencies are not able to analyze the underlying debt on a timely basis.

Pricing credit derivatives is also problematic because defaults are rare events. Especially, since a company typically only defaults once, empirical data on the default of a solvent company is typically unavailable. To overcome this obstacle, it is often assumed that companies in the same credit category and sector display similar default dynamics and properties.

In addition, there are numerous causes for default. There can be internal causes such as mismanagement, incompetence, or fraud; and external causes such as a recession or stiff competition. Typically default is caused by a combination of factors, whose correlation has to be integrated into the pricing model.

Furthermore, with credit derivatives, the counterparty risk is an important pricing element, since the default of the underlying debt typically leads to a large settlement payment of the protection selling counterparty. Ideally, the correlation between the default risk of the counterparty and the default risk of the underlying debt should be considered in the pricing process. Furthermore, the correlation between credit risk, market risk, and operational risk should be recognized when pricing credit derivatives. All this makes pricing credit derivatives not an easy venture.

Credit Derivatives Pricing Approaches

Various approaches to price credit derivatives exist. They can be categorized as (a) traditional models or (b) structural models – which are comprised of (b1) firm value models and (b2) first-time passage models, and (c) reduced form models.

Traditional models value credit risk based on historical data. A risky bond price is derived by observing default rates of past losses or downgrades of bonds with comparable credit rating and seniority. Beside the obvious constraint of projecting historical data into the future, these data-fitting approaches often abstract from the economic situation. This is problematic since default and recovery rates are dependent on the business cycle (i.e. whether the economy is in a recession or boom).

In this context, Altman (1989) in an often-cited article, found that investors appear to be highly risk-averse: When taking into account past default rates, the return of an investment in risky bonds was significantly higher than the return on Treasury bonds.[1] This implies that investors are not risk-neutral but require a high-risk premium when investing in risky bonds. Part of the low risky bond price can be explained by the lower liquidity of risky bonds and by the anticipation of a recession, which would increase downgrade and default probabilities.

Structural models derive the value of credit risk by analyzing the capital structure of the company. Robert Merton in 1974 in a seminal paper laid the groundwork for structural models. The Merton model is mathematically identical with the original Black-Scholes model,[2] however the variables are redefined.[3] The basic concept of the Black-Scholes-Merton model underlies structural credit risk models as well as *reduced form credit risk models*. Before we discuss the Black-Scholes-Merton as well as structural and reduced form models in detail, let's observe some simple approaches to generating the probability of default.

Simple approaches

Let's first have a closer look at some basic pricing features of credit derivatives. Table 5.1 shows the input variables for deriving the price of a credit derivative.

Integrating all of the issues in table 5.1 into a single pricing model is not a trivial task. Currently no pricing model is accepted as a benchmark model as, for example, the Black-Scholes model for standard options. In the following, let's look at the most popular credit derivative, a default swap and how the default swap premium is usually derived in trading practice.

The default swap premium derived from asset swaps

As already discussed in chapter 2, asset swaps and default swaps are quite similar instruments (compare figure 2.7). The asset swap spread reflects the credit quality difference between the underlying asset and a risk-free Libor flat asset. Equally, the default swap premium reflects the credit quality difference between the risky asset and a risk-free asset, expressed as a difference in the yields (compare equation (2.1a) and (2.1b) and figure 2.4).

Table 5.1: Inputs for deriving a credit derivatives price

	Input variables for deriving the price of a credit derivative
1)	Default probability and credit deterioration probability of the reference asset
2)	Default probability and credit deterioration probability of the credit derivatives seller
3)	Correlation between 1) and 2)
4)	Volatility of the underlying reference asset
5)	Volatility of the credit derivatives seller
6)	Correlation between 4) and 5)
7)	Maturity of the credit derivative
8)	Expected recovery rate of the reference asset
9)	Expected recovery rate of the credit derivatives seller
10)	Return of the reference asset (e.g. coupon of the reference bond)
11)	Risk-free interest rate term structure used to discount future cash flows
12)	Default probability of the credit derivatives buyer in case of periodic credit derivative premium[4]
13)	Expected recovery rate of the credit derivatives buyer in case of periodic credit derivative premium
14)	Correlation between the default probability of the credit derivatives buyer and the reference asset in case of periodic credit derivatives premium
15)	Market risks (as interest rate risk, currency risk, commodity risk, and stock price risk) and the correlation between market risk and credit risk
16)	Operational risks (e.g. legal risks, documentation risks, or settlement risks), which might endanger the enforceability of the payoff and the correlation between operational risk and credit risk
17)	Liquidity of the credit derivative
18)	Liquidity of the underlying reference asset
19)	BIS risk weight of the credit derivatives seller
20)	Urgency of protection (e.g. is an immediate credit deterioration expected or does the protection free up credit lines to enable further business with a client)
21)	Transaction costs

As a consequence, default swap traders often use the asset swap spread as a benchmark for deriving the default swap premium. The equality of the asset swap spread and the default swap premium can be shown with an arbitrage argument similar to the one in example 4.18.

In figure 5.1, 'x' represents the asset swap spread. Let's assume identical currency, maturity, and notional amounts for the funding, the investment in the A-rated asset, and the default swap. From figure 5.1 we can see that the no-arbitrage condition for the investor, assuming he finances at Libor flat, is $d = x$.

In the default swap market, the difference $d - x$ is called the basis. A long basis trade means buying the reference asset and buying protection as in figure 5.1. A short basis trade means shorting the asset and shorting default protection. If $d > x$, the basis is termed positive, if $d < x$, the basis is negative.

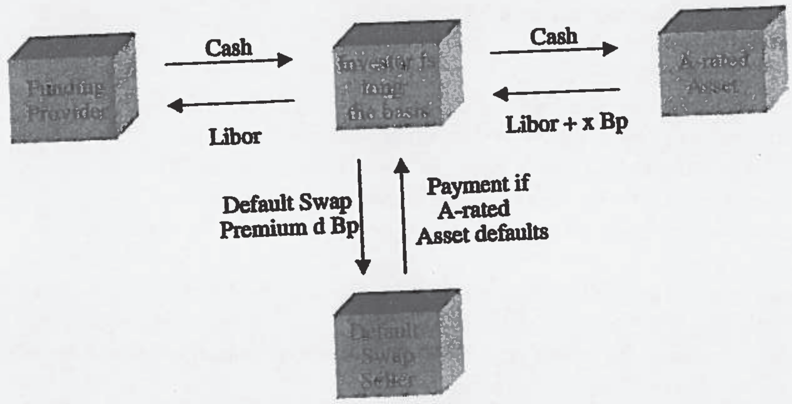

Figure 5.1: An investor buying an asset and hedging the credit risk with a default swap (Bp = basis points)

As pointed out, in an arbitrage-free environment, the no-arbitrage condition for Libor flat funding is d = x. However, in trading practice there are some features, which can increase or decrease the basis.

Features that increase the basis (default swap premium d – asset swap spread x)

1 *Natural market:* The default swap market has grown into the natural market to hedge credit risk. Especially for 3- to 5-year maturities many credit hedgers use the liquid default swap market rather than the asset swap market, driving default premiums up.

2 *Convertible bond arbitrage:* Hedge funds and other financial institutions strip the credit risk from the convertible bond and hedge it with default swaps to concentrate on managing the equity option.

3 *The delivery option:* In a default swap the delivery option allows the protection buyer to choose delivery from a pre-defined pool of assets, which increases the value of default swaps compared to asset swaps

4 *Default criteria:* Default criteria are clearly defined in the ISDA 1999 definitions, which facilitate trading. Also, default swap payments may be triggered by events, which do not constitute default in the cash market.

Features that decrease the basis (default swap premium d – asset swap spread x)

1 *Counterparty risk:* The default swap buyer is exposed to higher counterparty credit risk than the asset swap payer, since in an asset swap two cash flows of similar value are exchanged on a regular basis.

2 *Marking-to-market in default:* In case of default, it is typically quite difficult to mark-to-market an asset swap. Default swaps are designed to function in a default, so their marking-to-market is typically easier to achieve. This might drive asset swap spreads up, since the asset swap fixed rate payer, who will suffer a financial loss in the event of default, might want to be compensated for the higher uncertainty with receiving a higher spread.

Deriving the default swap premium using arbitrage arguments

We already derived an important arbitrage argument in chapter 2, which is used in trading practice to help determine the price of a default swap. The relationship was expressed in equation (2.1):

$$\text{Return on risk-free bond} = \text{Return on risky bond} - \text{Default swap premium (p.a.).} \tag{2.1b}$$

Solving equation (2.1b) for the default swap premium, we get:

$$\text{Default swap premium (p.a.)} = \text{Return on risky bond} - \text{Return on risk-free bond.} \tag{5.1}$$

Equation (5.1) can only serve as an approximation, as already discussed when examining equation (2.1) in chapter 2. Equation (5.1) abstracts from several features, which have to be included in the pricing of a defaults swap. We have listed these inputs in table 5.1. One of the most important points – and which is not included in equation (5.1) – is counterparty risk (i.e. the risk that the default protection seller defaults). As mentioned above, counterparty risk is an important feature, since in the case of default a typically large payment is due from the protection seller. In addition, the correlation between counterparty default risk and default risk of the underlying asset is of importance, since the default protection buyer will incur a loss in the amount of his reference asset value plus the default swap premium (minus the recovery rate of the reference asset issuer and the counterparty), if both the protection seller and the underlying asset default. These issues will be discussed later in this chapter.

"I price it where I can hedge it:" Pricing default swaps using hedging arguments

In the following, we will derive a price range for a default swap premium based on hedging considerations. Let's just recall the basic structure of a default swap, as seen in figure 5.2.

Since bank A has bought the default swap on a bond, it is short the credit risk of the bond (bank A's present value of the default swap will increase, if the bond price decreases due to credit deterioration). Thus to hedge the long default swap position, bank A has to go long the bond. If the funding for bank A is Libor + w and the bond pays Libor + x, the hedge of bank A can be seen in figure 5.3.

In figure 5.2, bank B is short the default swap, thus has a long bond position. Thus to hedge it, bank B has to short the bond. If bank B borrows the bond in the Repo market, the hedge can be seen in figure 5.4.

In a Repo the interest rate paid is usually sub-Libor, since the cash lender bank B has very little risk, since it receives the bond as collateral. This is why bank B only receives Libor – y.

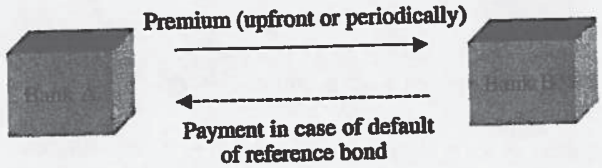

Figure 5.2: A standard default swap transaction

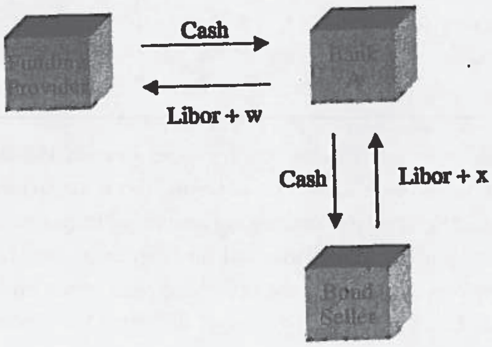

Figure 5.3: Bank A hedges a long default swap position by going long the underlying bond (Bank A's income is x − w)

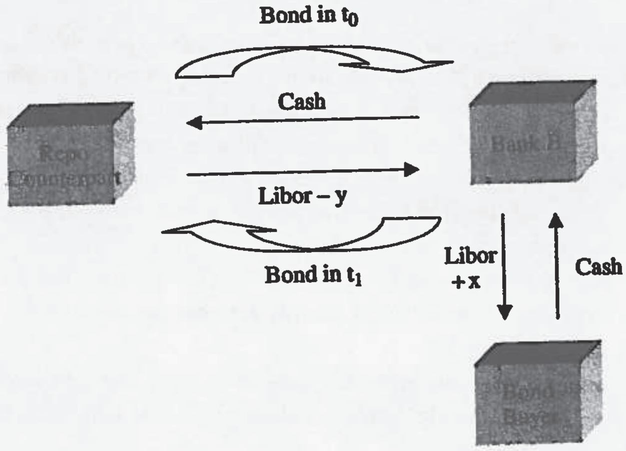

Figure 5.4: Bank B hedges a short default swap position by borrowing the bond in the Repo market and shorting it;[5] bank B's cost is $x - (-y) = x + y$

From figures 5.3 and 5.4 we can conclude that the income of the hedge for bank A is $x - w$. The cost for bank B is $x + y$. Hence, if the default swap price is derived on the basis of hedging costs, the price of the default swap lies between $x - w$ and $x + y$. Let's look at an example.

Example 5.1: Bank A is long a default swap, bank B is short the default swap. To hedge their position, bank A has to go long the underlying bond, bank B has to borrow the bond in the Repo market and short it.

Bank A's funding cost is Libor + 50. The bond pays Libor + 200. Bank B's interest rate received in the Repo is Libor − 30. Thus w = 50, x = 200 and y = 30. Therefore the default swap price lies between 200 − 50 = 150 basis point and 200 + 30 = 230 basis points.

Note that if the banks A and B agree on a price of 200 basis points, they will both lose money on the deal: Bank A pays 200 basis points for the default swap and receives 150 basis points in the hedge. Bank B receives 200 basis points from selling the default swap, but pays 230 basis points in the hedge.

The reasons why both banks might still enter into the default swap transaction, are the following:

- Bank A might be willing to pay a high price for the default swap, since the default swap might free credit lines to enable further credits to the client;
- Bank B might be a speculator and be willing to sell the default swap for 200 basis points;
- The default swap might increase the diversification for bank A and bank B;
- The low rating of bank A or bank B might make the default swap attractive relative to funding the transaction in the cash market;
- Due to low liquidity in the cash market, the default swap might still be more attractive than a cash deal;
- The off-balance-sheet feature might make the default swap more attractive than a cash transaction.

It is important to mention, that the profit of the hedged position for bank A and bank B depends on the funding cost and the interest rate paid in the Repo. In the above example 5.1, the breakeven default swap premium for both banks is 230 basis points if the funding cost of bank A is Libor − 30.

If the funding cost of bank A is even lower than Libor − 30 and the interest paid in the Repo is higher than Libor − 30, one or both banks can achieve a profit on their hedged position. For example, if the funding of bank A is Libor − 40 and the interest rate paid in the Repo is Libor − 20, a default swap premium of 230 basis points leads to a profit of 10 basis points for bank A and bank B.

Deriving the default probability and the upfront default swap premium on a binomial model

Deriving the probability of default of the underlying debt is one of the most important features when pricing credit derivatives. In the following, we will present a simple binomial model to find the risk-neutral probability of default.

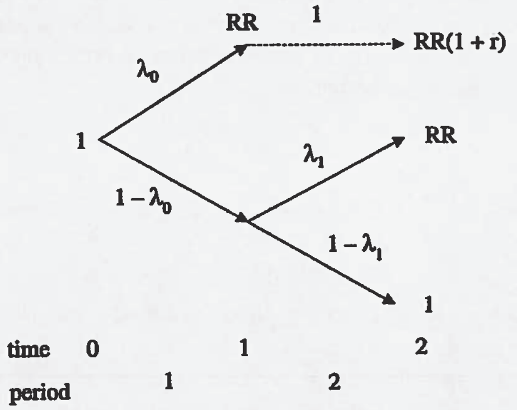

Figure 5.5: A two-period tree of risky debt with a notional amount of 1
λ_t = risk-neutral default probability in period t − 1;[6] RR = recovery rate (exogenous); r = risk-free interest rate.

Risk-neutrality is an important concept when pricing derivatives. If investors are risk-neutral, they do not require a compensation for taking risk. As a consequence, the expected return on all securities (including derivatives) is the risk-free interest rate. Hence, the present value of any security can be derived by discounting all future cash flows with the risk-free interest rate. The concept of risk-neutrality will be discussed in more detail in the section "Basic Properties of the Black-Scholes-Merton model," and "When to use martingale probabilities, when to use historical probabilities" in the presentation of the Jarrow-Lando-Turnbull 1997 model.

The binomial price tree of a two-period risky debt with a notional amount of 1 can be found in figure 5.5.

In figure 5.5 we can see that the value of the debt is set to 1 at time 0. At time 1 the debt has either defaulted with probability λ_0 and the debt will have a value of the recovery rate RR, or the debt will have not defaulted with probability $1 - \lambda_0$. It is assumed that the debt, if it has defaulted, will stay in default. Thus at time 2, the debt, if it has defaulted in time 1, will stay in default with a probability of 1 (dashed line in figure 5.5). Hence the received recovery rate RR at time 1 will increase to RR(1 + r) at time 2.

If the debt has not defaulted at time 1, it can either default at time 2 with probability λ_1, in which case the value of the debt is RR at time 2. If the debt does not default at time 2, the debt will mature with a value of 1 with a probability of $1 - \lambda_1$.

Let's now include the annual return of a risk-free debt r and an annual default swap premium of s in the tree. For a $1 investment in the risk-free debt an investor receives 1 + r at the end of period 1. Also, if we solve equation (2.1b), Return on risk-free asset r = Return on risky asset − Default swap premium (p.a.) s, for the risky bond return we get:

Return on risky asset = Return on risk-free asset r + Default swap premium (p.a.). s

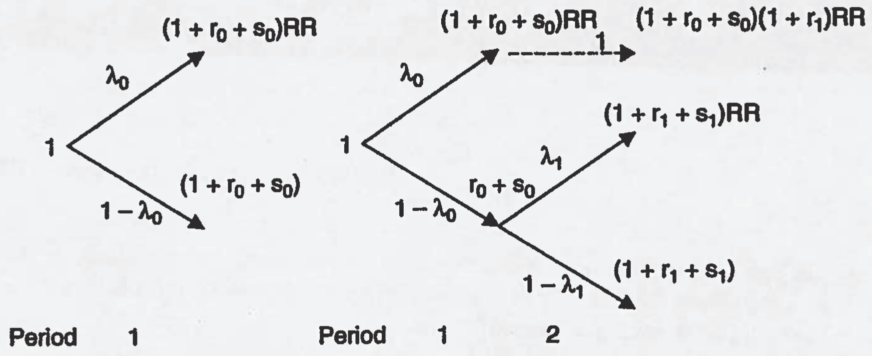

Figure 5.6: Cash flows of a one-period and a two-period risky debt issue with a notional amount of 1
λ_t: risk-neutral default probability in period $t-1$; RR: recovery rate (exogenous); r: risk-free interest rate; s: default swap premium.

Thus the risky investment of \$1 will grow to $1+r+s$ at the end of period 1. If we include these cash flows, we derive for a one-period debt and a two-period debt issue the binomial trees as in figure 5.6.

We can now find the risk-neutral probability of default during period 1, λ_0, by using the risk-neutral relationship that the expected return of the risk-free debt, $1+r_0$, must be equal to the expected return of the default probability weighted risky debt. From the one-period tree in figure 5.6, we derive the returns at the end of period 1:

$$1+r_0=\lambda_0 RR(1+r_0+s_0)+(1-\lambda_0)(1+r_0+s_0). \tag{5.2}$$

Dividing equation (5.2) by $(1+r_0)$, we find another intuitive interpretation from equation (5.2a):

$$1=[\lambda_0 RR(1+r_0+s_0)+(1-\lambda_0)(1+r_0+s_0)]/(1+r_0). \tag{5.2a}$$

Equation (5.2a) reflects the risk-neutral pricing principle: all expected cash flows $[\lambda_0$ RR $(1+r_0+s_0)+(1-\lambda_0)(1+r_0+s_0)]$ are discounted with the risk-free rate r_0, to derive the (given) present value of 1.

Solving equation (5.2) or (5.2a) for the risk-neutral probability of default in period 1, λ_0, we derive:

$$\lambda_0=\frac{s_0}{(1+r_0+s_0)(1-RR)} \tag{5.3}$$

The values of r_0 and s_0 can be found in the market. However, to derive λ_0, we have to also determine the recovery rate RR. This can be done by observing the recovery rate of previously defaulted debt with identical seniority in the same sector as the underlying debt. Let's look at deriving the probability of default in an example.

Example 5.2: The 1-year risk-free interest rate r = 5%, the 1-year default swap premium s = 3%, and the recovery rate is assumed to be RR = 60%. What is the risk-neutral probability of default in period 1? It is:

$$\lambda_0 = \frac{0.03}{(1+0.05+0.03)(1-0.6)} = 6.94\%$$

In order to derive λ_1, the risk-neutral default probability in period 2, we can equate the risk-free return at time 2, $(1 + r_0)(1 + r_1)$, to the probability weighted payoff of the risky debt in period 1 and 2. The reader should notice that r_1 is the *forward* interest rate from time 1 to time 2.[7] Correspondingly λ_1 is the default probability in period 2, so also a forward variable, which is realized at time 2. Thus, from the two-period tree in figure 5.6 we derive:

$$(1+r_0)(1+r_1) = \lambda_0 RR(1+s_0+r_0)(1+r_1)+$$
$$(1-\lambda_0)[(r_0+s_0)(1+r_1)+\lambda_1 RR(1+r_1+s_1)+(1-\lambda_1)(1+r_1+s_1)].$$

In the above equation derived from figure 5.6, we compare all values at time 2. Hence the first term λ_0 RR $(1 + s_0 + r_0)$ $(1 + r_1)$ reflects the fact that if default occurs at time 1, the investor receives RR $(1 + s_0 + r_0)$ with probability λ_0 and can invest this return at the risk-free interest rate r_1. The term $(r_0 + s_0)(1 + r_1)$ reflects the fact that looking from time 0, the proceeds at time 1 in case of no default $(r_0 + s_0)$ are certain and thus can be reinvested at the risk-free rate r_1.

Solving for the risk-neutral default probability in period 2, λ_1, we get:

$$\lambda_1 = \frac{\left(\frac{(1+r_0)(1+r_1)-\lambda_0 RR(1+r_0+s_0)(1+r_1)}{(1-\lambda_0)}\right)-(r_0+s_0)(1+r_1)-1-r_1-s_1}{RR(1+r_1+s_1)-1-r_1-s_1}. \quad (5.4)$$

Example 5.3: The 1-year risk-free interest rate r_0 is 5% and the forward risk-free interest rate r_1 from time 1 to time 2 is 6%. The default premium for the first year is 3% and the forward default swap premium from time 1 to time 2 is 3.5%. The recovery rate is assumed to be 60%. What is the probability of default in period 2? Following equation (5.4) it is

$$\lambda_1 = \frac{\left(\frac{(1+0.05)(1+0.06)-0.0694\times 0.6\times (1+0.05+0.03)(1+0.06)}{(1-0.0694)}\right)-(0.05+0.03)(1+0.06)-1-0.06-0.035}{0.6\times(1+0.06+0.035)-1-0.06-0.035}$$

$$=7.99\%.$$

See www.dersoft.com/ex53.xls for this example.

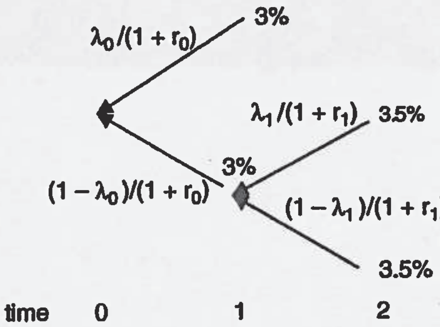

Figure 5.7: Probability weights and discount factors in a two-period tree

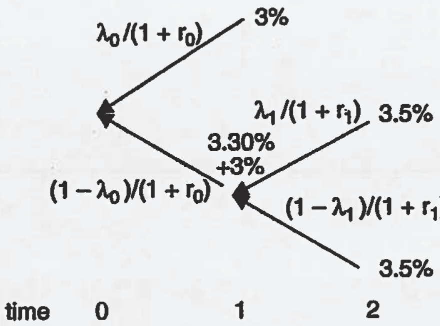

Figure 5.8: Probability weights and discount factors in a two-period binomial tree including the discounted value of the default swap premiums of time 2

Naturally, the risk-neutral default probabilities for a more than 2-period model can be derived by iteratively extending equations (5.3) and (5.4). A multi-period model can be found at www.dersoft.com/binomialdefaultmodel.xls.

To derive the upfront default swap premium, we have to weight the default swap premiums with their risk-neutral probability of occurrence and discount these with the risk-free interest rate. For our 2-period tree this can be seen in figure 5.7.

Following figure 5.7, we find the value of the default swap premiums at time 1 with $\lambda_1 = 7.99\%$ and $r_1 = 6\%$ as $3.5\% \times 0.0799 / (1 + 0.06) + 3.5\% \times (1 - 0.0799) / (1 + 0.06) = 3.30\%$. Integrating this value in figure 5.7 gives the result shown in figure 5.8.

Following figure 5.8, the upfront default swap premium at time 0 of the two-period default swap is $[(3\% + 3.30\%)(1 - 0.0694) / (1 + 0.05)] + [3\% \times 0.0694 / (1 + 0.05)] = 5.78\%$.

It should be mentioned that in figure 5.6 and equations (5.2) to (5.4) we assumed that the recovery rate RR is applied to the notional amount of 1 *and* the coupon r + s, hence to 1 + r + s. It is also reasonable to assume that the recovery rate only applies to the notional amount of 1. In this case figure 5.6 would simplify to figure 5.9.

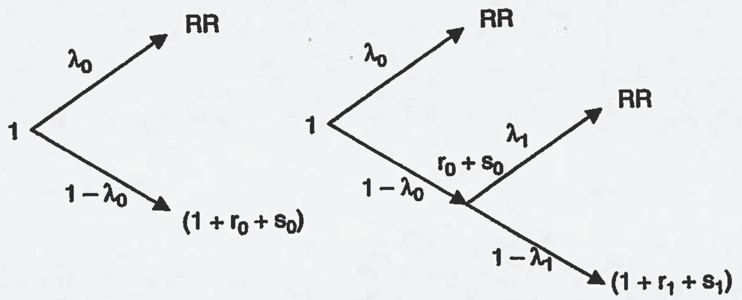

Figure 5.9: Cash flows of one-period and two-period binomial risky debt with a notional amount of 1 assuming the recovery rate is only paid to the notional amount

Equation (5.2) would then read (5.2b) $1 + r_0 = \lambda_0 \text{ RR} + (1 - \lambda_0)(1 + r_0 + s_0)$ and equation (5.3) would read (5.3a) $\lambda_0 = \frac{s_0}{(1+r_0+s_0-\text{RR})}$.

In the following, before we price credit derivatives in the Black-Scholes-Merton environment, let's look at some basic properties of this approach.

Basic Properties of the Black-Scholes-Merton model

A *stochastic process* describes the uncertain course that a variable follows through time. In 1973 Fischer Black, Myron Scholes, and Robert Merton transferred a stochastic process from physics to finance: the *generalized Wiener process.* According to this concept a variable grows with an average drift rate μ. Superimposed on this growth rate is a stochastic term, which adds volatility to the process.

If the relative change of a variable follows a generalized Wiener process, this is typically referred to as a *geometric Brownian motion.* Applied to stock prices, we derive that the relative change of a stock price S, dS/S, follows a path with an average expected growth rate μ, which is comprised of the expected stock price change plus the dividend. Superimposed on this growth rate is a noise term, which consists of the expected volatility of the stock σ multiplied with a Wiener process dz:

$$dS/S = \mu dt + \sigma dz \tag{5.5}$$

where

dS: change in the stock price S
μ: drift rate, which is the expected stock return (price change + dividends)
dt: infinitely short time period
σ: expected volatility of the stock price S
dz: $\varepsilon\sqrt{dt}$, where ε is a random drawing from a standardized normal distribution. All drawings are independent from each other.

μ dt in equation (5.5) is the expected return during period dt. σ dz is the stochastic part of the relative change of S.

The discrete version of equation (5.5) is

$$\Delta S/S = \mu \Delta t + \sigma \Delta z. \tag{5.6}$$

Let's look at equation (5.6) in an example.

Example 5.4: The present stock price is \$150, next year's expected return is 20%, the annual expected volatility is 30%. The sample drawing from a standardized normal distribution results in +1. What is the expected stock price in one day (= 1/365) due to the geometric Brownian motion?

Due to equation (5.6), the one-day change of the stock is $\Delta S = \$150 \times [(0.2 \times 1/365) + 0.3 \times 1 \times \sqrt{1/365}] = \$2.44$.

Thus, the stock price after one day is assumed to be \$150 + \$2.44 = \$152.44.

Since this stock price prediction is partly determined by a random drawing from a normal distribution, this prediction methodology is called a *random walk* process. If a price follows a random walk process, this means that due to the random nature of the process, principally no above market return trading strategy can be formulated.

This is consistent with the *efficient market hypothesis*, which says that all information about a stock is already incorporated in the current stock price. As a consequence, the past information of a stock price is irrelevant and the future process of a stock price depends only on its value at the beginning of the period. This property is also referred to as the *Markov property*. This property denies the essential hypothesis of technical analysis, which suggests that the future stock price can be derived from the past pattern of the stock price.

If a generalized Wiener process has no drift rate, thus $\mu = 0$, this is typically referred to as a *martingale*. A martingale has the convenient property that the expected value E of a random variable X at a future time t is equal to the current value of the variable: $E(X_t) = X_0$. Using this logic, a zero-coupon bond is not a martingale, since it will increase in time to its notional amount, assuming no default. Hence for a zero coupon bond we derive $E(X_t) > X_0$. It may also be argued that stock prices are not martingales, since stock prices increase on average in time. In the Black-Scholes-Merton environment the expected stock growth rate is the risk-free interest rate r, which is the growth rate of all assets, including derivatives.

Martingales are often compared to a "fair game." For example when playing roulette the probability of winning when betting on black is always 18/37 (assuming there are 18 black, 18 red, and 1 green (the zero) possibility) independent of what the previous outcome was. With the same logic the game blackjack is not a martingale, since the outcome of a game depends on the previous cards, assuming the cards are not put back into the stack.

Martingales also have convenient mathematical properties. Let's define θ as a *trading strategy* for security X, $\theta > 0$ representing a long position in X, and $\theta < 0$ representing a short position in X. We can now express a portfolio of θ units in X as a stochastic integral $\int_0^t \theta_s dx_s$. This is a martingale if X is a martingale, thus we can derive complex continuous martingales from simpler martingales. Since $\int_0^t \theta_s dx_s$ is a martingale, we can derive the convenient property from equation $E(X_t) = X_0$ that $E\left(\int_0^t \theta_s dX_s\right) = 0$.

We further assume that a *money market account* exists, which has a notional amount of \$1 at time zero and which grows with the risk-free interest rate r. Assuming reinvestment at r_k, the price of the money market bond for discrete time periods and discrete r is $M_t = \prod_{k=1}^{t-1}(1+r_k)$. For discrete time and with continuously compounded r, $M_t = \exp\left(\sum_{k=1}^{t-1} r_k\right)$.[8] For continuous time and a continuously compounded r we get $M_t = \exp\left(\int_0^t r(s)ds\right)$. The purpose of introducing the risk-free money market account is to fund the trading strategy θ. If an investor needs to borrow cash for the strategy, he can short the money market bond B_t to receive the cash. We will abstract from additional cost for borrowing B. The money market bond is often used as a *numeraire*, i.e. the unit in which profit and loss are measured. Principally any tradable asset with a strictly positive price process can serve as a numeraire.

While an investment in the money market grows with the risk-free interest rate r, a risky asset grows with the expected return of the risky asset μ, consider equations (5.5) and (5.6). The relationship between r and μ was expressed first by William Sharpe in his famous paper in 1966.[9] Sharpe stated that an investor wants to be compensated for taking risk, the risk being expressed as the volatility of the invested asset i, σ_i. The compensation for the risk is reflected in the excess return of the asset i, $\mu_i - r$:

$$\chi_i = \frac{\mu_i - r}{\sigma_i} \tag{5.7}$$

where χ is the Sharpe ratio, often referred to as the *market price of risk*. In a risk-neutral world, where investors are indifferent to risk, every asset grows with the same expected rate, which is the risk-free interest rate r. If we apply Sharpe's concept, the expected growth rate, solving equation (5.7) for r, is $\mu_i - \chi_i\sigma_i$.

Arbitrage is – as in chapter 4 – again narrowly defined as a risk-free profit. Formally, arbitrage exists if the trading strategy θ increases the wealth W with probability P of 1: If $W_0 = 0 \Rightarrow W_t > 0$ with $P = 1$ and $t > 0$. A trading strategy is called *self-financing* if the change in wealth W is solely derived by borrowing in the money market and/or from gains and losses of the trading strategy.

With the help of Ito's lemma,[10] Black, Scholes, and Merton found the famous partial differential equation (PDE) for valuing a derivative D:

$$D = \frac{\partial D}{\partial t}\frac{1}{r} + \frac{\partial D}{\partial S}S + \frac{1}{2}\frac{\partial^2 D}{\partial S^2}\frac{1}{r}\sigma^2 S^2 \tag{5.8}$$

where r is the risk-free interest rate, S is the price of the underlying asset (e.g. the stock price), and σ is again the volatility of the underlying asset.

For every derivative that satisfies the PDE (5.8), a dynamic, self-financing trading strategy can be created that replicates the derivative. For example, a long put can be replicated by selling the delta amount of the underlying. This property is also referred to as *completeness*. If the PDE (5.8) is satisfied, going long the derivative and short the portfolio or vice versa, is *arbitrage-free*, arbitrage in this case being defined narrowly as a risk-free profit.

Also – arguably luckily – the variable μ drops out during the process of creating the PDE. Therefore, no variable regarding the risk-preference of an investor is present. Thus, the PDE is *risk-neutral*. This means that the expected growth rate of all securities (including derivatives) under the risk-netural probability measure is the risk-free interest rate. If a security is expected to grow by more than the risk-free rate, investors will buy it and hence increase the price and reduce the rate of return to the risk-free rate, and vice versa.

Since a derivative is a *contingent claim* on an underlying security based on the specified input parameters, the Black-Scholes-Merton framework is called the arbitrage-free, risk-neutral, contingent claim pricing methodology.

One equation that satisfies the PDE, equation (5.8), is the famous Black-Scholes equation for valuing European style options.[11] For a call C:

$$C = SN(d_1) - Ke^{-rT}N(d_2) \tag{5.9}$$

where

$$d_1 = \frac{\ln\left[\frac{S}{Ke^{-rT}}\right] + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

where N: cumulative standard normal distribution; S: stock price; K: strike price; ln: natural logarithm; σ: volatility of the underlying asset, in this case the stock price; T: option maturity expressed in years; r: risk-free continuously compounded interest rate.

After having discussed the Black-Scholes-Merton framework, we will introduce two approaches to value credit-spread options on slightly modified Black-Scholes equations.

Valuing credit-spread options on a modified Black-Scholes equation where the credit-spread is modeled as a single variable

As discussed in chapter 2, a credit-spread option is an option on the difference between the yield of a risky asset and the yield of a risk-free asset. A credit-spread is defined as in equation (2.4):

Credit-spread = Yield of risky bond − Yield of risk-free bond.

We defined the payoff of a credit-spread put and a credit-spread call at option maturity T in equations (2.5) and (2.6) as:

$$\text{Payoff credit-spread put (T)} = \text{Duration} \times N \times \max(\text{Credit-spread (T)} - \text{Strike spread}, 0) \quad (2.5)$$

$$\text{Payoff credit-spread call (T)} = \text{Duration} \times N \times \max(\text{Strike spread} - \text{Credit-spread (T)}, 0) \quad (2.6)$$

where duration is defined in equation (2.8) as $D = -\frac{\partial B/B}{\partial y} = \sum_{t=1}^{T} tc_t e^{-yt}/B$, N is the notional amount of the swap, and the strike spread as in equations (2.5) and (2.6) is determined at option start.

One of the simplest approaches to value a credit-spread option is to model the credit-spread as a single variable S. We then apply a slight modification of the original Black-Scholes equation (5.9). In equation (5.9), the variable S grew with the risk-free interest rate r. This is not a reasonable assumption for a credit-spread. Setting the growth rate of the credit-spread to zero, we derive equation (5.9a)

$$C = e^{-rT}[SN(d_1) - KN(d_2)] \quad (5.9a)$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

where S is now the current credit-spread and σ is the annual volatility of this spread. All other variables are as defined in equation (5.9). Equation (5.9a) is also used to value options on futures where S in equation (5.9a) is the current futures price. It should be mentioned that the modified equation (5.9a) does not satisfy the PDE, equation (5.8). Hence, no self-financing replicating strategy can be created.

Let's look at an example of pricing a credit-spread option with equation (5.9a).

Example 5.5: Given is a current credit-spread of 3.30% and a strike spread of 3%. The notional amount is $1,000,000, the duration is 3.67, and the option maturity is 1 year. The risk-free interest rate is 5% and the annual volatility of the credit-spread is 150%. What is the credit-spread put option premium?

The reader should first recall that the payoffs in the credit-spread market are reversed, as expressed in equation (2.5) and (2.6) and discussed in chapter 2. So to value a put we have to use equation (5.9a) and we derive for

$$d_1 = \frac{\ln\left(\frac{0.033}{0.03}\right) + \frac{1}{2}1.5^2 \times 1}{1.5\sqrt{1}} = 0.8135 \quad \text{and} \quad d_2 = 0.8135 - 1.5\sqrt{1} = -0.6865.$$

N(0.8135) = 0.7921 and N(−0.6865) = 0.2462; see table A.1 in the appendix or use the Excel function normsdist(0.8135) = 0.7921 and normsdist(−0.6865) = 0.2462).

From equation (5.9a) we derive $e^{-0.05 \times 1}(0.033 \times 0.7921 - 0.03 \times 0.2462) = 0.0178$. Multiplying this with the constant factors duration and notional amount we derive the credit-spread put price as 0.0178 × 3.67 × $1,000,000 = $65,326. Compare www.dersoft.com/csobs.xls for this derivation as well as for the derivation of the credit-spread call price.

The simple approach of equation (5.9a) has some serious drawbacks. First, since the spread is modeled as a single variable, due to the log-normality assumption, the credit-spread cannot be negative. While this assumption is reasonable for a credit-spread between a risky asset and a risk-free asset, it is clearly not reasonable for a spread between two risky assets. Second, due to the single variable approach, the individual features of the underlying two yields such as yield level, yield volatility, and the yield correlation are not inputs of the model. Naturally, all drawbacks of the Black-Scholes approach apply, such as constant volatility, constant interest rates, and the inability to price American style options. Counterparty default risk is also not included in the approach.

Valuing credit-spread options on a modified Black-Scholes equation as an exchange option

Two counterparties may agree to simply exchange the yield of a risky asset and the yield of a risk-free asset. Hence the payoff is:

$$\max(y_2 - y_1, 0) \tag{5.10}$$

where

y_2: yield of the risky asset
y_1: yield of the risk-free asset.

The payoff in equation (5.10) reduces to $y_2 - y_1$, assuming that the risky bond cannot have a lower yield than the risk-free bond. The reader should note that since there is no strike in the payoff definition of equation (5.10), a distinction between call and put is not reasonable.

An option with a payoff of equation (5.10) is a frequently traded exotic option, termed an *exchange option*. William Margrave was the first to derive a closed form solution to value exchange options on a modified Black-Scholes equation.[12]

In order to value a credit exchange option on the basis of the Margrave model, we have to add the credit industry standard duration term of the risky asset 2, D(2) and the notional amount N. We then receive the payoff:

$$D(2)\,N\max(y_2 - y_1, 0)$$

and the resulting valuation equation for the credit exchange option $'E$ is:

$$'E = e^{-rT}D(2)N[y_2N(d_1) - y_1N(d_2)] \tag{5.11}$$

where

$$d_1 = \frac{\ln\left[\frac{y_2}{y_1}\right] + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

and σ: volatility of y_2/y_1, $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$, where

σ_1: yield volatility asset 1

σ_2: yield volatility of asset 2

ρ: correlation coefficient of the yields y_1 and y_2.

The exchange option value $'E$ has a negative dependence on the correlation coefficient ρ, $\partial E/\partial\rho < 0$. This is an expected result, since the more negative the correlation coefficient ρ, the higher will be the volatility of the difference between the two yields, and consequently the higher the expected payoff and option value. Furthermore, intuitively, the exchange option value $'E$ has a positive dependence on both the yield volatility of the asset 1 and asset 2, σ_1 and σ_2.

Example 5.6: The current yield of a risk-free asset 1, y_1, is 5% and the current yield of a risky asset 2, y_2, is 7%. The option maturity T is one year, the duration of the risky asset 2, D(2) is 3.67, and the notional amount N is \$1,000,000. The risk-free interest rate r is 4%, the yield volatility of asset 1, σ_1, is 20% and the yield volatility of asset 2, σ_2, is 50%. The yield correlation coefficient is 0.5. What is the price of a credit-spread option with a payoff D(2) N $\max(y_2 - y_1, 0)$?

Following equation (5.11), the volatility is $\sigma = \sqrt{0.2^2 + 0.5^2 - 2\times0.5\times0.2\times0.5}$ = 43.59%.

$$d_1 = \frac{\ln\left[\frac{0.07}{0.05}\right] + \frac{1}{2}0.4359^2 \times \sqrt{1}}{0.4359\times1} = 0.9899 \text{ and } d_2 = 0.9899 - 0.4359\times\sqrt{1} = 0.5540.$$

N(0.9899) = 0.8389 and N(0.5540) = 0.7102. Hence the credit exchange option value is:

$$e^{-0.05\times1}\times3.67\times\$1{,}000{,}000[0.07\times0.8389 - 0.05\times0.7102] = \$81{,}032.$$

A model which prices credit-spread options as an exchange option can be found at www.dersoft.com/csoex.xls.

The valuation approach of equation (5.11) is an improvement on equation (5.9a) since it includes the individual yield volatilities and the yield correlation. However, equation (5.11) does not include counterparty default risk and has, due to the Black-Scholes framework, constant volatilities and no mean reversion of interest rates.

Valuing credit-spread options on a term-structure based model

A term structure of interest rates describes the uncertain path that interest rates take through time. The interest rate that is modeled is the *short rate*, also called *instantaneous rate*, which applies to an infinitesimally short period of time. The short rate process is usually displayed in the form of a discrete or continuous binomial or trinomial model.

One of the first term structure models is the Cox-Ross-Rubinstein (CRR) (1979) model. The model is sometimes credited to Rendleman-Bartter (1980). Some sources (e.g. Smithson 1992) mention Sharpe (1978) as the original author, who outlined the basics of the concept of the CRR model.

The CRR model can be expressed as:

$$dr/r = \mu dt + \sigma dz \qquad (5.12)$$

where

r: short-term interest rate
μ: expected growth rate
σ: expected volatility of r
dz: Wiener process, as discussed in equation (5.5) and example 5.4.

Equation (5.12) states that the relative change in the short rate r, dr/r, is comprised of two terms. The first term μ dt represents the expected average growth rate of r. The second term, σdz, adds the volatility, also called "noise" to the process. The higher the volatility σ, the greater the possibility that dr/r will deviate from the expected growth path μ.

The attentive reader will notice that equation (5.5) expressing stock price behavior and equation (5.12) expressing interest rate behavior are mathematically identical. Hence Cox, Ross, and Rubinstein model the short rate the same way that stock prices are often modeled in finance.

Other more complex term structure models for the short rate are Vasicek (1977), Cox-Ingersoll-Ross (1985), or the no-arbitrage models of Ho-Lee (1986), Black-Derman-Toy (1990), and Hull-White (1990). Heath, Jarrow and Morton (1992) model an entire term structure of instantaneous forward rates, while Brace, Gatared, and Musiela (Libor market model) (1997) model an entire term structure of forward Libor rates.

We will now value a credit-spread option on the Hull-White trinomial model.[13] The model is expressed as:

$$dr = [\theta(t) - ar]dt + \sigma dz \qquad (5.13)$$

where

θ: function chosen so that the model fits the current term structure
a: mean reversion of r
other variables as defined in equation (5.12).

The key equations in the Hull-White model are:

$$P_{m+1} = \sum_{j=-n_m}^{n_m} Q_{m,j} e^{[-(\alpha_m + j\Delta R)\Delta t]}. \qquad (5.14)$$

Solving equation (5.14) for α_m we derive:

$$\alpha_m = \frac{\ln \sum_{j=-n_m}^{n_m} Q_{m,j} e^{-j\Delta R\Delta t} - \ln P_{m+1}}{\Delta t}. \qquad (5.15)$$

Furthermore:

$$Q_{m+1,j} = \sum_k Q_{m,k} q(k,j) e^{[-(\alpha_m + k\Delta R)\Delta t]} \qquad (5.16)$$

where

P_{m+1}: zero coupon bond maturing at time m + 1
n_m: number of nodes on each side of the central node at time mΔt
Q_m: present value of a security at time m that pays $1 if node (i,j) is reached and zero otherwise
R*: discrete interest rate for time Δt on a tree that is evenly spaced and has a zero slope
R: discrete interest rate for time Δt on a tree that matches the current term structure
α: variable that transforms an evenly spaced zero slope tree with interest rates R* into a tree that matches the upward (or downward) slope of the term structure with interest rates R – hence$\alpha(t) = R(t) - R^*(t)$
i: horizontal parameter of node (i,j)
j: vertical parameter of node (i,j)
q (k,j): probability of moving from node (m,k) to node (m + 1,j); the summation is taken over all values of k for which the probability is non-zero.

In an iterative, forward inductive process, first the $Q_{m,j}$ values are derived with equation (5.14) and the input of the market given zero bond price P_m. The value for α_m is then derived with equation (5.15). The value of α_m is then used to transform R into R* via$\alpha(t) = R(t) - R^*(t)$. In the next time step the values for $Q_{m+1,j}$ are derived, which give α_{m+1}, and so on.

The concept of a 3-period trinomial tree can be seen in figure 5.10. A model of the Hull-White trinomial tree can be found at www.dersoft.com/hwtri.xls.

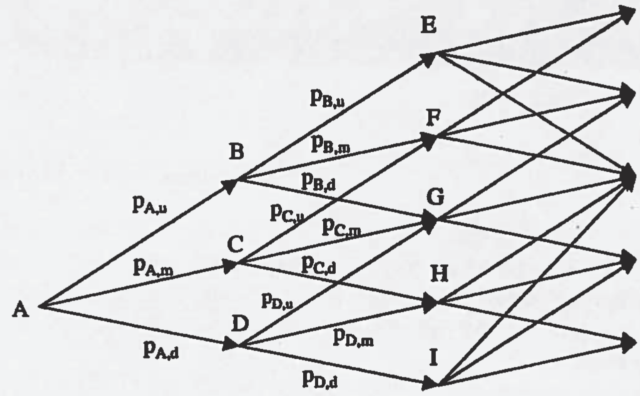

Figure 5.10: Three-period Hull-White trinomial tree
$p_{X,u}$, $p_{X,m}$, $p_{X,d}$: risk-neutral probability of moving from node X up, middle, and down, respectively; probabilities on the same level are identical, for example $p_{A,u} = p_{C,u} = p_{G,u}$; $p_{A,m} = p_{C,m} = p_{G,m}$ and $p_{A,d} = p_{C,d} = p_{G,d}$ (compare with figure 5.11).

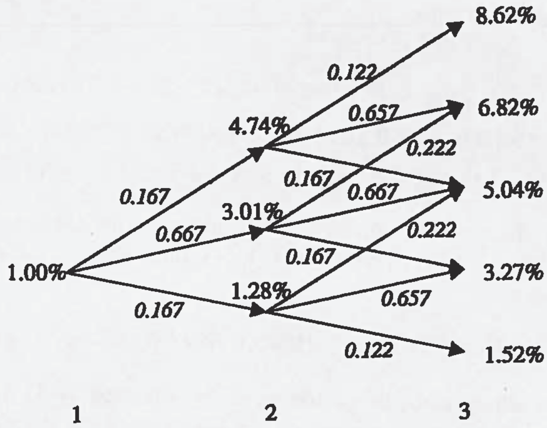

Figure 5.11: Hull-White two-period credit-spread tree with a 1%, 2%, and 3% credit-spread for year 1, 2, and 3 respectively and a mean reversion of 0.1 and volatility of 1% (spread rates are expressed as continuously compounded one-year rates)

Let's now evaluate a credit-spread option on the Hull-White tree. We first have to create a credit-spread tree. Let's assume an issuer has a bond with a 1-year credit-spread of 1%, the two-year credit-spread of 2%, and the 3-year credit-spread of 3%. Using these data in combination with the Hull-White trinomial model with $\Delta t = 1$ year, and the parameters 0.1 for the mean reversion a, and 1% for the volatility σ, we derive figure 5.11. The Hull-White trinomial model to evaluate credit-spread puts and calls can be found at www.dersoft.com/csohw.xls.

We can now use standard discrete option valuation techniques to value a credit-spread option.

Example 5.7: Given is a two-year credit-spread put option with a payoff as in equation (2.5): Duration × N × max[credit-spread (T) − strike spread, 0]. For the option price derivation we can currently ignore the constant factors Duration and N and implement them later.

Let's look at a credit-spread put option with a strike spread of 3% and an option maturity at time t = 2. At option maturity, the value of the option is just the intrinsic value: max[credit-spread (T) − strike spread, 0]. Hence for node E (see figures 5.10 and 5.11) we derive max(8.62% − 3%, 0) = 0.0562. For the other nodes we derive F = 0.0382, G = 0.0204, H = 0.0027 and I = 0. We now have to discount these values at time 2 weighted by their probabilities back to time 1. Let's assume the risk-free interest rate from time 1 to time 2 is 5%. (In a more sophisticated analysis we could apply an interest rate term structure tree to have different discount factors for each probability.)

$$\text{Node B} = e^{-(0.05\times1)} \times [0.0562 \times 0.122 + 0.0382 \times 0.657 + 0.0204 \times 0.222] = 0.0347$$

$$\text{Node C} = e^{-(0.05\times1)} \times [0.0382 \times 0.167 + 0.0204 \times 0.667 + 0.0027 \times 0.167] = 0.0194$$

$$\text{Node D} = e^{-(0.05\times1)} \times [0.0204 \times 0.222 + 0.0027 \times 0.657 + 0 \times 0.122] = 0.0060$$

Finally we have to discount the option values from time 1 back to time 0. Let's assume the interest rate from time 0 to time 1 is 4%. For the credit-spread put option value at time 0 we derive:

$$\text{Node A} = e^{-(0.04\times1)} \times [0.0347 \times 0.167 + 0.0194 \times 0.667 + 0.006 \times 0.167] = 0.0190 \text{ or } 1.90\%.$$

To derive the credit-spread put option price in equation (2.5) Duration × N × max[credit-spread (T) − strike spread, 0], we have to add the duration and notional amount. If the duration is 3.67 and the notional amount is $1,000,000, the credit-spread put option price is 3.67 × $1,000,000 × 0.0190 = $69,730.

See www.dersoft.com/ex57 for this example.

In order to value a credit-spread *call* option with a payoff as in equation (2.6), Duration × N × max[strike spread − credit-spread (T), 0] only a minor change to the evaluation in example 5.7 has to be done. The only difference is the intrinsic value at the last node, which in the case of a credit-spread call is max[strike spread − credit-spread (T), 0]. Then the same analysis as for a put (as in example 5.7) can be applied to derive the present value of the credit-spread call option. A credit-spread call with the inputs of example 5.7 comes out to 0.03%.

A simple spreadsheet for a put and call of example 5.7 can be found at www.dersoft.com/ex57.xls. As mentioned above, for a complete Hull-White trinomial model to evaluate credit-spread options, see www.dersoft.com/csohw.xls.

The advantage of using a term structure model is that we can incorporate the mean reversion of credit-spreads in the model. Also, the credit-spreads fit the initial term structure of credit-spreads in the market and we can evaluate American style credit-spread options. The drawback of the term structure approach is again that the individual features of the risky and risk-free bond, i.e. the individual yield level, yield volatility, and the yield correlation, are not inputs of the model.

After having derived the credit-spread option price with two modified Black-Scholes approaches and a term structure model, let's now look at the valuation of credit derivatives on a group of models termed *structural models*.

Structural Models

Structural models derive the probability of default by analyzing the capital structure of a firm, especially the value of the firm's assets compared to the value of the firm's debt. Structural models can be divided into *firm value models* and *first-time passage models*. In firm value models, bankruptcy occurs when the asset value of a company is below the debt value at the maturity of the debt T. In first-time passage models, bankruptcy occurs when the asset value drops below a pre-defined, usually exogenous barrier, allowing for bankruptcy before the maturity of the debt. Structural models have close ties to the Merton 1974 model. Let's have an in-depths look at it.

The original 1974 Merton model

In 1974 Robert Merton in a seminal paper created a firm value model to estimate a company's value of debt and the probability of default.[14]

The Merton call

Merton combined the simple equation, shareholders' equity (E) = company's assets (V) – company's liabilities (D), with the Black-Scholes option pricing framework. Merton's model is mathematically identical with the original Black-Scholes equation (5.9) for valuing a call. However, the variables are reinterpreted:

$$E_0 = V_0 N(d_1) - De^{-rT} N(d_2) \tag{5.17}$$

where

$$d_1 = \frac{\ln\left[\frac{V_0}{De^{-rT}}\right] + \frac{1}{2}\sigma_V^2 T}{\sigma_V \sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma_V \sqrt{T}$$

where E_0 is the current value of equity, V_0 is the current value of assets, D is the debt to be repaid at time T, N is the cumulative standard normal distribution, r is the risk-free continuously compounded interest rate, σ_V is the expected volatility of the asset, and T is the option maturity, measured in years.

Equation (5.17) states that equity holders have a claim on the assets of a company: If the asset value V increases, the equity value E will increase with unlimited upside potential; on the downside, if the debt D exceeds the assets V, the company will go bankrupt. In this case the equity holders will take the remaining assets to repay part of the debt, the equity value being zero. This unlimited upside potential and limited downside risk is an essential option criteria and is reflected in the time value as seen in figure 5.12.

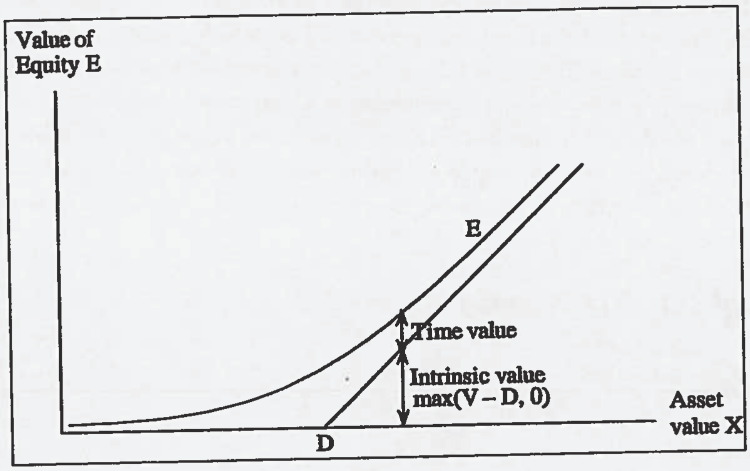

Figure 5.12: Intrinsic value and time value of Merton's credit model

In figure 5.12, the intrinsic value is max(V – D, 0). This intrinsic value plus the time value equals the value of the equity.

A well known property of the Black-Scholes model is that the risk-neutral probability of exercising a call option is $N(d_2)$. Therefore, the probability of not exercising the option is $N(-d_2)$. Not exercising the equity option means that the debt D is bigger than the assets V. This is the case of bankruptcy. Therefore, the probability of default in the Merton framework is $N(-d_2)$. Let's derive this default probability in a numerical example.

Example 5.8: The assets of company X are currently worth $1,300,000. In 90 days company X has to repay $1,000,000 in debt. The expected volatility of the assets is 30% and the risk-free interest rate is 5%. What is the probability of default in 90 days on the basis of the Merton model?

The probability of default is:

$$N(-d_2) = N\left(-\frac{\ln\left[\frac{V_0}{De^{-rT}}\right] + \frac{1}{2}\sigma_V^2 T}{\sigma_V\sqrt{T}} + \sigma_V\sqrt{T} \right)$$

$$= N\left(-\frac{\ln\left[\frac{1,300,00}{1,000,000 \times e^{-0.05 \times 90/365}}\right] + \frac{1}{2}0.3^2 \times 90/365}{0.3 \times \sqrt{90/365}} + 0.3 \times \sqrt{90/365} \right)$$

$$= N(-1.7695) = 3.84\%.$$ [15]

The Merton model can be found at www.dersoft.com/Mertonmodel.xls.

The Merton Put

The value of credit risk and the probability of a company's default in Merton's model can also be found by expressing credit risk with the help of a put option on the assets of the company: The equity holders can hedge the credit risk by buying a put on the assets with strike D, the put seller being the asset holders. In case of default, i.e. $V < D$, the equity holders will deliver the assets to the asset holders, the loss for the asset holders being $D - V$. Thus, the put option can be expressed as in equation (5.18)

$$P_0 = -V_0 N(-d_1) + De^{-rT} N(-d_2) \tag{5.18}$$

where

$$d_1 = \frac{\ln\left[\frac{V_0}{De^{-rT}}\right] + \frac{1}{2}\sigma_V^2 T}{\sigma_V \sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma_V \sqrt{T}$$

where P_0 is the current value of a put option on the company's assets V with strike D, and other variables are as defined in equation (5.9).

The equity holders will exercise the put option in equation (5.18) at time T if $D > V$. In the Merton model, this is the case of bankruptcy. Thus, the probability of exercising the put, which is $N(-d_2)$, is again the probability of default.

Rewriting equation (5.18) as $P_0 = \left(-\frac{N(-d_1)}{N(-d_2)} V_0 + De^{-rT}\right) N(-d_2)$ results in an intuitive interpretation of the default risk: The term $\frac{N(-d_1)}{N(-d_2)} V_0$ reflects the amount retrieved of the asset value V_0 in case of default, thus the recovery value. The term De^{-rT} is the present value of the debt, thus $\left(-\frac{N(-d_1)}{N(-d_2)} V_0 + De^{-rT}\right)$ is the present value of the loss in the event of default. Multiplying $\left(-\frac{N(-d_1)}{N(-d_2)} V_0 + De^{-rT}\right)$ with the probability of default $N(-d_2)$ gives the present value of the default risk, which equals the put value P_0.

The put option in equation (5.18) serves as a basis to find a closed form solution for the value of the underlying risky bond B. We can start by expressing B_0 as the debt D to be repaid at time T discounted by e^{-rT} minus the value of the credit risk, which is the put in equation (5.18):

$$B_0 = D_T e^{-rT} - [-V_0 N(-d_1) + D_T e^{-rT} N(-d_2)].$$

Using simple algebra and $1 - N(-d_2) = N(d_2)$ results in the value of the risky bond of:

$$B_0 = D_T e^{-rT} N(d_2) + VN(-d_1) \tag{5.19}$$

where

$$d_1 = \frac{\ln\left[\frac{V_0}{De^{-rT}}\right] + \frac{1}{2}\sigma_V^2 T}{\sigma_V \sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma_V \sqrt{T}.$$

The reader should keep in mind that deriving the risky bond price (and return) is crucial since we can derive the default swap premium from equation (5.1), Default swap premium (p.a.) = Return on risky bond – Return on risk-free bond, once we have derived the risky bond return. However, the reader should keep the restrictions of the simple equation (5.1) in mind, which are, among others, that counterparty risk is not included and we assume that interest rate changes affects the risky and risk-free bond to the same extent.

Merton's model using equity as proxy

One drawback of Merton's elegant model is that we need the asset value V and the asset volatility σ_V as inputs. Both parameters are not easily available in practice. However the equity value E and the equity volatility σ_E are observable. Using equation (5.17) and equation (5.20) derived from Ito's lemma:

$$E_0 = \frac{N(d_1)V_0\sigma_V}{\sigma_E} \tag{5.20}$$

we have two equations with two unknowns to solve for, V and σ_V.

Example 5.9: The equity of company X is currently worth \$2,000,000. In one year, company X has to repay \$1,800,000 in debt. The volatility of the equity is 80% and the risk-free interest rate is 5%. What is the probability of default in 1 year on the basis of the Merton model using equity volatility as a proxy?

Solving equations (5.17) and (5.20) iteratively, we derive the value of assets V_0 = \$3,693,544 and the volatility of the assets as σ_V = 44.45%. Inputting these values into equation (5.17), it follows that d_2 = 1.5073 and $N(-d_2) = 1 - N(d_2)$ = 6.59%. Thus, the probability of default in 1 year is 6.59%.

The reader may also verify equation (5.19) $B_0 = D_T e^{-rT} N(d_2) + VN(-d_1)$. The bond value B_0 comes out to be $1{,}800{,}000 \times e^{-0.05\times1} \times 0.9341 + 3{,}727{,}549 \times 0.0255$ = \$1,693,544. This is identical with the present value of the debt $D_0 = V_0 - E_0$ = \$3,693,544 – 2,000,000 = \$1,693,544.

The Merton model using equity volatility can be found at http://www.dersoft.com/Mertonequity.xls.

An interesting feature of the Merton model is that equity holders will benefit from an increase in volatility of the assets. In this case, the time value of equity will increase (see figure 5.12).

As mentioned above, the path-breaking Merton model serves as a basis for structural and reduced form models that value credit risk. However, the model is quite simple in a number of respects. It principally only allows default at the maturity of the debt T, and the debt can only take the form of zero-coupon bonds. Coupons as well as different seniorities cannot be handled. There is only one bankruptcy event, which occurs when the asset value falls below the value of the debt at maturity of the debt. Other bankruptcy events such as illiquidity, restructuring of debt, or a moratorium are not taken into account.

Due to the simplicity of the Merton model, it is not surprising that empirical testing of the default probability $N(-d_2)$ or the bond price equation (5.19) have overall not produced good results.[16] Nevertheless, the Merton model has served as an excellent basis for developing more realistic, complex models.

Extensions of the Merton model: first-time passage models

The Merton model uses the arbitrage-free Black-Scholes-Merton environment to derive the probability of default. The model principally only observes two points in time: today and option maturity. As a consequence the model can only evaluate European style options, i.e. options that can only be exercised at option maturity. The model does not apply to American style options, thus premature exercise is not accounted for. Therefore the original Merton model principally only evaluates the possibility of default at option maturity T, which corresponds to the maturity date of the debt.

This drawback was addressed by numerous authors and has led to the emergence of *first-time passage models*. These models define a typically exogenous default boundary in units of the asset value V, V_d. If the value of the assets V falls below the boundary V_d, the company is forced to restructuring or bankruptcy. Hence, default can occur at any time during the period of the debt. Let's have a closer look at first-time passage models.

The Black-Cox 1976 model

One of the first to discuss first-time passage models were Black and Cox in 1976.[17] Black and Cox suggest an exogenous exponential default boundary of $V_d = ke^{-\gamma(T-t)}$, where k and γ are exogenous constants. If the asset value V drops below V_d during time t to T, the asset holders can force the company into bankruptcy or restructuring. The mandatory bankruptcy or restructuring, expressed as a *safety covenant* of the asset holders, is an important feature of the model. It protects asset holders from further deterioration of the company's assets. In that sense a high value of k and a low value of γ forces early bankruptcy or restructuring and principally protects asset holders.

Besides safety covenants, Black and Cox also investigate subordination arrangements and restrictions for the equity holders to finance interest and dividend payments. All three provisions tend to increase the value of the risky bond.

Black and Cox also find a closed form solution for the risky bond B, which includes (continuous) dividends, a, to the stockholders:

$$B = Ne^{-rT}[N(z_1) - y^{2\theta-2}N(z_2)] + Ve^{-aT}[N(z_3) + y^{2\theta}N(z_4) + y^{\theta+\xi}e^{aT}N(z_5) + y^{\theta-\xi}e^{aT}N(z_6) - y^{\theta-\eta}N(z_7) - y^{\theta-\eta}N(z_8)] \quad (5.21)$$

where N is the notional amount of the bond, r the continuously compounded interest rate, T the maturity of the bond, and N cumulative normal standard distribution and:

$y = ke^{-\gamma T}/V$; k and γ exogenous

V: asset value

$\theta = (r - a - \gamma + 0.5\sigma^2)/\sigma^2$

a: continuously compounded dividends

σ^2: variance of the return of the firm

$$\delta = (r - a - \gamma - 0.5\sigma^2)^2 + 2\sigma^2(r - \gamma)$$

$$\xi = \sqrt{\delta}/\sigma^2$$

$$\eta = \sqrt{\delta - 2\sigma^2 a}/\sigma^2$$

$$z_1 = [\ln V - \ln PA + (r - a - 0.5\sigma^2)T]/\sqrt{\sigma^2 T}$$

$$z_2 = [\ln V - \ln PA + 2\ln y + (r - a - 0.5\sigma^2)T]/\sqrt{\sigma^2 T}$$

$$z_3 = [\ln V - \ln PA - (r - a - 0.5\sigma^2)T]/\sqrt{\sigma^2 T}$$

$$z_4 = [\ln V - \ln PA + 2\ln y + (r - a + 0.5\sigma^2)T]/\sqrt{\sigma^2 T}$$

$$z_5 = [\ln y + \xi\sigma^2 T]/\sqrt{\sigma^2 T}$$

$$z_6 = [\ln y - \xi\sigma^2 T]/\sqrt{\sigma^2 T}$$

$$z_7 = [\ln y + \eta\sigma^2 T]/\sqrt{\sigma^2 T}$$

$$z_8 = [\ln y - \eta\sigma^2 T]/\sqrt{\sigma^2 T}.$$

The asset process in the Black-Cox model is $dV/V = (\mu - c)dt + \sigma_1 dz_1$, where c is the payout of the risky bond. The underlying interest rate process and the recovery rate are rather simple. Interest rates do not follow a stochastic process but are assumed constant at a rate r, and the recovery rate is simply set to the asset value V at the time of default.

The Kim, Ramaswamy, and Sundaresan 1993 model

Kim, Ramaswamy, and Sundaresan (1993)[18] use a simpler default boundary but a more realistic stochastic interest rate process than Black and Cox. Default is triggered if the asset value drops below an exogenous constant w. The interest rate process follows the risk-neutral Cox-Ingersoll-Ross model:[19] $dr = a(b - r)dt + \sigma_1\sqrt{r}dz_1$, where r = interest rate, a = mean reversion factor, b = long term average of r, σ_1 = volatility of r, dz = Wiener process as defined in equation (5.5). In the Cox-Ingersoll-Ross model, interest rates mean-revert with rate a to the long-term average of rates, b. Since the interest rate r is taken to the square root, the model has the convenient property that interest rates cannot get negative.

The default boundary in the Kim-Ramaswamy-Sundaresan model is $1/(c\gamma)$, where c is the coupon rate and γ is the cash outflow of the firm. Thus the default boundary is endogenous but not time-dependent as in the Black-Cox model. The recovery rate is the minimum of the asset V and the face value of the debt D, if default occurs at maturity of the debt D: RR(T) = min(V, D). If however, default occurs before the debt maturity T, the recovery rate is the minimum of an exogenous recovery rate expressed in percent of a risk-free bond and the asset value: RR(t < T) = min(wP, V), where w is the exogenous recovery rate, P is the price of the risk-free bond.

Using a generalized Wiener process of the form $dV/V = (\mu - \gamma)dt + \sigma_2 dz_2$ for the asset value process V, where μ is the usual expected rate of return of V, and γ equals the payout of the firm, Kim, Ramaswamy, and Sundaresan derive a partial differential equation for coupon bonds B:

$$B=\frac{1}{2}\sigma_1^2V^2\frac{\partial^2 B}{\partial V^2}+\rho\sigma_1\sigma_2\sqrt{r}V\frac{\partial V}{\partial r\partial V}+\frac{1}{2}\sigma_2^2 r\frac{\partial^2 B}{\partial r^2}+a(b-r)\frac{\partial B}{\partial r}+(r-\gamma)V\frac{\partial B}{\partial V}-rB+c \tag{5.22}$$

where c is the coupon of the risky bond B.

Equation (5.22) has no closed form solution. However, Kim, Ramaswamy, and Sundaresan test it numerically and derive significantly better results in deriving realistic default swap premiums than the original Merton model.

The Longstaff-Schwartz 1995 model

In an often-cited paper, Longstaff and Schwartz (1995)[20] suggest a first-time passage model with an exogenous and constant default boundary k and an exogenous and constant recovery rate w. For the interest rate process, Longstaff and Schwartz use the well-known Vasicek model:[21] $dr = a(b - r)dt + \eta\sigma_1 dz_1$, where r = interest rate, a = mean reversion factor, b = long-term average of r, η = exogenous constant, σ_1 = volatility of r, and dz = Wiener process as defined in equation (5.5). The asset value also follows a generalized Wiener process as in equation (5.5): $dV/V = \mu dt + \sigma_2 dz_2$.

With the help of the closed form solution for a zero-coupon bond derived in the Vasicek model, Longstaff and Schwartz find a solution for the price of risky zero-coupon bonds and floating rate bonds. The equation for the risky zero-coupon bond B is:

$$B(V,k,r,T)=P(r,T)-wP(r,T)Q(V,k,r,T) \tag{5.23}$$

where B = Price of a risky bond; k = boundary for asset value V, if $V < k$ restructuring or default occurs; r = risk-free interest rate, T = maturity of risky bond B; P = Price of a risk-free bond; $w = 1$ − recovery rate; and

$$Q=\sum_{i=1}^{n}\left(N(\alpha_i)-\sum_{j=1}^{i-1}q_jN(\beta_{i,j})\right)$$

where

$$\alpha_i=\frac{-\ln X-M(iT/n,T)}{\sqrt{S(iT/n)}}$$

and

$$\beta_{ij}=\frac{M(jT/n,T)-M(iT/n,T)}{\sqrt{S(iT/n)-S(jT/n)}}$$

where

$$M(t,T) = \left(\frac{a-\rho\sigma\eta}{b} - \frac{\eta^2}{b^2} - \frac{\sigma^2}{2}\right)t + \left(\frac{\rho\sigma\eta}{b^2} + \frac{\eta^2}{2b^3}\right)e^{-bT}e^{(bt-1)}$$
$$+\left(\frac{r}{b} - \frac{a}{b^2} + \frac{\eta^2}{b^3}\right)(1-e^{-bt}) - \left(\frac{\eta^2}{2b^3}\right)e^{-bT}(1-e^{-bt})$$

$$S(t) = \left(\frac{\rho\sigma\eta}{b} + \frac{\eta^2}{b^2} + \sigma^2\right)t - \left(\frac{\rho\sigma\eta}{b^2} + \frac{2\eta^2}{b^3}\right)(1-e^{-\beta t}) + \left(\frac{\eta^2}{2b^3}\right)(1-e^{-2bt})$$

where a and b are parameters of the risk-free bond from the Vasicek model, ρ is the instantaneous correlation coefficient between the Wiener processes dz_1 and dz_2, and the passage of time integral Q(V,k,r,T) is the limit of Q(V,k,r,t,n) as $n \to \infty$.

Equation (5.23) is quite intuitive. P(r,T) is the value of the risk-free bond. Subtracted from P(r,T) is the discount for the risk of the bond B, which consists of two terms: wP(r,T) is the amount of the write-down in case of default, which is weighted with the probability of default Q(V,k,r,T).

However, the closed form solution (5.23) is obviously quite cumbersome and it is difficult to calibrate the parameters w, α, β, η, σ, and ρ so that the credit-spreads found in the market are matched.

Longstaff and Schwartz test their model with a simple linear regression of the form $\Delta s = a + b\Delta y + cI + \varepsilon$, where s is the credit-spread, y is the yield of the 30-year Treasury bond and I is the return of the firm's equity or asset index.

Key findings of Longstaff and Schwartz are that $b < 0$ and $c < 0$. $b < 0$ implies that credit-spreads decrease when the risk-free Treasury rate increases. This appears counterintuitive but can be explained by the fact that a higher interest rate means a higher growth rate μ of the asset value V. As a consequence of the higher asset value the probability of default is lower, and with it the credit-spreads.

The inverse relationship between long term risk-free interest rates and credit-spread is stronger for firms with lower credit quality. This is intuitive since a strong growth in the asset value V can improve the asset-liability relationship of a low rated firm to a significant degree.

$c < 0$ implies that the higher the value of assets or equity the lower the credit-spread. This is an anticipated result. Again the inverse relationship is higher for lower rated firms.

Drawbacks of the Longstaff-Schwartz model are the complex parameter calibration of the numerous parameters for the bond equations, and the fact that the underlying Vasicek model for interest rates is generally not arbitrage-free.

The Briys-de Varenne 1997 model[22]

In 1997, Briys and de Varenne addressed shortcomings of the Black-Cox, Kim-Ramaswamy-Sundaresan, and Longstaff-Schwartz models. In these models, the payoff to bondholders in case of bankruptcy may be larger than the firm's asset value. In this respect, payoff demands of the equity holders are not taken into account. Consequently, Briys and de Varenne suggest a default boundary and recovery rate, which guarantee that the payoff to bondholders at the time of default is realistic with respect to demands from the equity holders, and cannot be higher than the firm's asset value.

The default boundary is set at $V_d = kFP$, where k is an exogenous constant, F is the face value of the risky bond and P is the price of the risk-free bond. If $k = 1$, there is no risk for the bondholders since default is triggered at a value of FP. Hence, assuming $P = 1$, bond holders will receive the face value of the bond F. The other extreme is $k = 0$. In this case there is no exogenous default boundary V_d and the model corresponds to the original Merton model, where the default boundary is $V_T < D$.

To guarantee that bondholders can receive the payoff as defined in the model, Briys and de Varenne address the issue of the *strict priority rule*. The strict priority rule states that bondholders receive all of the remaining assets in case of bankruptcy and stockholders receive nothing. Let f_1 $(0 \le f_1 \le 1)$ and f_2 $(0 \le f_2 \le 1)$ be fractions of the remaining assets at default, f_1 denotes the fraction of the asset value if default occurs before maturity, f_2 denotes the fraction of the assets if default occurs at maturity. If $f_1 = f_2 = 1$, the strict priority rule applies, since the fraction paid to the bondholders is constant in time, i.e. there is no bargaining process between the equity holders and the asset holders in time. In reality, though, the strict priority rule often does not apply, thus $f_1 \neq f_2$.

Interest rates in the Briys-de Varenne model follow the stochastic process $dr = a(t)[b(t) - r_t]dt + \sigma_1(t)dz_1$. This resembles closely the extended Vasicek or Hull-White model of $dr = a(b - r_t)dt + \sigma_1 dz_1$. The difference lies in the fact that Briys and de Varenne model the mean reversion, a, the long term average, b, and the volatility, σ_1, as deterministic functions of time a(t) and $\sigma(t)$, which was discussed by Hull and White in 1990.[23]

For the assets of the company, Briys and de Varenne choose the stochastic process $\frac{dV}{V} = r\,dt + \sigma\left[\rho dz_1 + \sqrt{1-\rho^2}\,dz_2\right]$, where ρ is the correlation between the assets and the interest rates.

Following these assumptions, a closed form solution for the price of the risky bond B is derived:

$$B = FP(0,T)\Bigg[1 - \text{Put}(\ell_0, 1) + \text{Put}\left(q_0, \frac{\ell_0}{q_0}\right) - (1-f_1)\ell_0\left(N(-d_3) + \frac{N(-d_4)}{q_0}\right) - (1-f_2)\ell_0\left(N(d_3) - N(d_1) + \frac{N(d_4) - N(d_6)}{q_0}\right)\Bigg] \tag{5.24}$$

where

$$d_1 = \frac{\ln \ell_0 + \Phi(T)/2}{\sqrt{\Phi(T)}} = d_2\sqrt{\Phi(T)}$$

$$d_3 = \frac{\ln q_0 + \Phi(T)/2}{\sqrt{\Phi(T)}} = d_4\sqrt{\Phi(T)}$$

Table 5.2: Key features of the original Merton model in comparison with first-time passage models

Model	Asset process	Interest rate process	Closed form solution for risky bond	Default Boundary	Recovery Rate
Merton 1974	$dV/V = \mu\, dt + \sigma\, dz$	constant r	yes eq. (5.19)	$V_T = D_T$	$N(-d_1)/N(-d_2)$
Black-Cox 1976	$dV/V = (\mu - c)\, dt + \sigma_1\, dz_1$	constant r	yes eq. (5.21)	$V_d = ke^{-\gamma T}$	V
Kim, Ramaswamy & Sundaresan 1993	$dV/V = (\mu - \gamma)\, dt + \sigma_2\, dz_2$	CIR: $dr = a(b - r)dt + \sigma_1\sqrt{r}\, dz_1$	no, see eq. (5.22)	$V_d = c/\gamma$	before T: min(wP, V) at T: min (V, D)
Longstaff-Schwartz 1995	$dV/V = \mu\, dt + \sigma_2\, dz_2$	Vasicek: $dr = a(b - r)dt + \eta\sigma_1\, dz_1$	yes eq. (5.23)	$V_d = k$	w
Briys-de Varenne 1997	$dV/V = rdt + \sigma\,[\rho dz_1 + \sqrt{1-\rho^2}dz_2]$	Hull-White: $dr = a(t)[b(t) - r_t]\, dt + \sigma_1(t)\, dz_1$	yes eq. (5.24)	$V_d = kFP$	before T: f_1V_d at T: f_2V_T

$$d_5 = \frac{\ln q_0^2 + \Phi(T)/2}{\sqrt{\Phi(T)}} = d_6\sqrt{\Phi(T)}$$

$$\Phi(T) = \int_0^T \left[(\rho\sigma_V + \sigma_P(t,T))^2 + (1-\rho^2)\sigma_V^2\right]dt$$

$$\ell_0 = \frac{V_0}{FP(0,T)} \quad \text{and} \quad q_0 = \frac{V_0}{kFP(0,T)}$$

$$Put(\ell_0, 1) = -\ell_0 N(-d_1) + N(-d_2) \quad \text{and} \quad Put\left(q_0, \frac{\ell_0}{q_0}\right) = -q_0 N(-d_5) + \frac{\ell_0}{q_0} N(-d_6).$$

Equation (5.24) is quite intuitive: assuming the face value of the risky bond F = 1, the term FP(0,T) is the price of the risk-free bond. Deducted from FP(0,T) is the standard Merton put $(\ell_0, 1)$, which reflects the default risk of the bond B. Added to FP(0,T) is a $Put\left(q_0, \frac{\ell_0}{q_0}\right)$, which mirrors the safety covenant, i.e. that bondholders can trigger default in the event of $V_d = kFP(0,T)$. The terms including f_1 and f_2 reflect the strict priority rule. In case the strict priority rule applies, i.e. $f_1 = f_2 = 1$, the terms cancel out. Additionally, if $q_0 = \ell_0$, implying k = 1, the two put options cancel out and the bond B is risk-free, B = FP(0,T), as derived above.

Table 5.2 sums up the crucial features of the original Merton model and the first time passage models.

Critical appraisal of first-time passage models

The major achievement of first-time passage models is that unlike in the original Merton model, default before the maturity of the debt at time T is possible. However, several significant drawbacks remain. First, with the exception of the Kim-Ramaswamy-Sundaresan model, the default boundary involves an exogenous constant. Furthermore, the recovery rate of the models, with the exception of the Black-Cox model, also involves an exogenous constant. Consequently the default boundary and recovery rate are difficult to determine for practical purposes.

In addition, the closed form solutions for the risky bond price, equations (5.21) through (5.24) are quite complex and the calibration of the numerous parameters to match market credit-spreads is difficult in trading practice. Other shortcomings of the first-time passage models include the fact that some underlying stochastic processes for the asset value (e.g. CIR and Vasicek) are generally not arbitrage-free. Altogether, these drawbacks have so far limited the use of first-time passage models in credit risk practice.

Reduced Form Models

So far we have discussed two of the three basic concepts that derive the value of credit risk: traditional models (which use historical data) and structural models (which use the evolution of the asset-liability structure of a company to derive the value of credit risk).

Let's now discuss the third approach termed *reduced form models* also called *intensity models*. They are called reduced form, since they abstract from the explicit economic reasons for the default, i.e. they do not include the asset-liability structure of the firm to explain the default.

Rather, reduced form models use debt prices as a main input to model the bankruptcy process. Default is modeled by a stochastic process with an exogenous *default intensity* or *hazard rate*, which multiplied by a certain time frame, results in the risk-neutral default probability, also called pseudo- or martingale default probability. The value of hazard rate is derived by calibration of the variables of the stochastic process. Since reduced form models only model the timing of the default not the severity, the recovery rate is usually exogenous. Let's discuss several crucial reduced form models used in today's credit risk practice.

The Jarrow-Turnbull 1995 model[24]

Jarrow and Turnbull were one of the first to derive the value of credit risk and to price credit derivatives in the arbitrage-free reduced form model environment.

Jarrow and Turnbull combine a process for risk-free interest rates and a bankruptcy process of the risky debt to derive default probabilities and credit derivatives prices. The two processes are assumed to be independent from each other.

Let's define P as the price of the risk-free zero-coupon bond with notional amount 1 and maturity at time 2. π_0 is risk-neutral probability of an interest rate increase. This brings us to the interest rate tree in figure 5.13.

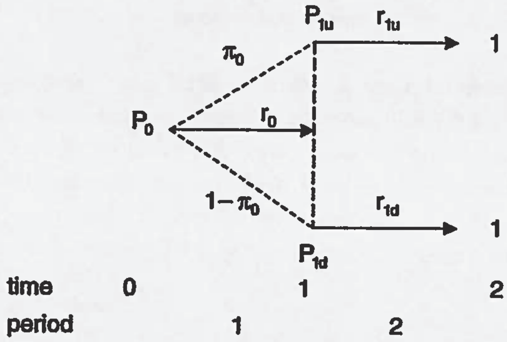

Figure 5.13: Risk-free interest rate tree in the Jarrow-Turnbull model
r = risk-free interest rate, P = risk-free zero-coupon bond price

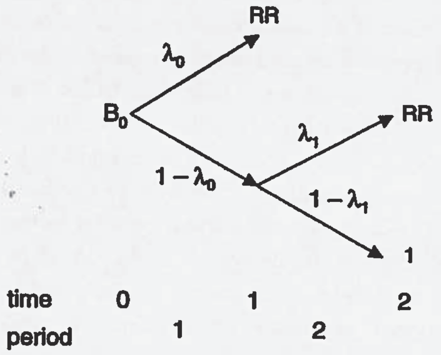

Figure 5.14: Bankruptcy process of risky bond B in the Jarrow-Turnbull model

The risk-free bond price at time t with maturity T, is $P_{t,T} = 1/(1 + r_{t,T})$. Since $r_{1u} > r_{1d}$, it follows that $P_{1d} > P_{1u}$.

Let B be the price of a risky zero-coupon bond with a notional amount of 1 and maturity at time 2. Let λ be the risk-neutral probability of default,[25] 1-λ the risk-neutral probability of survival, and RR the recovery rate in case of default. Thus, we derive the default process for the risky bond B in figure 5.14.

Combining figures 5.13 and 5.14, we get the quadruple tree in figure 5.15.

In figure 5.15, $B_{t,T}$ is the price of the risky bond at time t with maturity T. The recovery rate RR is exogenous and assumed independent from the bankruptcy process, and the interest rate process. r_1 is the risk-free forward interest rate from time 1 to time 2.

At time 1, the risky bond price takes the recovery value RR in case of default. If in default, it is assumed that the bond stays in default, thus the probability 1 from time 1 to time 2. The recovery rate RR is invested with the risk-free forward rate r_{u1} or r_{d1}, thus values $RR(1 + r_{1u})$ or $RR(1 + r_{1d})$ at time 2. In case of no default at time 1, the bond can take prices $B_{1,2,c}$ and $B_{1,2,d}$. $B_{1,2,c} < B_{1,2,d}$ since interest rates have increased in case of $B_{1,2,c}$ with probability π_0.

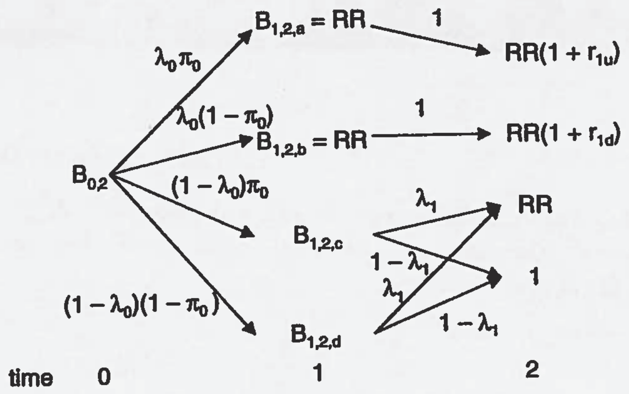

Figure 5.15: A combined interest rate and bankruptcy process

The reader should note that π_1 and $1 - \pi_1$ are not necessary in the 2-period tree, since the probabilities π_0 and $1 - \pi_0$ determine the interest rates from time 1 to time 2. The values for r and π can be generated by any short-rate model: Ho-Lee (1986); Cox-Ingersoll-Ross (1985); Vasicek (1977); Hull-White (1990); or Black-Derman-Toy (1990).

Jarrow and Turnbull show that their model is complete, i.e. that the derivatives can be replicated by primary products. In addition, the unique, risk-neutral or pseudo-probabilities λ and π guarantee that the prices P and B are martingales, thus the model is arbitrage-free. Furthermore, the Markov property allows displaying the combined interest rate and bankruptcy tree as a recombining tree.

Jarrow and Turnbull use a foreign exchange rate analogy to model the risky bond price B. The risky bond price at any time t with maturity T, $B_{t,T}$, is equal to the risk-free bond price $P_{t,T}$ multiplied with the "exchange rate" e, which is 1 in case of no default and equal to the recovery rate RR in case of default. Thus $B_{t,T} = P_{t,T}e_t$. If $E(e_T)$ is the expected payoff at time T, the risky bond price can be expressed as

$$B_{t,T} = P_{t,T}E(e_T). \tag{5.25}$$

Equation (5.25) states that the risky bond price is the expected payoff $E(e_T)$ discounted by the risk-free price $P_{t,T}$.

The probability of default

In the Jarrow-Turnbull model, the risk-neutral probability of default in period 1, realized at time 1, λ_0, can be derived separately from the interest rate process, since it is assumed that the interest rate process and the bankruptcy process are independent. Hence, from figure 5.14, for a one-period debt with a notional amount of \$1, we get:

$$B_{0,1} = P_{0,1}[\lambda_0 RR + (1 - \lambda_0)1]. \tag{5.26}$$

Equation (5.26) states that the risky bond price with maturity 1, $B_{0,1}$, is derived as the probability weighted values at time 1, discounted with the risk-free bond $P_{0,1}$. Solving equation (5.26) for λ_0 gives:

$$\lambda_0 = \frac{1 - \frac{B_{0,1}}{P_{0,1}}}{1 - RR}. \tag{5.27}$$

For the risk-neutral default probability in period 1, realized at time 2, λ_1, we receive from figure 5.14:

$$B_{0,2} = P_{0,1}\lambda_0 RR + P_{0,2}(1-\lambda_0)[\lambda_1 RR + (1-\lambda_1)].^{26} \tag{5.28}$$

Solving equation (5.28) for the risk-neutral default probability at time 2, λ_1, we get:

$$\lambda_1 = \frac{\frac{B_{0,2} - P_{0,1}\lambda_0 RR}{P_{0,2}(1-\lambda_0)} - 1}{RR - 1}. \tag{5.29}$$

This derivation of λ_0 and λ_1 is quite similar to that of the binomial tree in equations (5.3) and (5.4). The difference is that in the binomial tree in equations (5.3) and (5.4), the nature of the risky bond was incorporated via the swap premium s, whereas Jarrow and Turnbull use the bond price B to incorporate the risk. Also, in equations (5.3) and (5.4) all values were compared at time 2, not at time 0, as in equations (5.26) to (5.29). Also, in the binomial tree we used the spot interest rate r_0 and the forward rate r_1 to derive future values. In equation (5.26) to (5.29) we discount with the risk-free zero bond price P. Furthermore, in the binomial tree in equations (5.3) and (5.4), the swap premium s is effectively a coupon, whereas in the Jarrow-Turnbull equations (5.26) to (5.29) the underlying risky and risk-free bonds have no coupon.

Let's now derive the probability of default in the Jarrow-Turnbull model in an example.

Example 5.10: Let's assume that the probability of default in period 1 was derived with equation (5.27) as 4%. The risk-free bond prices for bonds with maturity 1 and 2 are 99 and 98 respectively. The risky bond price with maturity 2 is 91. The recovery rate is assumed to be 30%. What is the probability of default in period 2, realized at time 2, in the Jarrow-Turnbull model?

Following equation (5.29) it is:

$$\frac{\frac{91 - 99 \times 0.04 \times 0.3}{98 \times (1 - 0.04)} - 1}{0.3 - 1} = 6.48\%.$$

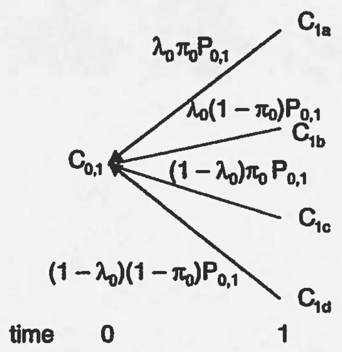

Figure 5.16: Deriving the call price at time 0 with maturity 1, $C_{0,1}$, in the Jarrow-Turnbull model

Pricing options in the Jarrow-Turnbull model

The call and put option price in the Jarrow-Turnbull model can easily be derived on the basis of the quadruple tree in figure 5.15. As in a standard binomial model, we first derive the underlying price tree, in this case the zero-coupon bond price tree. We then create an option price tree and discount back from the last node to find the present value of the option.

We have already derived the bond price tree for a bond with a notional amount of 1 and a maturity of 2 in figure 5.15. The bond prices $B_{1,2,a}$ and $B_{1,2,b}$ in case of default at time 1, regardless whether interest rates have gone up or down, are RR. The bond prices $B_{1,2,c}$ and $B_{1,2,d}$ in figure 5.15 are:

$$B_{1,2,c} = P_{1,2,u}[\lambda_1 RR + (1-\lambda_1)] \tag{5.30}$$

$$B_{1,2,d} = P_{1,2,d}[\lambda_1 RR + (1-\lambda_1)]. \tag{5.31}$$

$P_{1,2,u}$ is the risk-free forward bond price from time 1 to time 2 in case of an interest rate increase in period 1; $P_{1,2,d}$ is the risk-free forward bond price from time 1 to time 2 in case of an interest rate decrease in period 1. Recall that $P_{t,T} = 1/(1 + r_{t,T})$ and $P_{1,2,u} < P_{1,2,d}$.

Pricing a call: Having derived the bond prices, we can now build the option tree to derive the option price. Let's start with a call. The call price tree for a call with maturity 1 and an underlying bond with maturity bigger than time 1 is seen in figure 5.16.

Since time 1 is the maturity of the call, the call price at time 1 is simply the intrinsic value $\max(B_{t,T} - K, 0)$, where K is the strike price. Thus $C_{1,a} = \max(B_{1,2,a} - K, 0)$ as is $C_{1,b} = \max(B_{1,2,b} - K, 0)$, $C_{1,c} = \max(B_{1,2,c} - K, 0)$, and $C_{1,d} = \max(B_{1,2,d} - K, 0)$. From figure 5.16, we can see that the call price at time 0 is equal to the call at time 1, discounted with $P_{0,1}$ and weighted with the default probability λ_0 and the interest rate probability π_0:

$$C_{0,1} = [\lambda_0\pi_0 C_{1a} + \lambda_0(1-\pi_0)C_{1b} + (1-\lambda_0)\pi_0 C_{1c} + (1-\lambda_0)(1-\pi_0)C_{1d}]P_{0,1}. \tag{5.32}$$

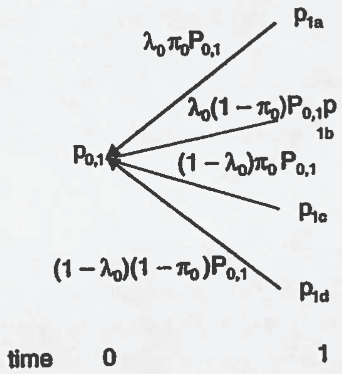

Figure 5.17: Deriving the put price at time 0 with maturity 1, $p_{0,1}$, in the Jarrow-Turnbull model

Example 5.11: Let's assume from the term-structure based model we have derived that $\pi_0 = 0.6$, $P_{0,1} = 0.94$, $P_{1,2,u} = 0.93$, and $P_{1,2,d} = 0.96$ where $P_{1,2,u}$ is the forward price from time 1 to time 2 in case of an upward move of interest rates, and $P_{1,2,d}$ is the forward price from time 1 to time 2 in case of a downward move of interest rates. Let's further assume that as in example 5.10, the probability of default in period 1, λ_0 is 4% and the probability of default in period 2 λ_1 is 6.48%. The recovery rate is assumed to be 40%. What is the price of a call with a strike of 0.85 and a maturity of time 1 on a bond with maturity at time 2 derived on the 2-period Jarrow-Turnbull binomial tree?

We first derive the prices $B_{1,2}$. From figure 5.15 we derive that $B_{1,2,a} = B_{1,2,b} = RR = 0.4$. From equation (5.30), $B_{1,2,c} = 0.93 \times (0.0648 \times 0.4 + (1 - 0.0648)) = 0.8938$, and from equation (5.31) $B_{1,2,d} = 0.96 \times (0.0648 \times 0.4 + (1 - 0.0648)) = 0.9227$. We can now derive $C_{1,a} = \max(0.4 - 0.85, 0) = 0$; $C_{1,b} = \max(0.4 - 0.85, 0) = 0$; $C_{1,c} = \max(0.8938 - 0.85, 0) = 0.0438$; and $C_{1,d} = \max(0.9227 - 0.85, 0) = 0.0727$. Following equation (5.25), we derive the call price as:

$$C_{0,1} = [0.04 \times 0.6 \times 0 + 0.04 \times (1-0.6) \times 0 + (1-0.04) \times 0.6 \times 0.0438 + (1-0.04) \times (1 - 0.6) \times 0.0727] \times 0.94 = 0.04995 \text{ or } 5.00\% \text{ of the notional amount of the bond of } 1.$$

Naturally, the Jarrow-Turnbull model can be extended to multi periods. A multi-period model of the Jarrow-Turnbull approach can be found at www.dersoft.com/jt.xls.

Pricing a put: Pricing a put within the Jarrow-Turnbull framework is similar to pricing a call. The only difference is the intrinsic value at maturity of the option. For a put, the intrinsic value is $\max(K - B_{t,T}, 0)$. Using the same technique as in figure 5.16 we can derive the tree shown in figure 5.17.

From figure 5.17 we derive equation (5.33) for a put p:

$$p_{0,1} = [\lambda_0 \pi_0 p_{1a} + \lambda_0(1-\pi_0)p_{1b} + (1-\lambda_0)\pi_0 p_{1c} + (1-\lambda_0)(1-\pi_0)p_{1d}]P_{0,1}. \quad (5.33)$$

Let's derive the put price in a numerical example.

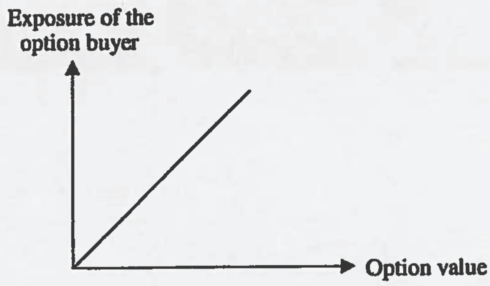

Figure 5.18: Credit exposure of an option buyer with respect to the option value

Example 5.12: Let's use the data from example 5.11. The probability of an upward move of interest rates in period 1 is $\pi_0 = 0.6$, $P_{0,1} = 0.94$, $P_{1,2,u} = 0.93$, and $P_{1,2,d} = 0.96$, where $P_{1,2,u}$ is the forward price from time 1 to time 2 in case of an upward move of interest rates, and $P_{1,2,d}$ is the forward price from time 1 to time 2 in case of a downward move of interest rates. The probability of default in period 1, λ_0 is 4% and the probability of default in period 2 λ_1 is 6.48%. The recovery rate is assumed to be 40%. What is the price of a put with a strike of 0.95 and a maturity of time 1 on a bond with maturity at time 2 derived in the 2-period Jarrow-Turnbull binomial tree?

From figure 5.11 we derive that $B_{1,2,a} = B_{1,2,b} = RR = 0.4$. From equation (5.30), $B_{1,2,c} = 0.93 \times (0.0648 \times 0.4 + (1 - 0.0648)) = 0.8938$ and from equation (5.31) $B_{1,2,d} = 0.96 \times (0.0648 \times 0.4 + (1 - 0.0648)) = 0.9227$. We can now derive $p_{1,a} = \max(0.95 - 0.4, 0) = 0.5500$; $p_{1,b} = \max(0.95 - 0.4, 0) = 0.5500$; $p_{1,c} = \max(0.95 - 0.8938, 0) = 0.0562$ and $p_{1,d} = \max(0.95 - 0.9227, 0) = 0.0273$. Following equation (5.33), we derive the put price as:

$$\begin{aligned} p_{0,1} &= [0.04 \times 0.6 \times 0.5500 + 0.04 \times (1-0.6) \times 0.5500 + \\ &\quad (1-0.04) \times 0.6 \times 0.0562 + (1-0.04) \times (1-0.6) \times 0.0273] \times 0.94 \\ &= 0.060963 \text{ or } 6.10\% \text{ of the notional amount of the bond of 1.} \end{aligned}$$

Pricing vulnerable options in the Jarrow-Turnbull model: In an option contract, the option buyer has counterparty risk, i.e. the risk that the option seller defaults. In that case, the option seller might not be able to meet his obligation to pay the intrinsic value to the option buyer. Figure 5.18 shows the credit exposure of the option buyer.

The option *seller* has no counterparty risk if the option premium is paid upfront. Hence, there is no future obligation from the option buyer to the option seller. An option that includes the aspect of default of the option seller in the valuation is termed *vulnerable option*.

The vulnerable option price in the Jarrow-Turnbull model can be derived easily: The value of the vulnerable option at time t with maturity T, $C_{t,T,V}$, can be expressed as the default-free option $C_{t,T}$ multiplied by the expected value of the bankruptcy process of the risky option seller, $E(e_{T,V})$. Thus we derive:

$$C_{t,T,V} = C_{t,T}E(e_{T,V}). \tag{5.34}$$

$e_{T,V}$ will be equal to 1 in case the vulnerable option seller has not defaulted at time T; $e_{T,V}$ will be equal to the recovery rate if the vulnerable option seller has defaulted at time T. Let's assume that the bankruptcy process of the payoff ratio e_V is independent from the risk-free interest rate process and the default process of the underlying asset. Hence, the option price at time 0 and with maturity 1 is simply the discounted price of the option at time 1:

$$C_{0,1,V} = (C_{1,1}/P_{0,1})E(e_{1,V})$$

or

$$C_{0,1,V} = C_{0,1}E(e_{1,V}).$$

Using equation (5.25) $B_{t,T} = P_{t,T}E(e_T)$, where $B_{t,T}$ is a zero-coupon bond issued by the option seller, we derive the value of the vulnerable option as:

$$C_{0,1,V} = (C_{0,1}B_{0,1})/P_{0,1}. \tag{5.35}$$

Since $B_{0,1}/P_{0,1} \leq 1$, the vulnerable option will always be lower than or equal to an option not including option seller default risk.

Example 5.13: The current bond price of the option seller with maturity 1 is at \$0.90 and the risk-free bond price with the same maturity is \$0.99. In example 5.11 we have derived a call price at time 0 with maturity time 1 excluding counterparty risk of 5.00%. What is the value of a corresponding vulnerable call? Following equation (5.35) with $B_{0,1} = 0.9$ and $P_{0,1} = 0.99$, we derive $C_{0,1,V} = 0.0500 \times 0.90/0.99 = 4.54\%$. Thus, the value of the default risk is 5.00% − 4.54% = 0.46%, or 0.46%/5.00% = 9.2% of the non-vulnerable call value.

Pricing non-vulnerable and vulnerable interest rate swaps in the Jarrow-Turnbull model

Contrary to an option, *both* swap counterparts in a swap have default risk. This is because both counterparts promise to make future payments. If company A has entered into a swap with company B, company A has default risk if the present value of the swap is positive from company A's point of view. Graphically this can be expressed as in figure 5.19.

If the present value of the swap is negative for company A, there is no credit risk for company A, since the swap represents a liability and not an asset.

Pricing non-vulnerable swaps: Let's start our analysis with pricing a non-vulnerable swap. We first have to find the discount factors to discount the future cash flows. In previous analyses, we have used the value of the risk-free bond P to discount. However, in a swap, it is more appropriate to find the discount factors from the swap curve.

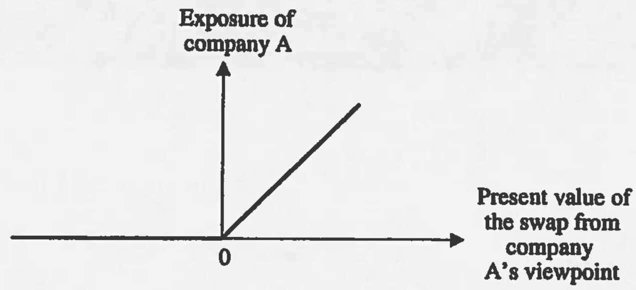

Figure 5.19: Credit exposure of a swap with respect to the value of the swap

Let's assume the 1-year swap rate is 4.60% and the two-year swap rate is 4.64%. The one year swap rate is a zero-rate (no coupons are paid until year 1), so we can derive the discount factor at time 1 easily as $df_1 = 1/(1 + r_1)$. With a 1-year swap rate of 4.60% we derive $df_1 = 1/(1 + 0.046) = 0.956023$. For the 2-year discount factor, assuming the swap rates are paid annually, we have to incorporate the fact that the rate is paid at time 1. We can use equation (5.36):[27]

$$df_n = \frac{df_0 - r_n \sum_{i=1}^{n-1} df_i (t_i - t_{i-1})}{1 + r_n (t_n - t_{n-1})}. \tag{5.36}$$

For the 2-year discount factor n = 2. With $r_2 = 4.64\%$ we derive:

$$df_n = \frac{1 - 0.0464 \times 0.956023 \times 1}{1 + 0.0464 \times 1} = 0.913265.$$

The forward swap rate from time 1 to time 2, $r_{1,2}$, can be derived from $df_1/df_2 - 1$. Thus we obtain $r_{1,2} = (0.956023/0.913265) - 1 = 4.68\%$.

Having derived all discount factors, we can now calculate the swap price. It is the difference between the discounted floating cash flows and the discounted fixed cash flows.

Example 5.14: The 1-year swap rate is 4.60% and the 2-year swap rate is 4.64%. What is the present value of a 2-year swap if the fixed rate is 4.64%? All rates are paid annually. The notional amount of the swap is 1. Figure 5.20 shows this diagrammatically.

In a standard swap, the fixing of the floating payment is done one period prior to the payment. Hence in figure 5.20, the payment of 4.6% at time 1 is fixed today, therefore known. (It is actually not precisely known, when the swap is down in the morning hours before the fixing.) The only stochastic cash flow in figure 5.20 is the floating cash flow at time 2.

Using the previously derived results, the fixed side of the swap has a present value

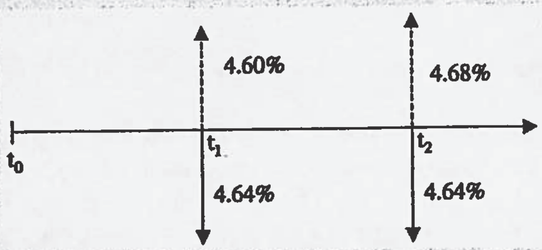

Figure 5.20: A swap with fixed cash flows of 4.64% and floating cash flows of 4.6% and an anticipated cash flow at time 2 of 4.68%

of $0.0464 \times 0.956023 + 0.0464 \times 0.913265 = 0.0867$. The present value of the floating side is $0.046 \times 0.956023 + 0.0468 \times 0.913265 = 0.0867$. Thus the swap has a present value of zero. This is the expected result, since the fixed rate in the swap 4.64% is equal to the swap rate of the maturity of the swap 4.64% (and the forward floating rate of 4.68% is the fair forward rate, which is assumed in swap valuation). This is also referred to as "par swaps value at par."

Pricing vulnerable swaps: A vulnerable swap is a swap which incorporates the possibility of default of the swap counterpart. Let's assume company A is paying a floating rate and is receiving a fixed rate in a swap with company B. If company B (or A) defaults, all future payments are null and void. Let e^* be the payoff in default, then e^* at time t, e^*_t, is zero with probability λ_{t-1}, and e^*_t is 1 with probability $1 - \lambda_{t-1}$. Let $E_0(e^*_t)$ be today's expected value of the payoff at time t. We can then write the value of the two-period vulnerable swap with annual payments and a notional amount of 1, V_{swap}, from the viewpoint of the floating rate payer company A as:

$$Vswap_0 = df_1[rfixed_1 - rfloating_1]E_0(e^*_1) + df_2[rfixed_2 - rfloating_2]E_0(e^*_2) \qquad (5.37)$$

where rfixed represents the fixed cash flows, rfloating represents the floating cash flows, and the df terms are the discount factors.

Example 5.15: Let's assume the 1-year swap rate is 4.60% and the 2-year swap rate is 4.64%. The fixed rate of the swap is 7%. All rates are expressed as annual rates and paid annually. The notional amount of the swap is 1. The probability of default of company B in period 1, λ_0, is 4% and the probability of default in period 2, λ_1, is 6.48%. The discount factors, as derived above, are $df_1 = 0.956023$ and $df_1 = 0.913265$. What is the value of the vulnerable swap for the viewpoint of the floating rate payer A? Following equation (5.37) it is:

$$Vswap_0 = 0.956023 \times (0.07 - 0.0460) \times (1 - 0.04) + 0.913265 \times (0.07 - 0.0468) \times (1 - 0.04) \times (1 - 0.0648) = 4.11\%.$$

Ignoring default risk of counterpart B, the swap would have a value of:

$$0.956023 \times (0.07 - 0.0460) + 0.913265 \times (0.07 - 0.0468) = 4.41\%.$$

Thus, the value of the credit risk is 0.30% of the notional amount. Hence, for a $100,000,000 swap, the value of credit risk is $300,000.

The Jarrow-Turnbull 1995 model in combination with an underlying Cox-Ross-Rubinstein (CRR) interest rate process can be found at www.dersoft.com/jt.xls.

Critical appraisal of the Jarrow-Turnbull 1995 model

The Jarrow-Turnbull 1995 model was one of the first reduced form models that incorporated credit risk in the pricing algorithms of derivatives in a no-arbitrage martingale framework. It is a path breaking article that serves as a basis for most, more elaborate reduced form models used in today's trading practice.

The shortcomings lie in the basic approach of the model: the direct economic reasons for default, i.e. the company's specific asset-liability structure or the company's liquidity are not part of the analysis. Rather, bond prices are the major input, assuming that bond prices can serve to reflect the credit risk of the debtor and to derive default probabilities. However, it has been shown that bond prices overestimate a company's probability of default quite substantially (see e.g. Altman 1989). In addition, bond prices are often quite illiquid, resulting in difficulties in determining a fair mid-market price.

Furthermore, it is assumed that the interest rate process and the default process are independent. Also, the default intensity is assumed constant, thus default is equally likely over the life of the debt. Last, the recovery rate of the model does not depend on the model variables, but is exogenous.

These shortcomings were addressed in extensions of the model, as in the Jarrow-Lando-Turnbull 1997 model.

The Jarrow-Lando-Turnbull 1997 model[28]

In 1997, Jarrow, Lando, and Turnbull derive default probabilities and valuation methods for credit derivatives not from rather illiquid bond prices, but on the basis of historical transition probabilities. The analysis is done within the arbitrage-free martingale framework. However, Markov properties are not mandatory since the martingale transition probabilities, also termed risk-neutral- or pseudo-probabilities, may depend on historical data up to the present. Let's first look at a historical default matrix, as shown in table 5.3.

We can deduce the annual default probability from table 5.3. We simply take the difference in the cumulative default probability for each entry. Doing so, we derive table 5.4.

From table 5.4 we can see that the historical default probability stays constant or increases slightly in time for highly rated credit. However, for low credits such as Caa, the probabil-

Table 5.3: Average global cumulative historical default rates with respect to time (numbers in %)

Year	1	2	3	4	5	6	7	8	9	10
Aaa	0	0	0	0.04	0.12	0.21	0.3	0.4	0.52	0.64
Aa	0.02	0.03	0.07	0.16	0.26	0.36	0.46	0.57	0.65	0.73
A	0.02	0.09	0.22	0.36	0.51	0.68	0.86	1.07	1.31	1.56
Baa	0.22	0.61	1.08	1.69	2.25	2.81	3.38	3.94	4.58	5.26
Ba	1.28	3.51	6.09	8.76	11.36	13.74	15.66	17.6	19.46	21.29
B	6.51	14.16	21.03	27.04	32.31	36.73	40.97	44.33	47.17	50.01
Caa	23.83	37.12	47.43	55.05	60.09	65.22	69.26	73.88	76.50	78.54

Source: Moody's Investor Service, April 2003

Table 5.4: Average global annual default rates with respect to time (numbers in %)

Year	1	2	3	4	5	6	7	8	9	10
Aaa	0	0	0	0.04	0.08	0.09	0.09	0.10	0.12	0.12
Aa	0.02	0.01	0.04	0.09	0.10	0.10	0.10	0.11	0.08	0.08
A	0.02	0.07	0.13	0.14	0.15	0.17	0.18	0.21	0.24	0.25
Baa	0.22	0.39	0.47	0.61	0.56	0.56	0.57	0.56	0.64	0.68
Ba	1.28	2.23	2.58	2.67	2.60	2.38	1.92	1.94	1.86	1.83
B	6.51	7.65	6.87	6.01	5.27	4.42	4.24	3.36	2.84	2.84
Caa	23.83	13.29	10.31	7.62	5.04	5.13	4.04	4.62	2.62	2.04

Source: Moody's Investor Service, April 2003

ity of a default decreases with increasing time. This is reasonable, since for a company with a bad rating, the coming years are the most crucial ones. Once they have passed, it can be assumed that the probability of default declines.

Tables 5.3 and 5.4 only express the probability of a certain credit to move to default, i.e. to move to credit state D. Jarrow, Lando, and Turnbull use a *transition matrix* in their analysis. A transition matrix Λ shows the historical transition probability of a credit in state i to move to a credit in state j, within a certain time frame, thus

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \ldots \lambda_{1D} \\ \lambda_{21} & \lambda_{22} \ldots \lambda_{2D} \\ \vdots & \\ \lambda_{D-1,1} & \lambda_{D-1,2} \ldots \lambda_{D-1,D} \\ 0 & 0 \quad\quad 1 \end{pmatrix}$$

where the transition probabilities $\lambda_{ij} \geq 0$ for all i, j. The probability of default for a certain credit state i, $\lambda_{i,D}$, is in the last column of Λ. The probability of survival for a bond in rating

Table 5.5: One-year historical transition matrix of year 2002 (numbers in %)

		Rating at year-end								
		Aaa	Aa	A	Baa	Ba	B	Caa	Default	WR
Initial Rating	Aaa	86.82	7.75	0	0	0	0	0	0	5.43
	Aa	1.38	82.23	12.12	0.14	0	0	0	0	4.13
	A	0	2.18	82.83	8.86	1.01	0.47	0.08	0.16	4.43
	Baa	0.17	0.17	2.46	79.47	7.55	2.04	1.87	1.19	5.09
	Ba	0	0.18	0.18	2.39	72.38	13.26	2.03	1.47	8.10
	B	0	0	0.14	0.41	2.71	72.9	9.76	4.88	9.21
	Caa	0	0	0	0	0.34	3.42	56.85	27.74	11.64

Source: Moody's Investor Service, April 2003. WR represents companies that had been rated initially but are not rated at year-end

class i, $Q_i = \sum_{j \neq D} q_{i,j} = 1 - \lambda_{i,D}$. The probability of remaining in the same credit state is on the diagonal and is $\lambda_{i,i} = 1 - \sum_{\substack{j=1 \\ i \neq j}} \lambda_{i,j}$.

The last row in Λ expresses that a credit that has defaulted stays in default. Hence, the transition probability 0, and the probability to stay in default is 1. Let's look at a transition matrix in practice, table 5.5.

In table 5.5, 82.83 reflects the probability of a credit, let's assume a bond, which is currently rated A to stay in A; 0.47 reflects the probability of a bond that is currently rate A to migrate to B; 0.14 is the probability of a bond currently rated B to move to A.

Transforming historical default probabilities into martingale probabilities

In the following we will show that it is necessary to transform the historical default probabilities derived from a transition matrix into risk-neutral martingale probabilities in order to satisfy no-arbitrage conditions. We will discuss when to use historical probabilities and when to use martingale probabilities and the associated problems at the end of this section.

Let's assume we have four rating classes, A, B, C, and default D. Let s_{01A}, s_{01B}, and s_{01C} be the credit-spread (the excess yield, so the difference between the yield of the risky bond and the yield of the risk-free bond) from time 0 to time 1 for a risky bond currently in rating class A, B, and C, respectively. Let's assume $s_{01A} = 0.01$, $s_{01B} = 0.015$, and $s_{01C} = 0.02$, hence in matrix form we can write:

$$s_{01} = \begin{pmatrix} 0.01 \\ 0.015 \\ 0.02 \end{pmatrix}.$$

Let s_{02A}, s_{02B}, and s_{02C} be the spread from time 0 to time 2 for a bond currently in rating class A, B, and C, respectively. Let's assume $s_{02A} = 0.02$, $s_{02B} = 0.025$, and $s_{02C} = 0.03$, hence:

$$s_{02} = \begin{pmatrix} 0.02 \\ 0.025 \\ 0.03 \end{pmatrix}.$$

Let's further assume the one-year historical transformation matrix is

$$\Lambda = \begin{pmatrix} & A & B & C & D \\ A & 0.7 & 0.15 & 0.10 & 0.05 \\ B & 0.1 & 0.6 & 0.2 & 0.2 \\ C & 0.05 & 0.15 & 0.65 & 0.15 \\ D & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence 0.7 is the probability of a bond currently rated A to stay in A; 0.2 is the probability of a bond currently rated B to be downgraded to C; 0.05 in the 2nd column and 4th row is the probability of a bond currently rated C to move to A. Let's further assume the risk-free continuously compounded interest rate from time 0 to time 1, $r_{01} = 5\%$ and the risk-free continuously compounded interest rate from time 0 to time 2, $r_{02} = 6\%$. The recovery rate RR is assumed to be 40%.

In a risk-neutral environment, we can express the risky zero-coupon bond price B at time t with maturity T and notional of \$1 as the value of the discounted expected future cash flow of 1. We discount with the risk-free interest rate r plus the swap spread s:

$$B_{t,T} = E_t\left[e^{-(r_{t,T}+s_{t,T})T}\right] \tag{5.38}$$

where E_t is the risk-neutral expectation value at time t, and s is the excess yield of the risky asset.

For a bond with a notional of \$1 that matures at time 1, the payoff at time 1 will be \$1 if the bond finishes in rating class A, B, or C. The payoff will be the recovery rate RR, if the bond defaults. Including the historical default probabilities from the transition matrix, we can express the bond price B at time 0 with maturity 1, which is rated A, B_{01A} as:

$$B_{01A} = e^{-(r_{01}+s_{01})} \equiv e^{-r_{01}} \begin{pmatrix} 1 & 1 & 1 & RR \end{pmatrix} \begin{pmatrix} A \to A \\ A \to B \\ A \to C \\ A \to D \end{pmatrix} \tag{5.39}$$

where $A \to A$ is the historical probability of a bond currently in rating class A to stay in A; $A \to B$ is the historical probability of a bond currently in rating class A to move to B; etc.

It is important to note that equation (5.39) is usually not satisfied in reality. With our numerical values above, we get:

$$B_{01A} = e^{-(0.05+0.01)} \neq e^{-r_{01}} \begin{pmatrix} 1 & 1 & 1 & 0.4 \end{pmatrix} \begin{pmatrix} 0.7 \\ 0.15 \\ 0.1 \\ 0.05 \end{pmatrix}$$

or

$$e^{-(0.05+0.01)} = 0.9418 \neq e^{-0.05} \begin{pmatrix} 1 & 1 & 1 & 0.4 \end{pmatrix} \begin{pmatrix} 0.7 \\ 0.15 \\ 0.1 \\ 0.05 \end{pmatrix}$$

$$= e^{-(0.05)} \times (1 \times 0.7 + 1 \times 0.15 + 1 \times 0.1 + 0.4 \times 0.05) = 0.9227.$$

Hence, in order to satisfy the no-arbitrage condition (5.39), we have to transform the historical transition probabilities into risk-neutral martingale probabilities, which satisfy condition (5.39).

In order to find the martingale probabilities λ_m, we have to adjust the historical probabilities λ with a factor η. η can be interpreted as a risk premium or risk adjustment. We can then rewrite equation (5.39) for a bond currently rated in class A as:

$$B_{01A} = e^{-(r_{01}+s_{01})} = e^{-r_{01}} \begin{pmatrix} 1 & 1 & 1 & RR \end{pmatrix} \begin{pmatrix} 1-(1-(A \to A))\eta_A \\ (A \to B)\eta_A \\ (A \to C)\eta_A \\ (A \to D)\eta_A \end{pmatrix}. \tag{5.40}$$

Generalizing the right side of equation (5.40) for a bond at time t with maturity T and solving for the risk adjustment of that bond in rating class i, η_i (we assume i = {A, B, C, D}), we get:

$$\eta_i = \left\{ 1 - \left(\frac{e^{r_{t,T}}}{e^{(r_{t,T}+s_{t,T})}} \right)^T \right\} \frac{1}{(1-RR)\lambda_{iD}} \tag{5.41}$$

where $\lambda_{i,D}$ is the probability of default of a bond in rating class i.[29]

Example 5.16: Given is the risk-free spot interest rate $r_{0t} = 0.05$, the risk spread of a risky bond in class A, $s_{01A} = 0.01$, class B, $s_{01B} = 0.015$, and class C, $s_{01C} = 0.02$. The recovery rate is assumed to be 40%. The historical transition matrix is given as:

$$
A = \begin{pmatrix} & A & B & C & D \\ A & 0.7 & 0.15 & 0.10 & 0.05 \\ B & 0.1 & 0.6 & 0.2 & 0.1 \\ C & 0.05 & 0.15 & 0.65 & 0.15 \\ D & 0 & 0 & 0 & 1 \end{pmatrix}.
$$

What are the risk-neutral martingale transition probabilities? We first have to derive the risk premiums λ_i. Following equation (5.41) we get:

$$
\eta_A = \left\{1 - \left(\frac{e^{0.05}}{e^{(0.05+0.01)}}\right)^1\right\} \frac{1}{(1-0.4)(0.05)} = 0.3317
$$

$$
\eta_B = \left\{1 - \left(\frac{e^{0.05}}{e^{(0.05+0.015)}}\right)^1\right\} \frac{1}{(1-0.4)(0.1)} = 0.2481
$$

$$
\eta_C = \left\{1 - \left(\frac{e^{0.05}}{e^{(0.05+0.02)}}\right)^1\right\} \frac{1}{(1-0.4)(0.15)} = 0.2200
$$

We now multiply the 1st row of the historical transition matrix with the risk adjustment for a bond in rating class A, 0.3317, the second row with the risk adjustment of a bond in class B, 0.2481, and the third row for the class C bond with 0.2200. On the diagonal we apply the adjustment to $1-(1-(i \rightarrow i))$. Hence, we derive the martingale transition matrix of:

$$
A_m = \begin{pmatrix} & A & B & C & D \\ A & 0.9005 & 0.0498 & 0.0322 & 0.0166 \\ B & 0.0248 & 0.9008 & 0.0496 & 0.0248 \\ C & 0.0110 & 0.0330 & 0.9230 & 0.0330 \\ D & 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Using these martingale probabilities, the no-arbitrage condition (5.39) is satisfied. For example, for the bond in class A we derive:

$$
B_{01A} = e^{-(0.05+0.01)} = e^{-0.05}(1 \quad 1 \quad 1 \quad 0.4) \begin{pmatrix} 0.9005 \\ 0.0498 \\ 0.0322 \\ 0.0166 \end{pmatrix} \text{ or}
$$

$$
e^{-(0.05+0.01)} = e^{-0.05} \times [1 \times 0.9005 + 1 \times 0.0498 + 1 \times 0.0322 + 0.4 \times 0.0166] = 0.9418.
$$

The reader may verify equation (5.40) for the bond in rating class B and C her/himself.

Martingale probabilities for period two

To calculate the risk adjustment and martingale probabilities for period 2, we first derive the historical transition matrix for period 2. Rating agencies usually provide transition probabilities just for a 1-year time frame. If we assume that time spent in one class is exponentially distributed and probabilities can be extrapolated, the historical transition probabilities from time 0 to time 2, Λ_{02}, are simply the transition matrix of period 1 multiplied by itself: $\Lambda_{02} = \Lambda_{01} \times \Lambda_{12}$. Using the values above we get:

$$\begin{pmatrix} & A & B & C & D \\ A & 0.7 & 0.15 & 0.10 & 0.05 \\ B & 0.1 & 0.6 & 0.2 & 0.1 \\ C & 0.05 & 0.15 & 0.65 & 0.15 \\ D & 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} & A & B & C & D \\ A & 0.7 & 0.15 & 0.10 & 0.05 \\ B & 0.1 & 0.6 & 0.2 & 0.1 \\ C & 0.05 & 0.15 & 0.65 & 0.15 \\ D & 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\Lambda_{02} = \begin{pmatrix} & A & B & C & D \\ A & 0.51 & 0.21 & 0.165 & 0.115 \\ B & 0.14 & 0.405 & 0.26 & 0.195 \\ C & 0.0825 & 0.195 & 0.4575 & 0.265 \\ D & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using equation (5.41), we derive the risk adjustments from time 0 to time 2:

$$\eta_{02A} = \left\{1 - \left(\frac{e^{0.06}}{e^{(0.06+0.02)}}\right)^2\right\} \frac{1}{(1-0.4)(0.115)} = 0.5683$$

$$\eta_{02B} = \left\{1 - \left(\frac{e^{0.06}}{e^{(0.06+0.025)}}\right)^2\right\} \frac{1}{(1-0.4)(0.195)} = 0.4168$$

$$\eta_{02C} = \left\{1 - \left(\frac{e^{0.06}}{e^{(0.06+0.03)}}\right)^2\right\} \frac{1}{(1-0.4)(0.265)} = 0.3663.$$

We now multiply the first row of the historical transition matrix Λ_{02} with the risk adjustment for a bond in rating class A, 0.5683, the second row with the risk adjustment of a bond in class B, 0.4168, and the third row for the class C bond with 0.3663. On the diagonal we apply the adjustment to $1 - (1 - (i \to i))$. Hence, we derive the martingale transition matrix for time 0 to time 2 of:

$$\Lambda_{02m} = \begin{pmatrix} & A & B & C & D \\ A & 0.7215 & 0.1193 & 0.0938. & 0.0654 \\ B & 0.0584 & 0.7520 & 0.1084 & 0.0813 \\ C & 0.0302 & 0.0714 & 0.8013 & 0.0971 \\ D & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The martingale probabilities in matrix Λ_{02m} guarantee that the no-arbitrage condition (5.39) is satisfied. For example for a bond currently in rating class B we derive:

$$B_{02B} = e^{-(r_{02}+s_{02})\times 2} = e^{-(r_{02}\times 2)}(1 \quad 1 \quad 1 \quad RR)\begin{pmatrix} B \to A \\ B \to B \\ B \to C \\ B \to D \end{pmatrix}.$$

This condition is satisfied since:

$$B_{02B} = e^{-(0.06+0.025)\times 2} = e^{-(0.06\times 2)} \times (1 \times 0.0584 + 1 \times 0.7520 + 1 \times 0.1084 + 0.4 \times 0.0813)$$
$$= 0.8437.$$

Pricing vulnerable derivatives using martingale probabilities

In the earlier section (equations 5.34 and 5.35), we derived the vulnerable option price with the help of the risky bond price $B_{t,T}$ as $C_{01V} = C_0 B_{01}/P_{01}$. The approach using risk-neutral martingale probabilities is quite similar. We can use the cumulative default probability of the debtor (which is implicitly incorporated in the risky bond price B), multiply it with the loss given default (1 – RR) and discount back to today. Deducting this from the non-vulnerable call price $C_{t,T}$, we find the vulnerable call price $C_{t,T,V}$ as:

$$C_{t,T,V} = C_{t,T} - \left[e^{-r_{t,T}T} \lambda_{t,T,D}(1-RR)\right] \qquad (5.42)$$

where $\lambda_{t,T,D}$ is the cumulative risk-neutral default probability of the counterparty from time t to T, and RR is the recovery rate. The term $\lambda_{t,T,D}(1 - RR)$ represents the default-probability weighted loss given default.

Example 5.17: Let's assume the non-vulnerable call price of a call with maturity T = 2 was derived as 8.00%. Let's further assume that the call seller is currently rated single B and his risk-neutral martingale default probability within 2 years is 0.0813 (see matrix Λ_{02m}). The recovery rate is assumed to be 40%. What is the value of a vulnerable call, if the 2-period risk-free spot interest rate r_{02} is 6%?

Following equation (5.42), the vulnerable call price is:

$$C_{02V} = 0.08 - \left[e^{-(0.06\times 2)} \times 0.0813 \times (1-0.4)\right] = 3.67\%$$

Hence the non-vulnerable call price is significantly reduced by 8.00% – 3.67% = 4.33% or 4.33%/8.00% = 54.08% of its no-default value.

The principle of equation (5.42) can be used for any derivatives such as forwards, futures, and swaps. Thus generalizing equation (5.42), we can write:

$$D_{t,T,V} = D_{t,T} - \left[e^{-r_{t,T}T} \lambda_{t,T,D}(1-RR)\right] \qquad (5.43)$$

where $D_{t,T,V}$ is any derivative such as an option, forward, future, or swap, in which the counterpart has, on a netted basis, a future obligation.

Equation (5.43) shows that the vulnerable derivative $D_{t,T,V}$ will be equal to the non-vulnerable derivative $D_{t,T}$ in case the default probability of the counterpart $\lambda_{t,T,D}$ is zero. The same logic applies to the recovery rate. In the (theoretical) case it is 100%, the vulnerable derivative $D_{t,T,V}$ will be again equal to the non-vulnerable derivative $D_{t,T}$, independently of the probability of default $\lambda_{t,T,D}$. The Jarrow-Lando-Turnbull 1997 model can be found at www.dersoft.com/jlt.xls.

When to use martingale probabilities, when to use historical probabilities

When discussing traditional models at the beginning of this chapter, we have already mentioned an inconsistency in the credit market. In practice, risky bond prices tend to overestimate the probability of default significantly (see e.g. Altman, 1989). This phenomenon can be partly explained by the illiquidity of risky bonds, especially when they are close to default. Also, the probability of a future recession may be incorporated in the risky bond valuation, thus lowering their price.

It can also be argued that investors are principally risk-averse, requiring a risk premium that is higher than that of the risk-neutral approach, which derives a risky bond price as the default-probability weighted, discounted value of all future cash flows (see condition 5.39). It can also be argued that many investors do not have the necessary information, i.e. the transition and the default probabilities, in order to derive the risk-neutral price of a risky asset.

So when should we use historical or historically based probabilities, and when should we use risk-neutral martingale probabilities? The answer depends on the nature of the analysis: In a credit VAR (value at risk)[30] analysis, which calculates potential future losses due to credit risk, we should apply historical default probabilities. When pricing and hedging credit derivatives, martingale probabilities should be used. This would be consistent with the general usage arbitrage-free pricing methodologies, which are employed by pricing desks in banking practice.

Critical appraisal of the Jarrow-Lando-Turnbull 1997 model

In the 1995 Jarrow-Turnbull model, default probabilities and credit derivatives prices were derived on the basis of rather illiquid bond prices. In their 1997 model, Jarrow, Lando, and Turnbull replaced bond prices as the main input and apply historical transition probabilities as the basis for their analyses. In today's practice, many investment banks and insurance companies apply the 1997 model and its extensions to price and hedge credit derivatives.

One specific shortcoming of the model is that the default probability $\lambda_{1,D}$ can become bigger than 1. This is especially the case for longer maturities T. Equation (5.41)

$$\eta_i = \left\{1 - \left(\frac{e^{r_{t,T}}}{e^{(r_{t,T}+s_{t,T})}}\right)^T\right\} \frac{1}{(1-RR)\lambda_{iD}},$$

reduces to

$$\lambda_{iD} = \left\{1 - \frac{1}{e^{s_{t,T}T}}\right\} \frac{1}{(1-RR)\eta_i}.$$

For this equation to be smaller than 1, we require that $\frac{1}{e^{s_t T}} > 1 + \eta_i(RR-1)$. This condition may not be satisfied for large s, T, η, and RR. (See www.dersoft.com/jlt.xls.)

General shortcomings of the model lie again in the fact that the ultimate reason of default, the asset-liability structure or the liquidity of a company, is not part of the analysis. Also, as in the 1995 model, the interest rate process and the bankruptcy process are assumed independent. Furthermore, the recovery rate RR is exogenously given.

Naturally, the nature of the transition matrix also bears problems. Jarrow, Lando, and Turnbull assume that bonds in the same credit class have the same yield spread. This is not necessarily the case as pointed out by Longstaff and Schwartz (1995). Rather, the rating–yield relationship is similar within sectors, which suggests conducting sector analyses, rather than aggregating data generally among counterparties.

A crucial problem is that ratings are often done infrequently and may not be recent enough to reflect current counterparty risk. In addition, Standard & Poors currently only rates about 8,000 companies in the US, and only about 1% of all companies worldwide. Nevertheless, the number of rated companies should increase in the future, allowing a widespread usage of the model and its extensions.

Other Reduced Form Models

Other significant reduced form models that have received recognition are Brennan-Schwartz (1980); Iben-Litterman (1991); Longstaff-Schwartz (1995); Das-Tufano (1996); Duffee (1996); Schoenbucher (1997); Henn (1997); Duffie-Singleton (1997); Brooks-Yan (1998); Madan-Unal (1998); Duffie-Singleton (1999); Duffee (1998); Das-Sundaram (2000); Hull-White (2000a); Wei (2001), Martin-Thompson-Brown (2001); Duffie-Lando (2001); and Jarrow-Yildirim (2002). For a survey article comparing the default swap evaluation equations of the Jarrow-Turnbull (1995), Brooks-Yan (1998), Duffee (1998), Das-Sundaram (2000), and Hull-White (2000a) models, see Cheng (2001).

Discussing all these models is beyond the scope of this book. Nevertheless let's look at crucial features of some of them.

Duffie and Singleton (1999)

Duffie and Singleton express the risky bond price B at time t with maturity T based on equation (5.38) $B_{t,T} = E_t[e^{-(r_{t,T}+s_{t,T})T}]$. In the Duffie-Singleton model, the swap spread $s_{t,T}$ equals approximately $\lambda_{t,T}(1 - RR)$. This result can be derived by a simple binomial tree for a zero-coupon bond with maturity at time 1 and a notional amount of $1, as shown in figure 5.21.

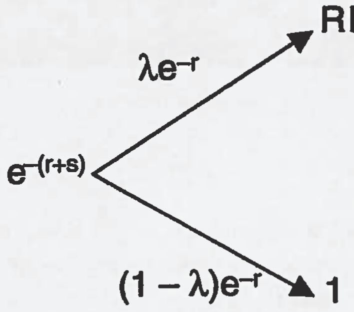

Figure 5.21: Deriving the swap spread s

In figure 5.21, r is the risk-free interest rate, s is the swap spread, λ is the *hazard rate*, which multiplied by time periods for default of 1 equals the risk-neutral probability of default. RR is the recovery rate.

From figure 5.21 we derive:

$$e^{-(r+s)} = \lambda e^{-r}RR + (1-\lambda)e^{-r}. \quad (5.44)$$

Solving equation (5.44) for s, using $e^x \approx 1 + x$, we get $s \approx \lambda(1 - RR) + \lambda r(1 - RR)$. Duffie and Singleton prove that the term $\lambda r(1 - RR)$ can be neglected for a continuous time setting. Hence, the interest rate process drops out and we can write for a default swap spread from time t to time T, $s_{t,T}$:

$$s_{t,T} \approx \lambda_{t,T}(1-RR) \quad (5.45)$$

where all variables are viewed at time t.

Equation (5.45) shows the intuitive approximate relationship between the swap spread s and the hazard rate λ: If the recovery rate RR is zero, $s_{t,T} \approx \lambda_{t,T}$. Hence the spread s approximately compensates the investor for the default risk λ. The relationship in equation (5.45) is often termed *credit triangle*, since two of the three variables are sufficient to generate the third.

The model may include a liquidity premium ℓ for the risky asset. In this case the swap spread is simply:

$$s_{t,T} \approx \lambda_{t,T}(1-RR) + \ell \quad (5.46)$$

where ℓ is a fractional value of the risky bond.

Duffie and Singleton show that any risky claim B with a notional amount N, for different interest rates r and swap spreads s at various times j, and time units of 1, with maturity $t + \Gamma$, can be expressed as:

$$B_{t,t+\Gamma} = E_t\left[e^{-\sum_{j=0}^{\Gamma-1}(r_{t+j}+s_{t+j})} N_{t+\Gamma}\right]. \quad (5.47)$$

Hence, one crucial finding of the Duffie-Singleton model is that any risky claim B can be priced by discounting the notional amount N with the default-adjusted process r + s. Equation (5.47) is an extension of equation (5.38).

In equations (5.44) to (5.46), the recovery rate RR is applied to the expected market value of the risky bond at the time of default, termed *recovery of market value RMV*, hence $E_d(RMV_{d+1}) = RR_d E_d(B_{d+1})$, where d + 1 is the time of default. In contrast, in the Jarrow-Turnbull 1995 and Jarrow-Lando-Turnbull 1997 model, the recovery value is a fraction of the risk-free bond price at the time of default. Brennan-Schwartz (1980), Longstaff-Schwartz (1995), and Duffee (1998) apply a simpler assumption with respect to the payoff in default. They assume that creditors at the time of default receive the recovery rate multiplied with the notional amount of the risky bond.

Das and Sundaram (2000)

Das and Sundaram express the risky bond price from time t to time $t + \Delta t$, with a notional amount of \$1, $B_{t,t+\Delta t}$ as:

$$B_{t,t+\Delta t} = e^{-(r_{t,t}+s_{t,t})\Delta t} = e^{-r_{t,t}\Delta t}[(1-\lambda_t)+\lambda_t RR] \tag{5.48}$$

where $r_{t,t}$ and $s_{t,t}$ are *instantaneous* rates (i.e. rates for infinitesimally small time periods) and λ_t is the risk-neutral default probability from time t to $t + \Delta t$. As pointed out earlier in this chapter, the hazard rate multiplied by a certain time frame, here Δt, results in the risk-neutral default probability.

In equation (5.48), for a risk-neutral default probability of $\lambda_t = 0$, the bond will pay the notional of \$1; for a risk-neutral default probability of 1, the bond will pay the recovery rate RR. Equation (5.48) is identical to equation (5.44) for $t = 0$ and $\Delta t = 1$. With respect to the recovery value at default, Das and Sundaram apply the recovery market value (RMV) approach of Duffie and Singleton. Hence the payoff in default, RMV, is the recovery rate multiplied with the expected value of the risky bond: $E_d(RMV_{d+1}) = RR_d E_d(B_{d+1})$, as above.

For the risk-free interest rate process, Das and Sundaram choose the Heath-Jarrow-Morton (HJM) (1992) term structure of forward rates. Hence, the risk-free forward interest rate viewed at a future time $t + \Delta t$, running from time T to $T + \Delta t$, $r_{t+\Delta t,T}$, is given by:

$$r_{t+\Delta t,T} = r_{t,T} + \alpha_{t,T}\Delta t + \sigma_{t,T} X_1 \sqrt{\Delta t} \tag{5.49}$$

where $0 \leq t \leq t + \Delta t \leq T$. The variable α is the drift rate (average growth rate) of the risk-free interest rate r, σ is the volatility of r, and X_1 is a random variable. Equivalent to equation (5.49), the equation for the forward swap spread s, viewed at a future time $t + \Delta t$, running from time T to $T + \Delta t$, $s_{t+\Delta t,T}$, is given by:

$$s_{t+\Delta t,T} = s_{t,T} + \beta_{t,T}\Delta t + \upsilon_{t,T} X_2 \sqrt{\Delta t}. \tag{5.50}$$

The variable β is the drift of the swap rate s, υ is the volatility of s, and X_2 is a random variable.

In the Das-Sundaram model the drift rate α is expressed as a function of the volatility σ. In addition, a recursive relation expressing α and β as functions of σ and υ can be derived. These functionalities facilitate the implementation of the model, which is expressed as a quadruple tree as in Jarrow-Turnbull (1995).

The Das-Sundaram model allows for four states of correlation between the risk-free interest rate r and the swap spread s at each node of the quadruple tree. This is attained by the variables X_1 and X_2 in equations (5.49) and (5.50). X_1 and X_2 can each take the values −1 and 1, hence the joint distribution of X_1 and X_2 is:

$$(X_1, X_2) = \begin{cases} (+1,+1), & \text{w.p.}(1+\rho)/4 \\ (+1,-1), & \text{w.p.}(1-\rho)/4 \\ (-1,+1), & \text{w.p.}(1-\rho)/4 \\ (-1,-1), & \text{w.p.}(1+\rho)/4 \end{cases} \tag{5.51}$$

where ρ is the correlation coefficient between X_1 and X_2. ρ is principally constant for the time frame of the credit derivative, but can vary at each node at the cost of higher computational complexity.

The recovery rate in the Das-Sundaram model is not exogenous as in most reduced form models, but derived as:

$$RR = \frac{1}{\lambda_a} e^{-(s_{t,t}-\eta_{t,t})\Delta t} - 1 + \lambda_a \tag{5.52}$$

where λ_a is the actual or historical probability, $s_{t,t}$ is the instantaneous swap spread and η is the *risk premium*, which is used to transform actual probabilities λ_a into risk-neutral probabilities λ, via:

$$\lambda = \lambda_a \left[\frac{1 - e^{-(s_{t,t})\Delta t}}{1 - e^{-(s_{t,t}-\eta_{t,t})\Delta t}} \right]. \tag{5.53}$$

The risk premium η is assumed to be a fraction of the swap spread s, hence $\eta_{t,t} = \pi s_{t,t}$, where π is a constant.

In the Heath-Jarrow-Morton model, as in any term structure model, a whole interest rate curve is represented at any node. We will denote the forward risk-free interest rate curve with R and the forward spread curve with S. In order to scale the risk-neutral probability λ to values between 0 and 1, Das and Sundaram choose a simple logit function:

$$\lambda(S, R) = \frac{1}{e^x + 1} \quad \text{where} \quad x = a + bF + cS. \tag{5.54}$$

Following equations (5.48) to (5.54), the quadruple tree shown in figure 5.22 is derived.

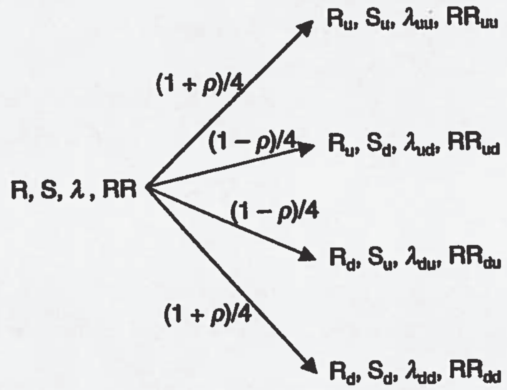

Figure 5.22: Quadruple tree of the Das-Sundaram model

In figure 5.22, R_u, R_d refer to $X_1 = 1$, $X_1 = -1$ respectively. S_u, S_d refer to $X_2 = 1$, $X_2 = -1$ respectively. λ_{uu} refers to the state R_u, S_u; λ_{ud}, λ_{du}, and λ_{dd} are defined respectively. R_{uu} refers to state R_u, S_u; RR_{ud}, RR_{du}, and RR_{dd} are defined respectively.

On the basis of equations (5.48) to (5.54), default swaps, European style and American style credit-spread options, as well as average credit-spread options can be priced. The inputs are R, S, the volatility of R and S, σ and υ, as well as the parameters a, b, c, π and the correlation coefficient ρ. The model (written in Visual Basic) can be found at www.dersoft.com/DasSundaram.exe.lnk. The reader should note that in the program we followed Das and Sundaram's notation of a credit-spread option. Here the payoff of a call is $\max(0, s_{T,T} - K)$ and the put payoff is $\max(K - s_{T,T}, 0)$, where K is the strike spread. These are the standard payoff definitions in the exotic options market for spread options. However, these payoffs are different from the payoff conventions in the credit derivatives market, where the call and put payoffs are opposite and a duration term is added; see equations (2.5) and (2.6).

Hull and White (2000)[31]

Hull and White derive a closed form solution for a default swap spread using swap valuation techniques. Hull and White incorporate the accrued interest of the reference asset, that a default swap buyer pays at the time of default in case of cash settlement. Also, the accrual payment of the default swap premium that a default swap buyer has to pay at the time of default is included in the analysis (for details on the accrued interest on the reference asset and the accrued interest on the swap premium, see chapter 2, "The default swap premium" and "Cash versus physical settlement").

In a standard default swap, the settlement amount of the default swap buyer is usually defined as the nominal amount N minus the recovery value RR, which may include the accrued interest, a, of the reference obligation. Hence we can express the expected settlement amount, also called claim amount, as:

$$N[1-(RR+RRa)]. \tag{5.55}$$

In example 2.1 in chapter 2, we had derived the claim amount. However, we had used equation N[1 − (RR + a)] instead of N[1 − (RR + RRa)]. Which of these two equations is more appropriate, is a question of the terms of the specific default swap contract. Hull and White apply equation N[1 − (RR + RRa)], since here the accrued interest, a, incorporates the default event in form of the recovery rate RR.

To derive the expected value of the claim amount of equation (5.55), we have to multiply by the risk-neutral probability of default λ, since the claim will be paid only in default. Discounting back to t_0 with the risk-free interest rate r, we derive the expected present value of the claim amount for a swap with a notional amount of N = 1 as:

$$\int_0^T (1-RR-RRa)\lambda_t e^{-rt}dt. \tag{5.56}$$

The value of the default swap *payments* of the default swap buyer can be expressed as the integral over all payments in case of default plus all payments in case of no default. Hence, we derive:

$$\int_0^T s\lambda_t(u_t+g_t)dt+s\left[1-\int_0^T \lambda_t dt\right]u_T \tag{5.57}$$

where s: swap premium; λ: risk-neutral probability of default of the reference asset; u: present value of all swap premium payments at rate $1 between zero and time t; g: present value of accrual payments of the swap premium s paid at time t; T: swap maturity.

From equation (5.56) and (5.57) we derive the value of the default swap from the view of the default swap buyer as:

$$\int_0^T (1-RR-RRa)\lambda_t e^{-rt}dt-\int_0^T s\lambda_t(u_t+g_t)dt-s\left[1-\int_0^T \lambda_t dt\right]u_T. \tag{5.58}$$

In order to find the equilibrium swap premium s_e, which gives the default swap a zero value, we have to set equation (5.58) to zero and solve for s. This will give us the value of s_e as:

$$s_e=\frac{\int_0^T [1-(RR+RRa)]\lambda_t e^{-rt}dt}{\int_0^T \lambda_t(u_t+g_t)dt+\left[1-\int_0^T \lambda_t dt\right]u_T}. \tag{5.59}$$

Hull and White (2001)

Hull and White extended equation (5.59) to include the possibility of default of either the default swap buyer or the default swap seller. As mentioned earlier, this type of default swap is also termed a *vulnerable* default swap. The reader should note that the default swap buyer usually has significantly higher counterparty exposure since the claim amount is typically much higher than the periodic default swap premiums. In case of an upfront default swap premium, only the default swap buyer has counterparty default risk, since the default swap buyer has no future obligation to the default swap seller.

To account for the possible default of the counterparty, only one term has to be added to equation (5.59), and two variables have to be redefined. Define:

λ_t^*: risk-neutral probability of default of the reference asset at time t and no earlier default of the counterparty
ϕ_t: risk-neutral probability of default of the counterparty at time t and no earlier default of the reference entity
π_t: risk-neutral probability of no default by the counterparty or the reference asset during the life of the swap

As it is standard in a default swap, it is assumed that there is an accrual payment, g, on the swap premium from the default swap buyer in case of default of the reference asset. However, there is no such payment in case the counterparty defaults. With these assumptions we can express the equilibrium value of a vulnerable default swap $s_{e,v}$ as:

$$s_{e,v} = \frac{\int_0^T [1-(RR+RRa)]\lambda_t^* e^{-rt} dt}{\int_0^T \left[\lambda_t^*(u_t+g_t)+\phi_t u_t\right] dt + \pi u_T}. \tag{5.60}$$

Note that for the case of no counterparty default risk i.e. $\phi_t = 0$, equation (5.60) is mathematically identical with (5.59). For the possibility of counterparty risk, $\phi > 0$, it follows from equations (5.59) and (5.60) that the swap value decreases, i.e. $s_{e,v} < s_e$.

Using default correlations to value default swaps

The approach above uses the risk-neutral probabilities λ_t^* and ϕ_t, which reflect the timing of default events, as well as π_t, which gives the joint probability of no default. A different approach of valuing a default swap is to derive the joint probability of default of the reference entity and the counterparty via a correlation coefficient and then alter the swap premium s. This approach will be discussed now.

The *default correlation* of the reference entity and the counterparty naturally is an important feature in the valuation of a default swap. The higher this correlation, the lower the value of the default swap: Only if both the reference entity and the counterparty default, will the default swap buyer be left with a huge loss. Hence, any default swap buyer should ensure that the default correlation between the reference entity and the default swap seller is low before entering into the default swap.

The probability of a joint default can be expressed easily. Let's first start with some basic statistics: If two companies' default probabilities are independent, the joint probability of default is simply the product of the individual default probabilities. So if the default probability of company r, λ^r, is 2% and the default probability of company c, λ^c, is 3%, the joint probability of default, in case the default probabilities of the companies are independent, is 0.006%. If two companies' default correlation (usually derived from the equity correlation) is $\rho(\lambda^r, \lambda^c)$, the joint probability of default $\lambda^r \cap \lambda^c$ can be shown to be:

$$\lambda(r \cap c) = \rho(\lambda^r, \lambda^c)\sqrt{[\lambda^r - (\lambda^r)^2][\lambda^c - (\lambda^c)^2]} + [\lambda^r \lambda^c]. \tag{5.61}$$

From equation (5.61) we can see that for a default correlation $\rho(\lambda^r,\lambda^c)$ of zero, the joint default probability $\lambda(r \cap c)$ is indeed the product of the individual default probabilities of λ^r and λ^c as stated above. We can also derive from equation (5.61), that the higher the correlation of default probabilities $\rho(\lambda^r, \lambda^c)$, the higher the joint probability of default $\lambda(r \cap c)$, thus the higher the expected loss. Therefore, assumptions about correlations are a key feature in the valuation of credit derivatives.

Hull and White (2001) derive an analytic approximation for a default swap premium, which incorporates the default correlation between the reference entity and the default swap counterparty. Define:

λ^r: Probability of default of reference entity r during the life of the default swap

λ^c: Probability of default of the counterparty c during the life of the default swap

$\lambda(r \cap c)$: Joint probability of default of the reference entity r and the counterparty c during the life of the default swap (derived from equation (5.61))

g_c: proportional reduction in the present value of the expected payoff (1 − RR − RRa) due to counterparty default

h_c: proportional reduction in the present value of the expected swap premium payments s due to counterparty default

s: default swap premium assuming no counterparty default risk

s_v: default swap premium including counterparty default risk (v for vulnerable)

If the reference entity defaults first, there will be the payoff of a standard default swap, 1 − RR − RRa. However, if the counterparty defaults first, there will be no payoff.

A small value of g_c and a high value of h_c increase the default swap value. Hence, the relationship between s and s_v can be expressed as:

$$s_v = s\frac{1-g_c}{1-h_c}. \tag{5.62}$$

In equation (5.62) s is the standard default swap premium, hence s incorporates the probability of default of the reference entity.

Let's first look at the proportional reduction of the expected payoff g. For g to occur, first the reference entity has to default (so that the payoff 1 − RR − RRa is due) and conditionally on this default, the counterparty has to default. The attentive statistics student recalls that this conditional default probability can be expressed as:

$$\lambda^c|\lambda^r = \lambda(r \cap c)/\lambda^r. \tag{5.63}$$

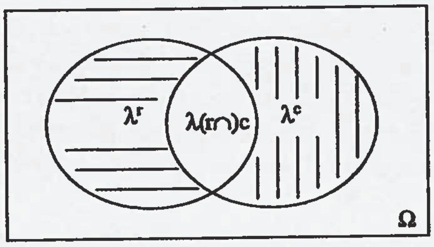

Figure 5.23: Risk-neutral default probability of reference entity r, λ^r, risk-neutral default probability of counterparty c, λ^c, and the joint default probability $\lambda(r \cap c)$

Equation (5.63) reads: The probability of counterparty c defaulting conditional on the default of the reference entity r, $\lambda^c \mid \lambda^r$, equals the joint probability of default of c and r, $\lambda(r \cap c)$, divided by the probability of default of r, λ^r. Hull and White assume that there is a 50% chance that the reference asset defaults first and then the counterparty defaults before having paid the payoff 1 − RR − RRa. Only in this case will there be a reduction of the payoff reflected by g. Hence we get:

$$g = 0.5\lambda(r \cap c)/\lambda^r. \tag{5.64}$$

Let's now look at the expected proportional reduction h of the swap payments of the default swap buyer due to counterparty default risk. This reduction h will occur if the counterparty defaults or the reference entity defaults. However, we have already incorporated the reference entity default risk in the standard swap premium s. Hence, we have to exclude the reference entity default risk and only incorporate counterparty default effects. Displaying first the default probabilities, we get figure 5.23.

The probability of counterparty c defaulting and not the reference entity r defaulting can be derived from the addition law of basic probability theory:

$$\lambda(r \cup c) = \lambda^r + \lambda^c - \lambda(r \cap c). \tag{5.65}$$

Equation (5.65) reads: The probability of the reference entity r or the counterparty c defaulting, $\lambda(r \cup c)$, equals the probability of default of r plus c, $\lambda^r + \lambda^c$, minus the joint probability of default $\lambda(r \cap c)$. Equation (5.58) can easily be verified from figure 5.23.

From equation (5.65) we can derive the probability of counterparty c defaulting but not the reference entity r defaulting, $\lambda(c \cap \bar{r})$ (vertically shaded area in figure 5.23) as:

$$\lambda(c \cap \bar{r}) = \lambda(r \cup c) - \lambda^r = \lambda^c - \lambda(r \cap c). \tag{5.66}$$

As mentioned above, we presently only consider the case where the counterparty defaults before the reference asset defaults. Only this case has to be considered here, since the reference asset defaulting first is already incorporated in the standard swap premium s. Assuming there is 50% chance that the counterparty defaults first, the term $\lambda^c - \lambda(r \cap c)$ reduces to $0.5\,[\lambda^c - \lambda(r \cap c)]$.

So far we have investigated $\lambda(c \cap \bar{r})$, the vertically shaded area in figure 5.23. We now have to additionally consider the term $\lambda(r \cap c)$, since it is part of the counterparty default risk. In this case, Hull and White assume that when both default with the counterparty defaulting first, the payments of the default swap buyer will reduce by one third. Altogether this results in a proportional reduction of the swap premium payments h of:

$$h = 0.5[\lambda^c - \lambda(r \cap c)] + 0.5\lambda(r \cap c)/3 = 0.5\lambda^c - \lambda(r \cap c)/3. \qquad (5.67)$$

Combining equations (5.62), (5.64), and (5.67), we derive the vulnerable swap premium s_v as:

$$s_v = s\frac{1 - 0.5\lambda(r \cap c)/\lambda^r}{1 - 0.5\lambda^c + \lambda(r \cap c)/3}. \qquad (5.68)$$

Example 5.18: The joint probability of default of the reference entity and the counterparty was derived (e.g. with equation (5.61)) as 10%. The default probability of the reference entity r is 20% and the default probability of the counterparty c is 30%. What is the default swap premium, assuming the default swap premium without counterparty default risk was derived as 5%?

Following equation (5.68), we derive:

$$s_V = 0.05\frac{1 - 0.5 \times 0.1/0.2}{1 - 0.5 \times 0.3 + 0.1/3} = 4.25\%.$$

Hence, the incorporation of counterparty default risk reduces the swap premium by $0.05 - 0.0425 = 0.75$ percentage points or $0.0075/0.05 = 15.00\%$ of its no-counterparty risk value.

Kettunen, Ksendzovsky, and Meissner (KKM) (2003)

KKM derive the default swap premium with a combination of two easily implementable discrete binomial trees. One tree represents the default swap premium payments, the other the default swap payoff in case of default.

In the following, the model will be presented in three parts:

1. The model excluding counterparty default risk
2. The model including counterparty default risk, which is not correlated to reference asset default
3. The model including reference entity–counterparty default correlation

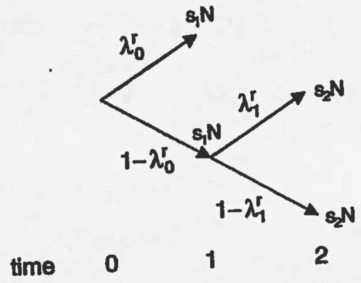

Figure 5.24: Discrete-time binomial model where the premium is paid at the end of a default period

1. The KKM model excluding counterparty default risk

Define:

λ_t^r: exogenous, risk-neutral probability of default of reference entity r, during time t to t + 1, which is expressed in years as $\Delta\tau_t$, viewed at time 0, given no earlier default of the reference entity r

s_t: default swap premium to be paid at time t

N: notional amount of the swap

r_t: risk-free interest rate from time 0 to time t + 1

τ_t: time between time 0 and time t, expressed in years

$\Delta\tau_t$: time between t and t + 1, expressed in years

RR_r: exogenous recovery rate of the reference entity

a: accrued interest on the reference obligation from the last coupon date until the default date.

A tree where the premium is paid at the end of a default period: Let's look first at the default swap premium tree. The discrete times t represent default swap premium payment dates. A simplified version of the default swap premium tree can be seen in figure 5.24.

The risk-neutral default probability λ^r is derived by calibration of the model. Including discount factors, we obtain the present value of the swap premium payments from figure 5.24 as:

$$[\lambda_0^r s_1 N + (1-\lambda_0^r) s_1 N] e^{-r_0 \tau_1} + \{(1-\lambda_0^r)[\lambda_1^r s_2 N + (1-\lambda_1^r) s_2 N]\} e^{-r_1 \tau_2} \quad (5.69)$$

where τ_t is the time between 0 and t expressed in years and r_t is the risk-free interest rate from time 0 to time t + 1.

Canceling several terms in equation (5.69) and generalizing for T periods we derive the present value of the default swap premiums as:

$$s_1 N e^{-r_0 \tau_1} + \sum_{t=2}^{T} \left[s_t N e^{-r_{t-1} \tau_t} \prod_{u=0}^{t-2} (1-\lambda_u^r) \right]. \quad (5.70)$$

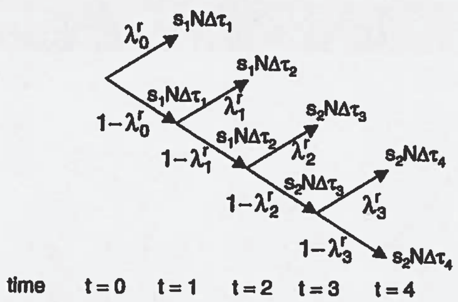

Figure 5.25: A tree with two premium payment dates at time t = 2 and t = 4 and four risk-neutral default probabilities λ_0 to λ_3

The first term of equation (5.70), s_1 N, is not weighted by a default probability. This is because in either default or no default at time 1, the premium payment s_1 N will be paid. It is typical in reality that in case of default, the default swap buyer will make a final (accrual) swap premium payment to the default swap seller.

A tree with more default periods than premium payment dates: In the simple tree in figure 5.24, the number of risk-neutral default probabilities is identical with the number of premium payment periods. The number of possible default periods can be easily increased. For example, if a user wants to double the number of default probabilities, the tree will look as in figure 5.25.

In figure 5.25, $\Delta\tau_t$ represents the time between t and t + 1, expressed in years. We assume that in case of default of the reference entity, the default swap buyer will make an accrual payment on his default swap premium $sN\Delta\tau_t$ at the end of the default period. Hence $sN\Delta\tau_t$ is paid at time t + 1, for default between time t and time t + 1. The exact time of default between times t will be determined by a random number generator in a programmed model, see section "The KKM model in combination with the Libor Market Model (LMM)," below.

In figure 5.25 we have 4 periods in which default can occur. If we have a two-year default swap with annual premium payments, it follows that $\Delta\tau_t = 0.5$. However, the length of the time periods in figure 5.25 may differ. Hence $\Delta\tau_x$ may be unequal to $\Delta\tau_y$.

As shown in figure 5.25, it is assumed that at the time of default, the default swap premium buyer will make a final accrual payment of the default swap premium $sN\Delta\tau_t$, which is typically the case in reality.

Integrating discount factors, we can derive the present value of the swap premium payments from figure 5.25 as:

$$\begin{aligned}&[\lambda_0^r s_1 N\Delta\tau_0 + (1-\lambda_0^r)s_1 N\Delta\tau_0]e^{-r_0\tau_1}\\ &\quad + \{(1-\lambda_0^r)[\lambda_1^r s_1 N\Delta\tau_1 + (1-\lambda_1^r)s_1 N\Delta\tau_1]\}e^{-r_1\tau_2}\\ &\quad + \{(1-\lambda_0^r)(1-\lambda_1^r)[\lambda_2^r s_2 N\Delta\tau_2 + (1-\lambda_2^r)s_2 N\Delta\tau_2]\}e^{-r_2\tau_3}\\ &\quad + \{(1-\lambda_0^r)(1-\lambda_1^r)(1-\lambda_2^r)[\lambda_3^r s_2 N\Delta\tau_3 + (1-\lambda_3^r)s_2 N\Delta\tau_3]\}e^{-r_3\tau_4}. \qquad (5.71)\end{aligned}$$

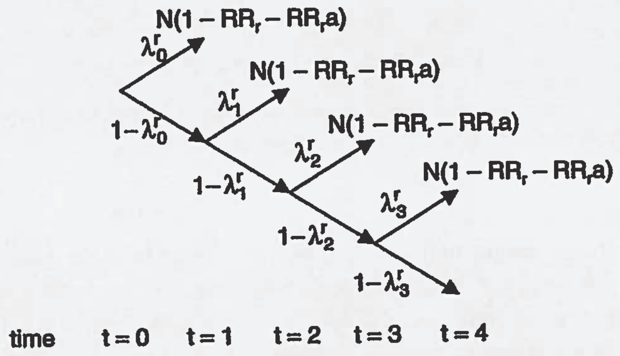

Figure 5.26: Binomial tree of the default swap payoff

If we set the swap premium s constant in time, i.e. $s_1 = s_2 = s_3 \ldots$, and cancel several terms in equation (5.71), we get for T time periods:

$$sN\Delta\tau_0 e^{-r_0\tau_1} + \sum_{t=2}^{T}\left[sN\Delta\tau_{t-1}e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}(1-\lambda_u^r)\right]. \tag{5.72}$$

Let's now look at the process for the payoff that the default swap buyer receives from the default swap seller in case of default of the reference asset. This payoff in practice is typically $N - NRR_r - NRR_r a$ or $N(1 - RR_r - RR_r a)$, where N is the notional amount, RR_r is recovery rate of the reference entity, and a is accrued interest on the reference obligation from the last coupon date until the default date.

The payoff is paid only in the event of default. Using the grid points from figure 5.25, we can build a tree as in figure 5.26.

Integrating discount factors, we derive the present value of the expected payoff from figure 5.26 as:

$$\begin{aligned}&\lambda_0^r N(1-RR_r-RR_r a)e^{-r_0\tau_1} + (1-\lambda_0^r)\lambda_1^r N(1-RR_r-RR_r a)e^{-r_1\tau_2}\\&+(1-\lambda_0^r)(1-\lambda_1^r)\lambda_2^r N(1-RR_r-RR_r a)e^{-r_2\tau_3}\ldots\end{aligned} \tag{5.73}$$

Generalizing equation (5.73), we get for the present value of the expected payoff:

$$\lambda_0^r N(1-RR_r-RR_r a)e^{-r_0\tau_1} + \sum_{t=2}^{T}\left[N(1-RR_r-RR_r a)_t\lambda_{t-1}^r e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}(1-\lambda_u^r)\right]. \tag{5.74}$$

Combining equations (5.72) and (5.74), we derive the present value of the default swap from the viewpoint of the default swap buyer as:

$$\begin{aligned}&\lambda_0^r N(1-RR_r-RR_r a)e^{-r_0\tau_1} + \sum_{t=2}^{T}\left[N(1-RR_r-RR_r a)_t\lambda_{t-1}^r e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}(1-\lambda_u^r)\right]\\&-sN\Delta\tau_0 e^{-r_0\tau_1} + \sum_{t=2}^{T}\left[sN\Delta\tau_{t-1}e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}(1-\lambda_u^r)\right].\end{aligned} \tag{5.75}$$

Setting equation (5.75) to zero and solving the equation for the default swap premium s, we get:

$$s = \frac{\lambda_0^r N(1 - RR_r - RR_r a) e^{-r_0 \tau_1} + \sum_{t=2}^{T} \left[N(1 - RR_r - RR_r a)_t \lambda_{t-1}^r e^{-r_{t-1}\tau_t} \prod_{u=0}^{t-2} (1 - \lambda_u^r) \right]}{N \Delta\tau_0 e^{-r_0 \tau_1} + \sum_{t=2}^{T} \left[N \Delta\tau_{t-1} e^{-r_{t-1}\tau_t} \prod_{u=0}^{t-2} (1 - \lambda_u^r) \right]}. \tag{5.76}$$

The swap premium s in equation (5.76) is the *fair* or *mid-market* default swap premium, since it gives the swap a value of zero. The fair default swap premium s multiplied with $\Delta\tau$ is paid at each time t, starting at t + 1 until T or default, whichever occurs sooner.

Example 5.19: Given is a default swap with a notional amount N of $1,000,000, and an assumed recovery rate of the reference entity RR_r of 40%. The swap terminates in 1 year (time 2). The default swap premiums are paid annually and the probability of default λ in 6 months (time 1) is 10% and in 1 year (time 2) is 30%. The accrued interest, a, of the underlying bond from the last bond coupon date to time 1 will be 1% and to time 2, 4%. The 6-month and 1-year interest rates are 5% and 6%, respectively. What is the annual default swap premium s?

Following equation (5.76), the numerator is:

$$1{,}000{,}000 \times (1 - 0.4 - 0.4 \times 0.01) \times e^{(-0.05 \times 0.5)} \times 0.1 + 1{,}000{,}000 \times (1 - 0.4 - 0.4 \times 0.04) \times e^{(-0.06 \times 1)} \times 0.3 \times (1 - 0.1) = 206{,}626$$

and the denominator is

$$1{,}000{,}000 \times 0.5 \times e^{(-0.05 \times 0.5)} + 1{,}000{,}000 \times 0.5 \times e^{(-0.06 \times 1)} \times (1 - 0.1) = 911{,}449$$

Hence the swap premium is 206,626/911,449 = 22.67%.

See www.dersoft.com/ex519.xls for this example.

2. The KKM model including counterparty default risk, which is not correlated to reference asset default

Counterparty default risk is the risk that a counterparty does not honor its obligation. As mentioned earlier, this type of default swap is also termed a *vulnerable* default swap. Counterparty default risk can be easily integrated in the model as derived so far. In the default swap (consider figure 2.2) the default swap buyer has counterparty default risk, since the default swap seller has a potential future obligation in the amount of $N(1 - RR_r - RR_r a)$ to the default swap buyer. This type of risk is included in the Kettunen, Ksendzovsky, and Meissner model.

In a standard default swap (consider figure 2.2), the default swap seller also has counterparty default risk, if the swap premium s is paid periodically, and additionally if the default swap premium s is an above market premium. (If the swap premium s is a below market premium, the default swap will have a negative present value for the default swap seller, so there is no risk; consider figure 5.19.) This type of counterparty default risk is not included in the Kettunen, Ksendzovsky, and Meissner model. Naturally, if the default swap premium is paid upfront, the default swap seller has no counterparty default risk, since the default swap buyer has no future obligation.

In this section we are excluding the correlation between the default of the reference asset and the counterparty. If the default probability of two entities is not correlated, the joint probability of default is simply the multiplication of the individual default probabilities. So if the default probability of the reference entity r, λ^r, is 2% and the default probability of counterparty c, λ^c, is 3%, the joint probability of default, in the case where the default probabilities are independent, is 0.006%. Formally: $\lambda(r \cap c) = \lambda^r \times \lambda^c$. This property will be applied in this section.

In order to include counterparty default risk, both trees, the payoff tree as well as the swap premium payment tree, will expand to a quadruple tree. Let's start with the payoff tree.

The default swap payoff tree: Define:

λ_t^c: exogenous, risk-neutral probability of default of counterparty c, during time t to t + 1, which is expressed in years as $\Delta\tau_t$, viewed at time 0, given no earlier default of the counterparty c

RR_c: Exogenous recovery rate of the counterparty

$S_f(t)$: Fair value of the default swap at time t excluding counterparty risk (i.e. the swap value including the notional amount that gives the default swap a present value of zero).

We assume that if both reference entity and the counterparty default, with probability $\lambda^r\lambda^c$, the standard payoff in case of default of the reference asset will be reduced by the recovery rate of the counterparty. Hence the payoff will be $N(1 - RR_r - RR_r a)RR_c$. There will be no payoff if neither the reference entity nor the counterparty default, probability $(1 - \lambda^r)(1 - \lambda^c)$. There will be the standard payoff $N(1 - RR_r - RR_r a)$ if only the reference entity defaults, probability $\lambda^r(1 - \lambda^c)$. We assume that if only the counterparty defaults, probability $(1 - \lambda^r)\lambda^c$, the counterparty will pay the time t value of the default swap, $S_f(t)$, multiplied by the recovery rate of the counterparty, $S_f(t)\ RR_c$. This can be represented as in figure 5.27.

Including discount factors, we derive from figure 5.27 the present value of the payoff of a two-period default swap as:

$$\begin{aligned}
&[\lambda_0^r\lambda_0^c N(1-RR_r-RR_r a)RR_c+(1-\lambda_0^r)(1-\lambda_0^c)N\times 0+\lambda_0^r(1-\lambda_0^c)N(1-RR_r-RR_r a)\\
&\quad+(1-\lambda_0^r)\lambda_0^c S_f(1)RR_c]e^{-r_0\tau_1}\\
&\quad+(1-\lambda_0^r)(1-\lambda_0^c)[\lambda_1^r\lambda_1^c N(1-RR_r-RR_r a)RR_c+(1-\lambda_1^r)(1-\lambda_1^c)N\times 0\\
&\quad+\lambda_1^r(1-\lambda_1^c)N(1-RR_r-RR_r a)+(1-\lambda_1^r)\lambda_1^c(S_f(2)RR_c)]e^{-r_1\tau_2}.
\end{aligned} \tag{5.77}$$

Generalizing equation (5.77) for T periods, we derive:

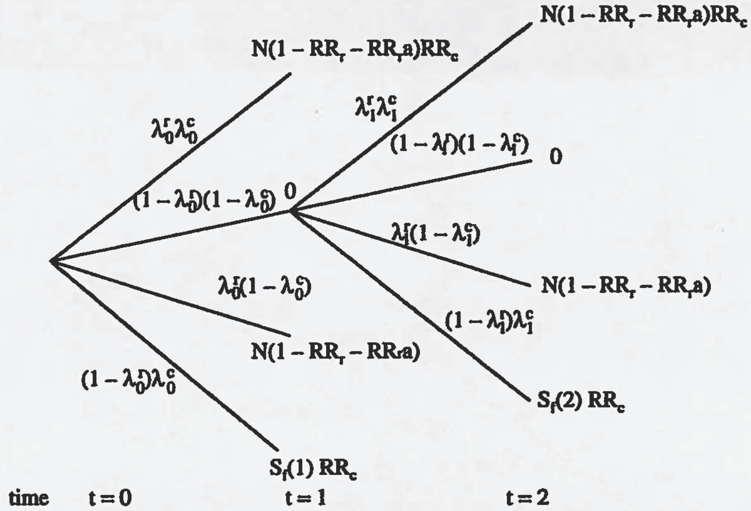

Figure 5.27: Two-period payoff tree of a default swap including counterparty default risk

$$\sum_{t=1}^{T}\{[\lambda_{t-1}^r\lambda_{t-1}^c N(1-RR_r-RR_r a)RR_c+\lambda_{t-1}^r(1-\lambda_{t-1}^c)N(1-RR_r-RR_r a)+$$

$$(1-\lambda_{t-1}^r)\lambda_{t-1}^c S_f(t)RR_c]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}(1-\lambda_u^r)(1-\lambda_u^c)\}. \tag{5.78}$$

The default swap premium payment tree: Let's now look at the default swap premium payment tree. In the case of no default of the reference entity or the counterparty, probability $(1-\lambda^r)(1-\lambda^c)$, the standard swap premium payment $sN\Delta\tau_t$ will apply. The same swap premium $sN\Delta\tau_t$ will be paid in case of the default of the reference asset and no default of the counterparty, probability $\lambda^r(1-\lambda^c)$.

In case of both the reference entity and the counterparty defaulting, probability $\lambda^r\lambda^c$, the final swap premium payment of the default swap buyer depends on the national bankruptcy law and the specific terms of the default swap contract. Principally three scenarios exist.

Scenario 1: The default swap buyer makes no final accrual payment and receives the payoff $N(1-RR_r-RR_r a)RR_c$.

Scenario 2: The default swap buyer makes a final accrual payment of the minimum of his obligation and the payment of the default swap seller: min $[sN\Delta\tau, N(1-RR_r-RR_r a)RR_c]$. This scenario nets the obligations in case of $sN\Delta\tau \geq N(1-RR_r-RR_r a)RR_c$ and gives a payoff of $N(1-RR_r-RR_r a)RR_c - sN\Delta\tau$ in case of $N(1-RR_r-RR_r a)RR_c \geq sN\Delta\tau$.

Scenario 3: The default swap buyer makes a final accrual payment of $sN\Delta\tau$. However, this payment may be higher than the reduced, recovery rate dependent, final payment of the default swap seller $N(1-RR_r-RR_r a)RR_c$.

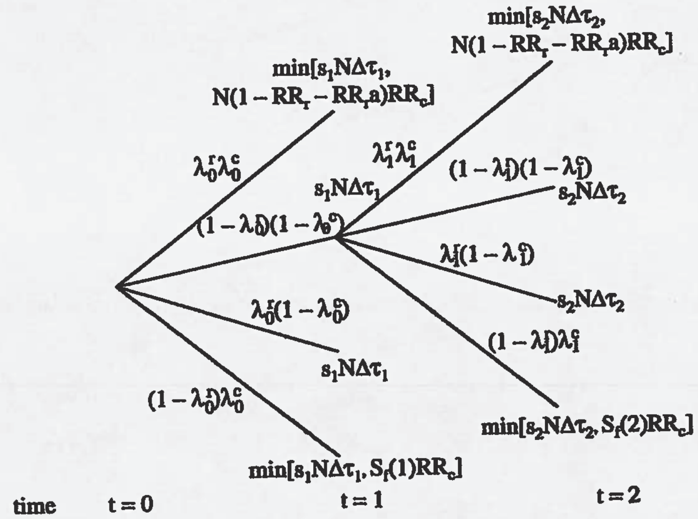

Figure 5.28: Two-period swap premium tree of a default swap including counterparty default risk

In case of the counterparty defaulting but not the reference entity, probability $(1-\lambda^r)\lambda^c$, the final swap premium payment of the default swap buyer depends again on the national bankruptcy law and the specific terms of the default swap contract. Principally the modified three scenarios are now:

Scenario 1: The default swap buyer makes no final accrual payment and receives the payoff $N(1-RR_r-RR_ra)RR_c$.

Scenario 2: The default swap buyer makes a final accrual payment of the minimum of his obligation and the payment of the default swap seller: min $[sN\Delta\tau, S_f(t)\,RR_c]$. This scenario nets the obligations in case of $sN\Delta\tau \geq S_f(t)\,RR_c$ and gives a payoff of $S_f(t)\,RR_c - sN\Delta\tau$ in case of $Ns_f(t)\,RR_c \geq sN\Delta\tau$.

Scenario 3: The default swap buyer makes a final accrual payment of $sN\Delta\tau$. However, this payment may be higher than the reduced, recovery rate dependent, final payment of the default swap seller $S_f(t)\,RR_c$.

Of all scenarios, scenarios 2 and 3 reflect best the international and US bankruptcy law. Scenario 1 is based on a "walk away clause," that allows the solvent party to cease payments but receive the recovery rate of the defaulting party. Most derivatives contracts do not include such a clause. Scenario 2 is based on netting agreements, which are a standard provision of the legal documentation of OTC derivatives, making scenario 2 a realistic choice. However, also scenario 3 can be considered realistic: Often solvent parties honor their obligation to their defaulting counterparty, even though the defaulting counterparty cannot honor its obligations. This is done for public relations reasons.

In the following analysis, we will apply scenario 2: Hence, we derive the swap premium payment tree as seen in figure 5.28.

From figure 5.28 we get for the present value of the swap premium payments:

$$\{\lambda_0^r\lambda_0^c \min[sN\Delta\tau_0, N(1-RR_r-RR_r a)RR_c]+(1-\lambda_0^r)(1-\lambda_0^c)s_1N\Delta\tau_0+\lambda_0^r(1-\lambda_0^c)s_1N\Delta\tau_0$$
$$+(1-\lambda_0^r)\lambda_0^c \min[sN\Delta\tau_0, S_f(1)RR_c]\}e^{-r_0\tau_1}$$
$$+(1-\lambda_0^r)(1-\lambda_0^c)\{\lambda_1^r\lambda_1^c \min[s_1N\Delta\tau_1, N(1-RR_r-RR_r a)RR_c]+(1-\lambda_1^r)(1-\lambda_1^c)s_1N\Delta\tau_1$$
$$+\lambda_1^r(1-\lambda_1^c)s_1N\Delta\tau_1+(1-\lambda_1^r)\lambda_1^c \min[s_1N\Delta\tau_1, S_f(2)RR_c]\}e^{-r_1\tau_2}. \quad (5.79)$$

Assuming a constant swap premium s, i.e, $s_1 = s_2 = s_3 \ldots$, generalizing equation (5.79) for T periods and simplifying the notation by using $\min[sN\Delta\tau_{t-1}, N(1 - RR_r - RR_r a)RR_c] \equiv \min[x_t]$ and $\min[sN\Delta\tau_{t-1}, S_f(t)RR_c] \equiv \min[y_t]$, we derive:

$$\sum_{t=1}^{T}\{[\lambda_{t-1}^r\lambda_{t-1}^c \min[x_t]+(1-\lambda_{t-1}^r)(1-\lambda_{t-1}^c)sN\Delta\tau_{t-1}+\lambda_{t-1}^r(1-\lambda_{t-1}^c)sN\Delta\tau_{t-1}$$
$$+(1-\lambda_{t-1}^r)\lambda_{t-1}^c \min[y_t]]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}(1-\lambda_u^r)(1-\lambda_u^c)\}. \quad (5.80)$$

From equations (5.78) and (5.80) we derive the value of the default swap from the viewpoint of the default swap buyer as:

$$\sum_{t=1}^{T}\{[\lambda_{t-1}^r\lambda_{t-1}^c N(1-RR_r-RR_r a)RR_c+\lambda_{t-1}^r(1-\lambda_{t-1}^c)N(1-RR_r-RR_r a)$$
$$+(1-\lambda_{t-1}^r)\lambda_{t-1}^c S_f(t)RR_c]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}(1-\lambda_u^r)(1-\lambda_u^c)\}$$
$$-\sum_{t=1}^{T}\{[\lambda_{t-1}^r\lambda_{t-1}^c \min[x_t]+(1-\lambda_{t-1}^r)(1-\lambda_{t-1}^c)sN\Delta\tau_{t-1}+\lambda_{t-1}^r(1-\lambda_{t-1}^c)sN\Delta\tau_{t-1}$$
$$+(1-\lambda_{t-1}^r)\lambda_{t-1}^c \min[y_t]]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}(1-\lambda_u^r)(1-\lambda_u^c)\}. \quad (5.81)$$

Setting equation (5.81) to zero and solving for the fair default swap premium s, which gives the default swap a value of zero, we derive:

$$s=\frac{\displaystyle\sum_{t=1}^{T}\{[\lambda_{t-1}^r\lambda_{t-1}^c N(1-RR_r-RR_r a)RR_c+\lambda_{t-1}^r(1-\lambda_{t-1}^c)N(1-RR_r-RR_r a)+(1-\lambda_{t-1}^r)\lambda_{t-1}^c S_f(t)RR_c]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}(1-\lambda_u^r)(1-\lambda_u^c)\}}{\displaystyle\sum_{t=1}^{T}\{[\lambda_{t-1}^r\lambda_{t-1}^c \min[x_t]/s+(1-\lambda_{t-1}^r)(1-\lambda_{t-1}^c)N\Delta\tau_{t-1}+\lambda_{t-1}^r(1-\lambda_{t-1}^c)N\Delta\tau_{t-1}+(1-\lambda_{t-1}^r)\lambda_{t-1}^c \min[y_t]/s]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}(1-\lambda_u^r)(1-\lambda_u^c)\}}. \quad (5.82)$$

Equation (5.82) with $\lambda^c = 0$ is identical to equation (5.76), which is the equation for the fair default swap premium excluding counterparty default risk. Equation (5.82) with RR_c

$= 0$ and $\lambda^c > 0$ results in a default swap premium that is lower than the default swap premium without counterparty risk of equation (5.76), which satisfies the no-arbitrage condition with respect to counterparty default risk. A high recovery rate of the counterparty RR_c can, however, result in a default swap premium s that is higher than the default swap premium excluding counterparty default risk. This is the case in scenario 1, because the swap premium payments cease, but due to the high recovery rate of the counterparty the payoff will increase and with it the default swap value and consequently the premium s.

It is also interesting to note that the default swap premium s is negatively related to the recovery rate of the reference entity, hence $\partial s / \partial RR_r \leq 0$. This is because the recovery rate is deducted from the payoff, which is $N(1\text{-}RR_r\text{-}RR_r a)$. Hence with a higher recovery rate RR_r, the value of the default swap and with it the default swap premium s decreases. However, the swap premium has a positive dependence on the recovery rate of the counterparty: $\partial s / \partial RR_c \geq 0$. This is because a default swap buyer is willing to pay a higher default swap premium s, if the payoff will be higher due to a higher recovery rate RR_c.

Let's explain equation (5.82) in an example.

Example 5.20: Let's alter example 5.19 and include counterparty default risk. In example 5.19 we had a default swap with a notional amount N of $1,000,000 and an assumed recovery rate of the reference asset RR_r of 40%. The swap terminates in 1 year (time 2). The default swap premiums are paid annually and the probability of default λ of the reference asset in six months (time 1) is 10% and in 1 year (time 2) is 30%.

The accrued interest, a, of the underlying bond from the last bond coupon date to time 1 will be 1% and to time 2, 4%. The 6-month and 1-year interest rates are 5% and 6%, respectively.

In order to include counterparty default, we first derive the fair value of the above default swap without counterparty default risk at time 1 of 22.67% (see www.dersoft.com/ex520.xls). Furthermore we assume that the probability of default λ^c of the counterparty in 6 months (time 1) is 20% and in 1 year 40% (time 2). What is the default swap premium s including counterparty default risk for an assumed recovery rate of the counterparty RR_c is 5%?

Following equation (5.82) the numerator is:

$$[0.1\times0.2\times1{,}000{,}000\times(1-0.4-0.4\times0.01)+0.1\times(1-0.2)\times1{,}000{,}000\times$$
$$(1-0.4-0.4\times0.01)+(1-0.1)\times0.2\times1{,}000{,}000\times0.2267\times0.05]\times e^{-0.05\times0.5}$$
$$+\{[0.3\times0.4\times1{,}000{,}000\times(1-0.4-0.4\times0.01)+0.3\times(1-0.4)\times1{,}000{,}000\times$$
$$(1-0.4-0.4\times0.01)+(1-0.3)\times0.4\times1{,}000{,}000\times0.3504\times0.05]\times e^{-0.06\times1}\}$$
$$(1-0.1)\times(1-0.2)=126{,}055$$

Following equation (5.82) we derive for period 1, min[x] = min[100,000, 29,800] = 29,800 and min[y] = min[100,000, 5,960] = 5,960 and for period 2, min[x] = min[100,000, 29,200] = 29,200 and min[y] = min[100,000, 17,520] = 17,520 (see www.dersoft.com/ex520.xls).

Hence, the denominator in equation (5.82) is:

$$
\begin{aligned}
&[0.1\times 0.2\times 29{,}800/0.2000+(1-0.1)\times(1-0.2)\times 500{,}000+0.1\times(1-0.2)\times 500{,}000\\
&+(1-0.1)\times 0.2\times 5{,}960/0.2000]\times e^{-0.05\times 0.5}+[0.3\times 0.4\times 29{,}200/0.2000\\
&\times(1-0.3)\times(1-0.4)\times 500{,}000+0.2\times(1-0.4)\times 500{,}000+(1-0.2)\times 0.4\\
&\times 17{,}920/0.2000]\times e^{-0.06\times 1}(1-0.1)\times(1-0.2)=630{,}195
\end{aligned}
$$

Hence, the default swap premium including counterparty default risk is 126,055/630,195 = 20.00%. So including counterparty default risk, the default swap premium has decreased from 22.67% (see example 5.19) to 20.00% or by (22.67 – 20.00) / 22.67 = 11.77%. For scenario 1, the decrease of the default swap premium is 6.32% and for scenario 3, 32.74%.

3. The KKM model including reference entity–counterparty default correlation

So far we have assumed that the risk-neutral default probability of the reference entity λ^r and the risk-neutral default probability of the counterparty λ^c are not correlated. As stated above, the joint probability of two entities defaulting is simply the multiplication of the individual default probabilities, if the companies' default is not correlated. Formally: $\lambda(r \cap c) = \lambda^r \times \lambda^c$. Also note that $\lambda(\bar{x}) = 1 - \lambda^x$.

If the default probabilities of the reference entity r and the counterparty c are correlated, we apply equation (5.61):

$$\lambda(r\cap c)=\rho(\lambda^r,\lambda^c)\sqrt{[\lambda^r-(\lambda^r)^2][\lambda^c-(\lambda^c)^2]}+[\lambda^r\lambda^c] \qquad (5.61)$$

where $-1 \le \rho(\lambda^r, \lambda^c) \le 1$ is the correlation coefficient, which can be derived from equity correlation. The individual risk-neutral default probabilities λ^r and λ^c are derived by calibrating the model.

The probability of neither the reference asset nor the counterparty defaulting, i.e. $\lambda(\bar{r} \cap \bar{c})$ can be derived from the addition law of basic probability theory (see figure 5.23):

$$\lambda(r\cup c)=\lambda^r+\lambda^c-\lambda(r\cap c). \qquad (5.65)$$

From equation (5.65) it follows that:

$$\lambda(\bar{r}\cap\bar{c})=1-\lambda(r\cup c)=1-[\lambda^r+\lambda^c-\lambda(r\cap c)]. \qquad (5.83)$$

From equation (5.65) we can also derive the probability of the reference entity r defaulting and not the counterparty c, $\lambda(r \cap \bar{c})$ (see figure 5.23 horizontally shaded area):

$$\lambda(r\cap\bar{c})=\lambda^r-\lambda(r\cap c). \qquad (5.84)$$

Note that equation (5.84) with zero correlation, is equal to $\lambda^r(1 - \lambda^c)$ from part 2 of this section, which excluded correlation altogether. In case of zero correlation $\rho(\lambda^r, \lambda^c) =$

0, and from equation (5.61) it follows that $\lambda(r \cap c) = \lambda^r\lambda^c$. Substituting this into the right side of equation (5.84), we derive $\lambda^r - \lambda^r\lambda^c$ or $\lambda^r(1 - \lambda^c)$.

Naturally, the probability of the counterparty c defaulting and not the reference entity r, $\lambda(c \cap \bar{r})$, (see vertically shaded area in figure 5.23), is:

$$\lambda(c \cap \bar{r}) = \lambda^c - \lambda(r \cap c). \tag{5.85}$$

Example 5.21: Let's alter example 5.20 to include a default correlation between the reference entity and the counterparty. The default probability of the reference asset for the next year is 10% and for the counterparty 20%. Let's assume from equity correlation, the default correlation $\rho(r,c)$ is 0.5.

For period 1 we derive: From equation (5.61) the probability of joint default $\lambda(r \cap c) = 0.5 \times \sqrt{[0.1-(0.1)^2][0.2-(0.2)^2]} + [0.1 \times 0.2] = 8\%$. From equation (5.83) the probability of both the reference entity and the counterparty not defaulting is $\lambda(\bar{r} \cap \bar{c}) = 1 - \lambda(r \cup c) = 1 - [0.1 + 0.2 - 0.08] = 78\%$. The probability of the reference entity r defaulting and not the counterparty c, $\lambda(r \cap \bar{c})$, is, following equation (5.84), $\lambda(r \cap \bar{c}) = 0.1 - 0.08 = 2\%$. The probability of counterparty c defaulting and not the reference entity r, $\lambda(c \cap \bar{r})$, is $\lambda(c \cap \bar{r}) = 0.2 - 0.08 = 12\%$.

For period 2, we have from example 5.19, $\lambda^r = 30\%$ and $\lambda^c = 40\%$. We assume the same correlation coefficient as in period 1, 0.5. Using the same equations (5.61), (5.83) to (5.85), we derive for period 2: $\lambda(r \cap c) = 23.22\%$, $\lambda(\bar{r} \cap \bar{c}) = 53.22\%$, $\lambda(r \cap \bar{c}) = 6.78\%$, and $\lambda(c \cap \bar{r}) = 16.78\%$.

Using these probabilities instead of the non-correlated default probabilities $\lambda^r\lambda^c$, $(1 - \lambda^r)(1 - \lambda^c)$, $(1 - \lambda^r)\lambda^c$ and $\lambda^r(1 - \lambda^c)$, we derive for scenario 2 a default swap premium of 6.81%, and 8.20% and 5.89% for scenarios 1 and 3, respectively (see www.dersoft.com/ex521.xls). When excluding this reference entity–counterparty default correlation (example 5.20) we had derived a default swap premium of 20.00% for scenario 2, and 21.24% and 15.25% for scenario 1 and 3 respectively. Hence, we realize the significant impact of the default correlation.

See www.dersoft.com/ex521.xls for this example.

The KKM model in combination with the Libor Market Model (LMM)

The Kettunen-Ksendzovsky-Meissner (KKM) model can be easily combined with any interest rate term structure based model. Before we show how it is combined with the Libor Market Model (LMM), let's first discuss basic properties of the LMM model.

The Libor Market Model

The Libor Market Model[32] falls into the framework of the Heath-Jarrow-Morton (HJM) 1992 term structure model.[33] The main weakness of the HJM model is that interest rates are expressed as instantaneous rates, i.e. for infinitesimally short periods of time. These rates are not observable in the market. In the LMM model, interest rates can be conveniently expressed as discrete forward rates.

Hull and White (2000a) show that a one-factor Libor Market Model can be discretized as

$$F_k(t_{j+1}) = F_k(t_j)\exp\left[\left(\sum_{i=j+1}^{k}\frac{\Delta\tau_i F_i(t_j)\Lambda_{i-j-1}\Lambda_{k-j-1}}{1+\Delta\tau_i F_i(t_j)} - \frac{\Lambda_{k-j-1}^2}{2}\right)\Delta\tau_j + \Lambda_{k-j-1}\in\sqrt{\Delta\tau_j}\right] \quad (5.86)$$

where

$F_k(t)$: forward interest rate between time k and k + 1, seen at time t, with compounding of $\Delta\tau_i$

$\Delta\tau_i$: time between horizontal nodes i and i + 1, expressed in years

ε: random drawing from a standard normal distribution

Λ_k: forward rate volatility term for time t_k to t_{k+1}. Assuming $\Delta\tau$ is constant, Λ can be derived iteratively using:

$$\Lambda_{k-1} = \sqrt{k\sigma_k^2 - \sum_{v=0}^{k-2}\Lambda_v^2} \quad (5.87)$$

where σ_k is a caplet's volatility between time t_k and t_{k+1}.

Figure 5.29 shows a non-recombining one-factor five-period LMM model. At each node, an entire interest rate curve is generated. The number displayed highest at each node represents the spot rate, the number displayed second highest represents the one-period forward rate, etc.

Pricing Default Swaps on the KKM model on the basis of the LMM model

Kettunen, Ksendzovsky, and Meissner (2003) derive the value of the European-style default swap including reference asset–counterparty default correlation using a Monte-Carlo implementation of the LMM model via the following steps:

1 The default probability of the reference asset is simulated by a one-factor LMM model. Hence, in equations (5.86) and (5.87), the forward interest rates and their volatilities are replaced by the forward default probabilities of the reference asset and its volatilities.

2 The default probability of the counterparty is also simulated by a one-factor LMM model. Hence, in equations (5.86) and (5.87), the forward interest rates and their volatilities are replaced by the forward default probabilities of the counterparty and its volatilities.

3 The default correlation between the reference asset and the counterparty is integrated into the model with equations (5.61), (5.83), (5.84), and (5.85).

4 A random number between 0 and 1 is generated to dictate which of the four different default scenarios occurs: No default $\lambda(\bar{r} \cap \bar{c})$, only reference asset default $\lambda(r \cap \bar{c})$, only counterparty default $\lambda(c \cap \bar{r})$, or both reference asset and counterparty default $\lambda(r \cap c)$. If either the reference asset or the counterparty or both default, a random number between 0 and 1 is generated and multiplied with the length of the time step to simulate the exact default time in the specific time step.

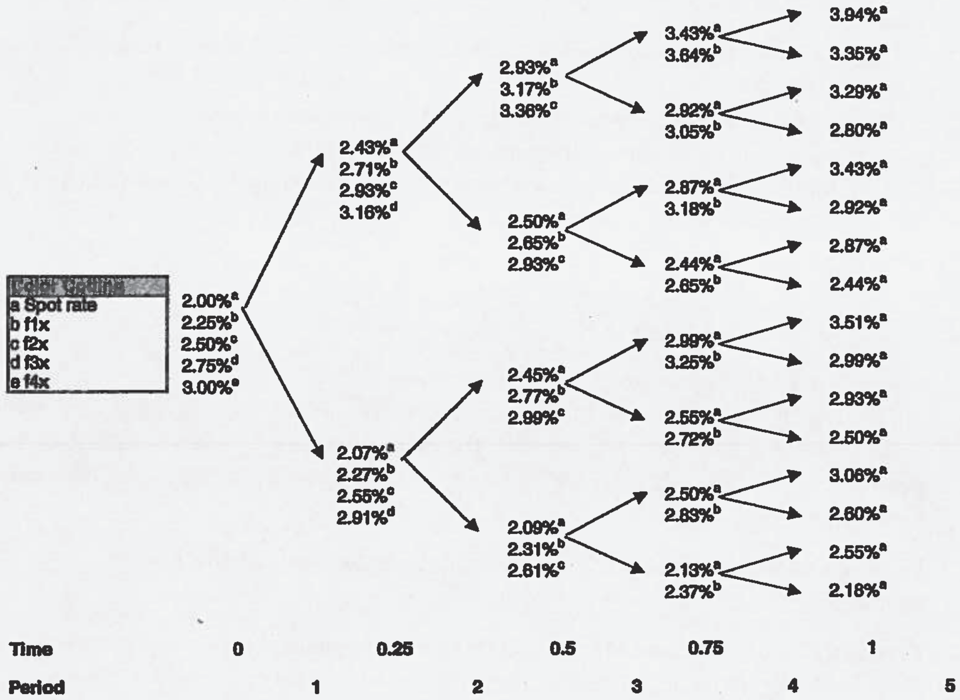

Figure 5.29: One-factor LMM model for 3-month interest rates with a 2% spot rate input and forward rate inputs of 2.25%, 2.5%, 2.75%, and 3% for periods 2, 3, 4, 5, respectively; caplet volatility inputs are 16%, 17%, 16%, and 15% for periods 2, 3, 4, 5, respectively
The model can be found at www.dersoft.com/lmmtree.xls

5 The payoff and the premium payment of the default swap are calculated taking into account the specific default scenario. Table 5.6 presents the payoffs and premium payments when the observed time step is one year, the coupon of the reference asset is paid annually, and the premium of the default swap is paid annually.[34]

6 The payoff (if other than zero) is discounted back to time zero using the interest rate term structure (that is modeled using LMM). The premium payment of the default swap is also discounted back to time zero using the interest rate term structure (that is modeled using LMM) and added to the previously discounted premium payments.

7 If there is no default during the time step and the maturity of the default swap is not reached, steps 1 to 6 are repeated until maturity or default is encountered. (See table 5.6 column "How to proceed.")

8 The accumulated discounted payoffs are divided by the accumulated discounted premium payments to derive the swap premium s, following equation (5.82), which includes the default correlation between the reference asset and the counterparty of equations (5.61) and (5.83) to (5.85). The swap premium s is derived as the average of all trials.

9 The steps 1 to 8 are repeated until the desired accuracy is achieved (recommended at least 100,000 times).

Table 5.6: Payoffs and premium payments for the four default scenarios

Default scenario	Payoff	Premium payment	Paid at	How to proceed
$\lambda(\bar{r} \cap \bar{c})$	–	sN	t_{k+1}	Continue to next time step (if not maturity)
$\lambda(r \cap \bar{c})$	$(1 - RR_r - RR_r cT_d)N$	sNT_d	T_d	Stop trial
$\lambda(c \cap \bar{r})$	$S_f(T_d)\, RR_c$	$Min(sNT_d, S_f(T_d)RR_c)$	T_d	Stop trial
$\lambda(c \cap r)$	$(1 - RR_r - RR_r cT_d)RR_c N$	$Min[sNT_d, (1 - RR_r - RR_r cT_d)\, RR_c N]$	T_d	Stop trial

where

RR_c: Exogenous recovery rate of the counterparty

RR_r: Exogenous recovery rate of the reference entity

T_d: Randomly simulated default time between the last node and the consecutive node, expressed in years

$S_f(T_d)$: Fair value of the default swap from the time the default swap was issued until the time of reference asset default without the possibility of counterparty default. $S_f(T_d)$ includes the notional amount N

a: Accrued interest on the reference asset from the last coupon date until the default date, hence $a = cT_d$, where c = coupon of the reference asset

s: Default swap premium

N: Notional amount of the swap

A visual basic program that follows the previously mentioned nine steps is available at www.dersoft.com/dslmmkkm.xls. The program requires the inputs:

- Forward interest rates and caplet volatilities;
- Forward default probabilities of the reference asset and reference asset volatilities;
- Forward default probabilities of the counterparty and counterparty volatilities;
- Default correlation between the reference asset and counterparty;
- Recovery rate of the reference asset;
- Recovery rate of the counterparty;
- Maturity of the default swap;
- Coupon of the reference asset and coupon payment frequency;
- Default swap premium payment frequency;
- Length of time step (0.25, 0.5, and 1 year are currently available).

The program provides a 95% confidence interval for the simulated result so that the accuracy of the result can be evaluated.

It is interesting to note that the model derives a zero value for the default swap premium, if the correlation coefficient $\rho(\lambda^r, \lambda^c)$ is 1, the default probabilities and the default volatilities of the reference asset and the counterparty are identical, and the recovery rates are zero. In this case, if the reference asset defaults, the counterparty will default, making the default swap useless. The reader can easily verify this feature when using the model.

Pricing TRORs

So far we have discussed several models that evaluate default swaps and credit-spread options. We have not explicitly mentioned how to price TRORs. The reason is simple. TRORs can be evaluated on the basis of default swaps. In chapter 2 we had derived equations (2.2a) and (2.2b):

Receiving in a TROR = Short a default swap + Long a risk-free asset
Paying in a TROR = Long a default swap + Short a risk-free asset

Consequently, in order to price a TROR, we can simple derive the price of a default swap and add a long or short position of a risk-free asset.

Further research in valuing credit derivatives

In the options market, the Black-Scholes equation and its modifications have been widely accepted as the benchmark model. No such model currently exists for pricing and hedging credit derivatives. Currently structural models and reduced form models compete to establish dominance in the market. An approach such as the one of Duffie and Lando (2001) that includes structural as well as reduced form elements could prove successful. Kamakura has already launched a hybrid Jarrow-Merton default probability model, which includes elements of structural and reduced form models.[35]

Further research will also focus on integrating most or all of the crucial input variables (see table 5.1) in credit derivatives models. Research will also focus on relaxing many of the restrictions that exist with current credit derivatives models. In a recent article Jarrow and Yildirim (2002) derive a model in which the process for interest rates and the process for default are correlated. Valuing exotic default swaps such as yield curve swaps, differential swaps, or Libor in arrears swaps and index amortizing swaps, as well as exotic types of options such as barrier options, lookback options, or average options also await exploration.

SUMMARY OF CHAPTER 5

Numerous input variables and their correlations are necessary to price a credit derivative. Among the most crucial inputs are the default probability of the reference asset and the default probability of the counterparty, and the reference asset–counterparty default correlation.

The two main approaches to value credit derivatives are *structural models* and *reduced form models* (also termed *intensity models*). Both models have their roots in the seminal Merton 1974 contingent claim methodology, though structural models have closer ties to Merton.

Structural models endogenize the bankruptcy process by modeling assets and liabilities of a company as in the Merton model. Structural models can be divided into *firm value models* and *first-time passage models*. In firm value models bankruptcy occurs when the asset value of a company is below the debt value at the maturity of the debt. In first-time passage models, bankruptcy occurs when the asset value drops below a pre-defined, usually exogenous barrier, allowing for bankruptcy before the maturity of the debt.

Reduced form models abstract from the explicit economic reasons for the default (i.e. they do not assess the asset–liability structure of the firm to explain the default). Rather, reduced form models use debt prices as a main input to model the bankruptcy process. Default is modeled by a stochastic process with an exogenous *default intensity* or *hazard rate*, which, multiplied by a certain time frame, results in the risk-neutral default probability also called *pseudo-* or *martingale default probability*. The value of the hazard rate is derived by calibration of the variables of the stochastic process.

Numerous types of structural and reduced form models exist, focusing on different aspects of the default process. Current research concentrates on the creation of a coherent combination of structural and reduced form models.

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Questions and Problems

Answers, available for instructors, are on the Internet. Please email gmeissne@aol.com for the site.

5.1 *Explain why the pricing of credit derivatives is more difficult than the pricing of derivatives in the equity, commodity, foreign exchange, or fixed income markets.*

5.2 *What are the three main approaches for pricing credit derivatives? Characterize and criticize them briefly.*

5.3 *Of the numerous input factors, which ones do you believe are the most important for pricing a credit derivative? Discuss.*

5.4 *Explain why in trading practice the default swap premium is often derived from the asset swap spread.*

5.5 *Derive the price range of a default swap using hedging arguments.*

5.6 *Derive the probability of default for period 1 in a simple 1-step binomial tree. Using this result, derive the probability of default for period 2 in a 2-step binomial model.*

5.7 *Explain the market price of risk equation:* $\chi_i = \frac{\mu_i - r}{\sigma_i}$. *Give a numerical example showing that the market price of risk decreases for an asset that has a higher market price of risk than the arbitrage-free market price of risk* χ_M.

5.8 *Explain why a zero-coupon bond is not a martingale. Do you think a stock price is a martingale?*

5.9 *Show that the Black-Scholes equation for a put* $P = -SN(-d_1) + Ke^{-rT}N(-d_2)$ *with*

$$d_1 = \frac{\ln\left(\frac{S}{Ke^{-rT}}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \quad \textit{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

satisfies the PDE:

$$D = \frac{\partial D}{\partial t}\frac{1}{r} + \frac{\partial D}{\partial S}S + \frac{1}{2}\frac{\partial^2 D}{\partial S^2}\frac{1}{r}\sigma^2 S^2.$$

5.10 *Do you think it is reasonable to value a credit-spread option on a modified Black-Scholes equation, where the spread is modeled as a single variable? What are the drawbacks?*

5.11 *Discuss the original Merton equation* $E_0 = V_0N(d1) - De^{-rT}N(d_2)$ *where*

$$d_1 = \frac{\ln\left(\frac{V_0}{De^{-rT}}\right) + \frac{1}{2}\sigma_V^2 T}{\sigma_V\sqrt{T}} \quad \textit{and} \quad d_2 = d_1 - \sigma_V\sqrt{T}.$$

What is the probability of default in the Merton model? Explain.

5.12 *The probability of default in the original Merton model can also be derived via a put option. Explain.*

5.13 *Discuss the derivation of the probability of default in first-time passage models. Why have first-time passage models not been too successful in trading practice?*

5.14 *Discuss the vulnerable option pricing approach in the Jarrow Turnbull 1995 model.*

5.15 *Why is it necessary to transform historical transition probabilities into risk-neutral (martingale) transition probabilities in the derivatives pricing process? When should historical probabilities, and when should risk-neutral probabilities be applied?*

5.16 *Why is the reference asset–counterparty default correlation important when pricing default swaps? Discuss how the reference asset–counterparty default correlation can be incorporated in a default swap pricing process.*

5.17 *Show how TRORs can easily be evaluated on the basis of default swap pricing.*

Notes

1 Altman, E., "Measuring Corporate Bond Mortality and Performance," *Journal of Finance*, 44, 1989, pp. 909–21.

2 Black, F. and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, May–June 1973, pp. 637–54.

3 Merton had outlined the basic principle of arbitrage-free derivatives pricing. See Merton, R. "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, vol. 4, no. 1, Spring 1973, pp. 141–83.

4 See Meissner, G., "Pricing Default Swaps – Which Default Probabilities, Which Default Correlations Should Be Included?" Hawaii University Working Paper, 2004.

5 For reasons of simplicity we have not included arrows for the bond sale from bank B to the bond buyer at time t_0 and no arrows for the cash returns at time t_1.

6 In many reduced form models such as Jarrow and Turnbull (1995), Lando (1998), and Duffie and Singleton (1999), a "hazard rate" (also called default intensity) is used to model default. The hazard rate h, multiplied by a certain time period results in the default probability. The λ_t

used in figure 5.5 and throughout this book already incorporates the time period. Hence λ_t is the risk-neutral default probability for period t to t + 1. Naturally, if the time period for the possible default is 1, the hazard rate and the default probability are identical.

7 For an explanation of forward rates see Hull, J., "Options, Futures and Other Derivatives," Prentice Hall, 2002, 4th edn, pp. 98ff and Meissner, G., *Trading Financial Derivatives*, Simon and Schuster, 1998, pp. 84ff.

8 Earlier in the book we used the notation e^x (e = Euler's number). For convenience we are now using $\exp(x) = e^x$.

9 W. Sharpe, "Evaluating Mutual Fund Performance," *Journal of Business*, (39) 1966, pp. 138–99.

10 See K. Ito, "On Stochastic Differential Equations," *Memoirs, American Mathematical Society*, 4, 1951, pp. 1–51.

11 For a proof see http://www.dersoft.com/bspdeproof.doc.

12 Margrave, W., "The value of an option to exchange one asset for another," *Journal of Finance* 33, 1978, pp. 177–86.

13 See Hull, J. and A. White, "Pricing Interest Rate Derivative Securities," *Review of Financial Studies*, 3, 4 1990, pp. 573–92, and Hull J. and A. White, "Numerical procedures for implementing term structure models I: single factor models," *The Journal of Derivatives*, 2, 1994, pp. 7–16.

14 Merton, R., "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates," *Journal of Finance* 29, 1974, pp. 449–70.

15 The value of N(−1.7695) = 3.84% can be found in table A.1 in the appendix or with Excel function normsdist(−1.7695) = 3.84%.

16 See for example P. Crosbie, "Modeling Default Risk," in *Credit Derivatives: Trading & Management of Credit and Default Risk*, Risk Books; Jones E., S. Mason and E. Rosenfeld, "Contingent Claim Analysis of Corporate Structures: An Empirical Investigation," *Journal of Finance*, 1984, 39(3), pp. 611–28, and Kamakura, "Comparison of the Merton and Jarrow Credit Models for Pricing Risky Debt," *Internal Kamakura paper* to be received at www.Kamakuraco.com.

17 Black, F. and J. Cox, "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions," *The Journal of Finance*, 31, 1976, pp. 351–67.

18 Kim, J., K. Ramaswamy, and S. Sundaresan, "Does Default Risk in Coupons Affect the Valuation of Corporate Bonds?: A Contingent Claim Model," *Financial Management*, 22(3), 1993, pp. 117–31.

19 Cox, J., J. Ingersoll and S. Ross, "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53 (1985), pp. 385–407.

20 Longstaff, F. and E. Schwartz, "A Simple Approach to Valuing Risky Fixed and Floating Rate Debt," *The Journal of Finance*, no. 3, July 1995, pp. 789–819.

21 Vasicek, O., "An Equilibrium Characterization of the Term Structure," *Journal of Economics*, 5, 1977, pp. 177–88.

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23 Hull, J. and A. White, "Pricing Interest Rate Derivative Securities," *The Review of Financial Studies*, 3(4), 1990, pp. 573–92.

24 Jarrow, R. and S. Turnbull, "Pricing Derivatives on Financial Securities Subject to Credit Risk," *Journal of Finance*, vol. L, no. 1, March 1995, pp. 53–85.

25 Jarrow and Turnbull use $\lambda\mu$, where μ represents a time period, and call $\lambda\mu$ pseudo- or martingale probability. We will drop the variable μ, hence our λ_t already incorporates the length of the time period t to t + 1, in which default occurs.

26 Jarrow and Turnbull derive the bond price as $B_{0,2} = P_{0,2} \{\lambda_0 RR + (1 - \lambda_0)[\lambda_1 RR + (1 - \lambda_1)]\}$ (equation 28, p. 66 of Jarrow, R., and S. Turnbull, "Pricing Derivatives on Financial Securities Subject to Credit Risk," *Journal of Finance*, vol. L, no. 1, March 1995, pp. 53–85). This is slightly inconsistent since the weighted default value at time 1, λ_0 RR, is discounted with the risk-free bond price with maturity 2.

27 For the derivation of equation (5.36) see G. Meissner, *Trading Financial Derivatives – Futures, Swaps and Options in Theory and Application*, Simon and Schuster, 1997.

28 Jarrow, R., D. Lando, and S. Turnbull, "A Markov Model for the Term Structure of Credit Risk Spreads," *The Review of Financial Studies*, 1997, vol. 10, no. 2, pp. 1–42.

29 The derivation of equation (5.41) can be found at www.dersoft.com/541.doc.

30 A credit value at risk analysis addresses the question: What is the maximum amount we can lose due to a certain type of credit risk, within a certain time frame, with a certain probability? See chapter 6 for a detailed analysis of VAR.

31 Hull, J. and A. White, "Valuing Credit Default Swaps I: No Counterparty Default Risk," *Journal of Derivatives*, Fall 2000, vol. 8, issue 1, pp. 29–41.

32 The Libor Market Model is credited to three groups of authors: Brace, A., D. Gatarek, and M. Musiela, (BGM) "The Market Model of Interest Rate Dynamics," *Mathematical Finance*, 7, no. 2, 1997, pp. 127–55; Jamshidian, F., "Libor and Swap Market Models," *Finance and Stochastics*, 1 1997, pp. 293–330; and Miltersen, K., K. Sandmann, and D. Sondermann, "Closed Form Solutions for Term Structure Derivatives with LogNormal Interest Rates," *Journal of Finance*, 52, no. 1, March 1997, pp. 409–30. In the following, we will use the notation of Hull, J. and A. White, "Forward Rate Volatilities, Swap Rate Volatilities, and Implementation of the LIBOR Market Model," *Journal of Fixed Income*, September 2000a, vol. 10, issue 2, pp. 46–63 and Hull, J., *Options, Futures and Other Derivatives*, Prentice Hall, 2002.

33 Heath, D., R. Jarrow, and A. Morton, "Bond Pricing and the Term Structure of Interest Rates, A New Methodology," *Econometrica*, 60, no. 1, 1992, pp. 77–105.

34 Calculations for premium payments and payoffs are more complicated when the observed time step is different to the time interval of premium payments or the time interval of coupon payments of the reference asset. The reason is that it is necessary to keep track of the accrued interest and the accumulated premium payment.

35 See Duffie, D. and D. Lando, "Term Structures of Credit Spreads with Incomplete Accounting Information," *Econometrica*, 69(3), pp. 633–64; see also Kamakura's hybrid Jarrow-Merton model, KPD-JM, at www.kamakuraco.com. See also Mashal, R., M. Naldi, "Pricing Multiname Credit Derivatives: Heavy Tailed Hybrid Approach," *Columbia University Working Paper*, www.columbia.edu/rm586/.