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Jarrow + Turnbull

INTEREST RATE CONTRACTS

13.0 INTRODUCTION

In our examination of the pricing and hedging of options and futures in Chapter 5 and in Chapter 8 through Chapter 12, we made the strong assumption that interest rates are constant through time. We imposed this assumption in order to simplify our discussion of pricing and hedging. This assumption implies that all default-free securities, and options or futures on default-free securities, are identical from an economic perspective. That is, they are all riskless, and they all earn the same constant interest rate.

The assumption of constant interest rates is obviously an unrealistic assumption. But for short-dated options and futures on underlying assets distinct from and uncorrelated with interest rates (like common stock), it is an acceptable first approximation. For these derivatives, interest rates are only of secondary importance to the analysis. Yet, for options and futures on the term structure of interest rates, this assumption is unreasonable and unacceptable, even as a first approximation.

The next step in our examination of derivatives—Chapter 13 through Chapter 17—is to study the relaxation of this constant interest rate assumption.

Our first task in this chapter is to examine the basic types of traded default-free securities making up the term structures of interest rates, that is, Treasury bills, Treasury notes, Treasury bonds, and the futures contracts written on these instruments. Also necessary is an understanding of related interest rate instruments: Eurodollar forward rate agreements and futures. In subsequent chapters we will examine other interest rate contracts and the pricing and hedging of interest rate derivatives.

13.1 ZERO-COUPON BONDS

First we discuss default-free zero-coupon bonds. We recall that a **zero-coupon bond** is a bond that has no coupon payments. Profits from owning such a security come solely from price appreciation. Zero-coupon bonds are sometimes called **discount bonds** because they are sold at a discount, a price lower than the par or face value of the bond that is paid at maturity.

U.S. Treasury bills are zero-coupon bonds issued by the U.S. government. Because their payment is guaranteed by the taxing power of the U.S. government, they are generally considered to be default-free. Treasury bills (T-bills) are short-term instruments with maturities of a year or less. New Treasury bills are regularly issued by the U.S. government via competitive auctions. Every Thursday the Treasury auctions new 91-day (13-week) and 182-day (26-week) T-bills, and every fourth Thursday it auctions new 364-day (52-week) T-bills. The minimum face value of a Treasury bill is \$10,000. Quotes for T-bills are given in terms of discount rates.

Discount Rates

For U.S. T-bills, the **discount rate**[1], i_d, is defined by

$$B(0, T) = [1 - i_d(T/360)], \tag{13.1}$$

where $B(0, T)$ is the date-0 value of a T-bill with a dollar payoff at maturity T. The maturity is expressed in days, and it is assumed that there are 360 days in the year. Table 13.1 gives some examples of bid/asked quotes for T-bills.

TABLE 13.1 *Treasury Bill Quotes, Thursday, May 28, 1998**

TREASURY BILLS

Maturity	Days to Mat.	Bid	Asked	Chg.	Ask Yld.
Jun 04 '98	6	4.39	4.35	−0.05	4.41
Jun 11 '98	13	4.44	4.40	−0.06	4.47
Jun 18 '98	20	4.46	4.42	−0.06	4.49
Jun 25 '98	27	4.53	4.49	−0.06	4.57
Jul 02 '98	34	4.65	4.61	−0.05	4.69
Jul 09 '98	41	4.67	4.63	−0.04	4.72
Jul 16 '98	48	4.71	4.67	−0.05	4.76
Jul 23 '98	55	4.79	4.75	−0.04	4.85
Jul 30 '98	62	4.81	4.79	−0.03	4.90
Aug 06 '98	69	4.86	4.84	−0.04	4.95
Aug 13 '98	76	4.90	4.88	−0.03	5.00
Aug 20 '98	83	4.94	4.92	−0.03	5.05
Aug 27 '98	90	4.92	4.91	−0.03	5.04
Sep 03 '98	97	4.96	4.94	−0.03	5.08
Sep 03 '98	**97**	**4.97**	**4.96**	**+0.01**	**5.10**
Sep 10 '98	104	4.99	4.97	−0.03	5.11
Sep 17 '98	111	5.00	4.98	−0.02	5.13
Sep 24 '98	118	4.98	4.96	−0.03	5.11
Oct 01 '98	125	5.03	5.01	−0.02	5.17
Oct 08 '98	132	5.04	5.02	−0.01	5.19
Oct 15 '98	139	5.08	5.06	−0.02	5.23
Oct 22 '98	146	5.07	5.05	−0.01	5.23
Oct 29 '98	153	4.95	4.93	−0.01	5.11
Nov. 05 '98	160	5.11	5.09	−0.01	5.28
Nov 12 '98	167	5.13	5.11	....	5.31
Nov 19 '98	174	5.12	5.10	−0.01	5.30
Nov 27 '98	182	5.13	5.12	....	5.33
Dec 03 '98	**188**	**5.13**	**5.12**	**−0.01**	**5.33**
Dec 10 '98	195	5.13	5.11	....	5.32
Jan 07 '99	223	5.11	5.09	....	5.30
Feb 04 '99	251	5.13	5.11	+0.01	5.33
Mar 04 '99	279	5.14	5.12	....	5.36
Apr 01 '99	307	5.14	5.12	+0.01	5.37
Apr 29 '99	335	5.14	5.12	+0.01	5.38
May 27 '99	363	5.14	5.13	+0.01	5.41

*Source: *The Wall Street Journal*, May 29, 1998.

[1]A good reference for interest rate calculations is Stigum (1981).

For securities such as stocks, the ask price is higher than the bid price. This is not the case for the T-bill quotes, however, due to the inverse relation between the T-bill's price and the discount rate. Thus the ask rate is less than the bid rate. Referring to Table 13.1, consider the Treasury bills that mature on August 20. These T-bills have a maturity of 83 days. The bid rate is 4.94, which exceeds the ask rate of 4.92.

From these rates, however, we can show that the ask price on T-bills does exceed the bid price. Indeed, the bid is a discount rate of 4.94 percent, implying a price of

$$\begin{aligned}\text{Bid price} = 10{,}000\,B(0,T) &= 10{,}000\left[1 - \frac{4.94}{100} \times \frac{83}{360}\right]\\ &= \$9{,}886.11,\end{aligned}$$

given that Treasury bills trade with a face value (payoff) of $10,000. The ask discount rate is 4.92, implying a price of

$$\begin{aligned}\text{Ask price} &= 10{,}000\left[1 - \frac{4.92}{100} \times \frac{83}{360}\right]\\ &= \$9{,}886.57.\end{aligned}$$

In terms of dollars, the ask price is greater than the bid price. This completes our discussion of discount rates.

Simple Interest Rates

Here we discuss simple interest rates (see also Chapter 1). The **simple interest rate**, i_S, is defined as

$$B(0,T) \equiv \frac{1}{[1 + i_S(T/365)]}, \tag{13.2}$$

assuming a 365-day year. In some cases a 360-day year is used.

The difference between discount rates and simple interest rates is greater the higher the discount rate and the longer the time to maturity. For example, consider a discount rate of 4 percent and a maturity of 30 days. The T-bill price with face value 100 is

$$100\,B(0,T) = 100\left[1 - \frac{4}{100} \times \frac{30}{360}\right] = 99.6667.$$

This implies an equivalent simple rate of

$$\begin{aligned}i_S &= 1\left[\frac{1 - B(0,T)}{B(0,T)}\right]\frac{365}{T}\\ &= 4.069 \text{ percent,}\end{aligned}$$

a difference of 0.069 percent. If a 360-day year had been used, the simple interest rate would be 4.013 percent, a difference of only 0.013 percent.

Continuously Compounded Interest Rates

The continuously compounded annual interest rate, r, is defined by

$$B(0,T) = \exp[-r(T/365)], \tag{13.3}$$

assuming a 365-day year.

Referring to the previous example in which

$$B(0,T) = 0.996667,$$

and $T = 30$ days, then

$$\begin{aligned} r &= \{-\ln [B(0,T)]\}(365/T) \\ &= 4.062 \text{ percent.} \end{aligned}$$

The three different interest rates are different ways of expressing dollar prices. They are each important, and each has its own use. For example, discount rates are used in quoting Treasury bill prices and in the T-bill futures markets. Simple interest rates are used in the Eurodollar markets, swaps markets, and foreign currency markets. Continuously compounded rates are used primarily in academic articles. There are two reasons for this difference. First, the use of continuously compounded interest rates avoids a lot of minor problems with respect to market conventions. Second, much theoretical work is in continuous time for which it is convenient to use continuously compounded rates.

13.2 COUPON BONDS

Let us now discuss default-free coupon bonds. Coupon bonds are bonds with regular interest payments, called coupons, plus a principal repayment at maturity. The principal is called the face value of the bond.

Intuitively, the value of a coupon-bearing bond is determined by summing the present value of all its coupon payments and the present value of the terminal face value. To be precise, we need to introduce some notation. Suppose that the coupon bond makes coupon payments c at dates $t = 1, \ldots, T$, where T is the maturity date of the bond. Let the face value of the bond F be paid at the maturity date T. This section only considers coupon bonds with no default risk. Bonds with default risk are studied in Chapter 18.

Pricing

First we examine the pricing of default-free coupon bonds. Let $B(0, t)$ denote the present value of receiving one dollar at date t with no risk of default. This amount is the price of a zero-coupon bond. The value of the coupon-bearing bond is equal to

$$B_c(0) = \sum_{t=1}^{T} cB(0,t) + FB(0,T). \tag{13.4}$$

This expression states that the coupon bond's value, B_c, equals the sum of the present value of all the coupon payments and the face value. This is an arbitrage-free pricing relation. The right side of Expression (13.4) represents the cost of constructing a synthetic coupon bond with identical coupon payments and face value as the traded coupon bond. The synthetic coupon bond consists of c zero-coupon bonds of maturity t for $t = 1, \ldots, T$ plus F additional zero-coupon bonds of maturity T.

If the left side exceeds the right side of Expression (13.4), then the arbitrage opportunity would be to sell short the coupon bond and go long the synthetic, pocketing the difference, with no future obligations. If the right side exceeds the left side, then changing the signs of the previous trading strategy generates arbitrage profits. The only condition consistent with no arbitrage is an equality.

We now illustrate the use of Expression (13.4) to value a coupon bond.

EXAMPLE

Valuing a Coupon Bond

This example illustrates the use of Expression (13.4) to value a coupon bond. Consider a coupon bond that matures in twenty-two months' time. The next coupon payment of \$3.50 per \$100 face value occurs in four months' time. After that, coupon payments of \$3.50 occur every six months. The final payment includes principal. Details are shown in Table 13.2.

The present value of the coupon bond, which is trading above par, is \$104.5071. This represents the sum of the present value of all the remaining coupon payments plus principal. ■

TABLE 13.2 *Pricing of a Bond*

Maturity (Months)	Pricing of a Zero-Coupon Bond	Coupon	Principal	Present Value of Coupon Plus Principal
4	0.9835	3.50	0	3.4422
10	0.9592	3.50	0	3.3572
16	0.9355	3.50	0	3.2743
22	0.9124	3.50	100	94.4334
				104.5071

Yield-to-Maturity

The **yield-to-maturity** is defined as an interest rate per annum that equates the present value of future cash flows to the current market value. We can express this definition algebraically.

A bond that pays interest annually

For a bond that pays interest annually, the yield-to-maturity, y, is defined as

$$B_c(0) \equiv \sum_{t=1}^{T} \frac{c}{(1+y)^t} + \frac{F}{(1+y)^T}, \tag{13.5}$$

where B_c denotes the price of the coupon-bearing bond, c is the coupon at date t, F is the face value, and T is the total number of coupon payments.

In general, solving Expression (13.5) for the yield-to-maturity is a difficult task, since it is a nonlinear equation. Consequently, the yield-to-maturity is most often calculated numerically.

EXAMPLE **Yield-to-Maturity**

This example illustrates the calculation of a bond's yield-to-maturity. Let the market value of a bond be 104.52. Suppose that the bond has a maturity of two years and a coupon of 7 percent, which is paid on an annual basis. By definition, the yield-to-maturity, y, is defined to be the solution to

$$104.52 = \frac{7}{(1+y)} + \frac{7}{(1+y)^2} + \frac{100}{(1+y)^2}.$$

Let us guess at the solution, setting $y = 0.05$. The right side is 103.72, implying our initial guess was too high. If we try $y = 0.045$, the right side is 104.68, implying the yield-to-maturity is too low. Thus, the correct value is between 4.5 and 5.0 percent. By repeating this process, we will find that $y =$ 4.584 percent. ■

A bond that pays interest semiannually

For a bond that pays interest semiannually, the yield-to-maturity, y, is defined as

$$B_c(0) \equiv \sum_{t=1}^{2T} \frac{c}{(1+y/2)^t} + \frac{F}{(1+y/2)^{2T}}. \tag{13.6}$$

In this expression, there are twice the compounding periods ($2T$), and the yield over each semiannual period is divided by 2 ($y/2$). Because of the ability to reinvest the semiannual coupon, the effective annual yield-to-maturity, y_A, is

$$y_A \equiv (1+y/2)^2 - 1. \tag{13.7}$$

EXAMPLE

Semiannual Yield-to-Maturity

In this example we illustrate the use of Expression (13.6). Consider a bond with a maturity of two years that pays a coupon on a semiannual basis. The coupon is 7.00 percent per annum, implying that every six months of the dollar value of the coupon payment is \$3.50 per \$100 face value. Let the market value of the coupon bond be \$103.79. The yield-to-maturity, y, is defined by

$$103.79 \equiv \sum_{t=1}^{4} \frac{3.50}{(1 + y/2)^t} + \frac{100}{(1 + y/2)^4}.$$

Again, by a trial and error basis, we find that $y/2 = 2.49$ percent. The effective annual yield-to-maturity is

$$\begin{aligned} y_A &\equiv (1 + 0.0249)^2 - 1 \\ &= 5.04 \text{ percent.} \end{aligned}$$

■

The yield-to-maturity is the holding period return per year on the coupon bond only if the coupons can be reinvested at the same rate as the yield-to-maturity.

To see this result, consider a bond that matures in one year and has a coupon of \$3.00 that is paid semiannually. Let the bond sell at par. The yield-to-maturity, y, is defined as

$$100 = \frac{3}{(1 + y/2)} + \frac{3}{(1 + y/2)^2} + \frac{100}{(1 + y/2)^2},$$

implying that $y = 0.06$.

Let us compute the holding period return on this bond assuming reinvestment at the yield-to-maturity. At the end of the first six months, the investor receives the coupon and the calculation assumes that the investor can reinvest this coupon at the yield-to-maturity rate. At the end of one year when the bond matures, the investor receives the final coupon payment and principal, giving a total of \$103 and the proceeds from reinvesting the first coupon. The total value is

$$103 + 3\,(1 + 0.03) = 106.09.$$

The holding period return on this investment is

$$\frac{106.09 - 100}{100} = 6.09 \text{ percent,}$$

which agrees with the effective annual yield-to-maturity in Expression (13.7):

$$\begin{aligned} y_A &= (1 + 0.06/2)^2 - 1 \\ &= 6.09 \text{ percent.} \end{aligned}$$

The important point is that the effective yield implicitly assumes reinvestment of the coupon payments at the yield-to-maturity over the life of the bond. We should question the validity of this assumption in a world with changing interest rates.

Continuous compounding

The yield-to-maturity using continuous compounding, y_c, is defined by

$$B_c(0) = \sum_{t=1}^{T} c \exp(-y_c \times t) + F \exp(-y_c \times T). \tag{13.8}$$

If $t = 1, \ldots, T$ are expressed in years, then y_c is an annual continuously compounded yield-to-maturity.

EXAMPLE

Continuously Compounded Yield to Maturity

This example illustrates the computation of the continuously compounded yield-to-maturity. Let the market value of a bond be 104.52. Let the bond have a maturity of two years and a coupon of 7 percent, which is paid on an annual basis. The annual continuously compounded yield-to-maturity is defined by

$$104.52 = 7 \exp(-y_c) + 7 \exp(-2y_c) + 100 \exp(-2y_c),$$

implying that $y_c = 4.48$ percent, which, as would be expected, is slightly lower than the discretely compounded yield-to-maturity of 4.58 percent. ■

Quotes

U.S. Treasury bonds and notes are coupon-bearing bonds issued by the U.S. government. Both are normally considered to be default-free. Treasury notes are issued with maturities up to ten years, and Treasury bonds have maturities greater than ten years. All Treasury notes and bonds have semiannual coupon payments. Face value denominations range from $1,000 to $1 million. Table 13.3 gives some examples of market quotes for Treasury notes and bonds.

The convention is that prices are quoted assuming a face value of $100. An explanation of bid/ask quotes is best given via an example.

EXAMPLE

Bid/Ask Quotes

This example illustrates the computation of bid/ask prices from bid/ask quotes on Treasury bonds and notes. Suppose we have a bid or ask quote of 100:05. The figures on the right-hand side of the colon refer to 32nds of a dollar. Thus the dollar value is $100^5/_{32}$ or 100.15625.

Bond prices are quoted "flat." This means that the price does not include accrued interest. For example, consider the $11^5/_8$ November 2004 bond. Using

TABLE 13.3 *Treasury Note and Bond Quotes, Thursday, May 28, 1998**

TREASURY BONDS, NOTES & BILLS

Thursday, May 28, 1998

Representative and Indicative Over-the-Counter quotations based on $1 million or more.

Treasury bond, note and bill quotes are as of mid-afternoon. Colons in bond and note bid-and-asked quotes represent 32nds; 101:01 means 101 1/32. Net changes in 32nds. Treasury bill quotes in hundredths, quoted in terms of a rate discount. Days to maturity calculated from settlement date. All yields are based on a one-day settlement and calculated on the offer quote. Current 13-week and 26-week bills are boldfaced. For bonds callable prior to maturity, yields are computed to the earliest call date for issues quoted above par and to the maturity date for issues quoted below par. n-Treasury note. i-Inflation-indexed. wi-When issued. iw-Inflation-indexed when issued; daily change is expressed in basis points.

Source: Dow Jones/Cantor Fitzgerald.

U.S. Treasury strips as of 3 p.m. Eastern time, also based on transactions of $1 million or more. Colons in bid-and-asked quotes represent 32nds; 99:01 means 99 1/32. Net changes in 32nds. Yields calculated on the asked quotation. ci-stripped coupon interest. bp-Treasury bond, stripped principal. np-Treasury note, stripped principal. For bonds callable prior to maturity, yields are computed to the earliest call date for issues quoted above par and to the maturity date for issues below par.

Source. Bear, Stearns & Co. via Street Software Technology Inc.

GOVT. BONDS & NOTES

Rate	Maturity Mo/Yr	Bid	Asked	Chg.	Ask Yld.
5 3/8	May 98n	99:30	100:00	....	5.24
6	May 98n	99:31	100:01	....	0.30
5 1/8	Jun 98n	99:31	100:01	....	4.67
6 1/4	Jun 98n	100:02	100:04	....	4.71
8 1/4	Jul 98n	100:13	100:15	....	4.48
5 1/4	Jul 98n	100:00	100:02	+ 1	4.81
6 1/4	Jul 98n	100:06	100:08	+ 1	4.71
5 7/8	Aug 98n	100:03	100:05	....	5.06
9 1/4	Aug 98n	100:28	100:30	....	4.73
4 3/4	Aug 98n	99:26	99:28	....	5.19
6 1/8	Aug 98n	100:06	100:08	....	5.06
4 3/4	Sep 98n	99:24	99:26	....	5.27
6	Sep 98n	100:06	100:08	....	5.20
7 1/8	Oct 98n	100:22	100:24	+ 1	5.07
4 3/4	Oct 98n	99:20	99:22	....	5.49
5 7/8	Oct 98n	100:04	100:06	+ 1	5.39
5 1/2	Nov 98n	99:31	100:01	....	5.42
8 7/8	Nov 98n	101:18	101:20	....	5.25
5 1/8	Nov 98n	99:26	99:28	....	5.38
5 5/8	Nov 98n	100:02	100:04	....	5.37
5 1/8	Dec 98n	99:24	99:26	....	5.44
5 3/4	Dec 98n	100:04	100:06	....	5.41
6 3/8	Jan 99n	100:16	100:18	....	5.44
5	Jan 99n	99:19	99:21	....	5.51
5 7/8	Jan 99n	100:06	100:08	....	5.48
5	Feb 99n	99:19	99:21	....	5.48
8 7/8	Feb 99n	102:10	102:12	− 1	5.42
5 1/2	Feb 99n	99:30	100:00	....	5.49
5 7/8	Feb 99n	100:06	100:08	− 1	5.52
5 7/8	Mar 99n	100:08	100:10	....	5.48
6 1/4	Mar 99n	100:18	100:20	....	5.47
7	Apr 99n	101:07	101:09	....	5.48
6 3/8	Apr 99n	100:26	100:28	....	5.38
6 1/2	Apr 99n	100:26	100:28	− 1	5.51
6 3/8	May 99n	100:23	100:25	....	5.53
9 1/8	May 99n	103:10	103:12	....	5.47
6 1/4	May 99n	100:24	100:26	− 1	5.41
6 3/4	May 99n	101:04	101:06	....	5.52
6	Jun 99n	100:13	100:15	− 1	5.54
6 3/4	Jun 99n	101:07	101:09	....	5.51
6 3/8	Jul 99n	100:27	100:29	....	5.53
5 7/8	Jul 99n	100:08	100:10	− 1	5.59
6 7/8	Jul 99n	101:12	101:14	− 1	5.58
6	Aug 99n	100:13	100:15	− 1	5.59
8	Aug 99n	102:24	102:26	− 1	5.56
5 7/8	Aug 99n	100:09	100:11	− 1	5.58
6 7/8	Aug 99n	101:16	101:18	− 1	5.56
5 3/4	Sep 99n	100:04	100:06	− 2	5.60
7 1/8	Sep 99n	101:29	101:31	− 1	5.57
6	Oct 99n	100:15	100:17	− 2	5.59
5 5/8	Oct 99n	99:30	100:00	− 2	5.62

Rate	Maturity Mo./Yr.	Bid	Asked	Chg.	Ask Yld.
7 1/2	Oct 99n	102:16	102:18	− 2	5.59
5 7/8	Nov 99n	100:10	100:12	− 2	5.60
7 7/8	Nov 99n	103:04	103:06	− 2	5.57
5 5/8	Nov 99n	99:31	100:01	− 2	5.60
7 3/4	Nov 99n	103:01	103:03	− 2	5.58
5 5/8	Dec 99n	99:31	100:01	− 2	5.60
7 3/4	Dec 99n	103:05	103:07	− 2	5.60
6 3/8	Jan 00n	101:03	101:05	− 3	5.62
5 3/8	Jan 00n	99:18	99:20	− 2	5.61
7 3/4	Jan 00n	103:10	103:12	− 3	5.60
5 7/8	Feb 00n	100:11	100:13	− 3	5.62
8 1/2	Feb 00n	104:20	104:22	− 3	5.59
5 1/2	Feb 00n	99:24	99:26	− 3	5.61
7 1/8	Feb 00n	102:14	102:16	− 3	5.60
5 1/2	Mar 00n	99:24	99:25	− 4	5.62
6 7/8	Mar 00n	102:04	102:06	− 3	5.60
5 1/2	Apr 00n	99:24	99:26	− 3	5.60
5 5/8	Apr 00n	99:30	100:00	− 4	5.62
6 3/4	Apr 00n	101:30	102:00	− 4	5.63
6 3/8	May 00n	101:10	101:12	− 4	5.62
8 7/8	May 00n	105:30	106:00	− 4	5.60
5 1/2	May 00n	99:28	99:29	....	5.55
6 1/4	May 00n	101:07	101:09	....	5.57
5 7/8	Jun 00n	100:14	100:16	− 4	5.61
6 1/8	Jul 00n	100:31	101:01	− 3	5.61
6	Aug 00n	100:23	100:25	− 4	5.62
8 3/4	Aug 00n	106:13	106:15	− 4	5.60
6 1/4	Aug 00n	101:08	101:10	− 4	5.62
6 1/8	Sep 00n	101:01	101:03	− 4	5.61
5 3/4	Oct 00n	100:08	100:10	− 3	5.61
5 3/4	Nov 00n	100:08	100:10	− 3	5.61
8 1/2	Nov 00n	106:17	106:19	− 3	5.59
5 5/8	Nov 00n	99:31	100:01	− 3	5.61
5 1/2	Dec 00n	99:21	99:23	− 3	5.62
5 1/4	Jan 01n	99:02	99:04	− 2	5.60
5 3/8	Feb 01n	99:12	99:13	− 2	5.61
7 3/4	Feb 01n	105:09	105:11	− 2	5.60
11 3/4	Feb 01	115:11	115:15	− 3	5.53
5 5/8	Feb 01n	100:00	100:02	− 2	5.60
6 3/8	Mar 01n	101:29	101:31	− 2	5.61
6 1/4	Apr 01n	101:21	101:23	− 1	5.60
5 5/8	May 01n	100:05	100:06	− 1	5.55
8	May 01n	106:08	106:10	− 6	5.65
13 1/8	May 01	120:02	120:08	− 7	5.61
6 1/2	May 01n	102:12	102:14	− 1	5.61
6 5/8	Jun 01n	102:24	102:26	− 1	5.62
6 5/8	Jul 01n	102:26	102:28	− 1	5.62
7 7/8	Aug 01n	106:17	106:19	− 1	5.60
13 3/8	Aug 01	122:18	122:24	− 2	5.55
6 1/2	Aug 01n	102:17	102:19	− 1	5.61
6 3/8	Sep 01n	102:06	102:08	− 1	5.62
6 1/4	Oct 01n	101:27	101:29	− 2	5.63

Mat.	Type	Bid	Asked	Chg.	Ask Yld.
7 1/2	Nov 01n	105:25	105:27	− 1	5.62
15 3/4	Nov 01	131:18	131:24	− 2	5.54
5 7/8	Nov 01n	100:23	100:25	− 1	5.63
6 1/8	Dec 01n	101:17	101:19	− 1	5.63
6 1/4	Jan 02n	101:31	102:01	− 2	5.63
14 1/4	Feb 02	128:17	128:23	− 2	5.58
6 1/4	Feb 02n	102:00	102:02	− 1	5.63
6 5/8	Mar 02n	103:10	103:12	− 1	5.63
6 5/8	Apr 02n	103:19	103:21	− 2	5.57
7 1/2	May 02n	106:16	106:18	− 2	5.63
6 1/2	May 02n	103:07	103:09	− 2	5.57
6 1/4	Jun 02n	102:05	102:07	− 1	5.63
3 5/8	Jul 02i	98:25	98:26	− 2	3.94
6	Jul 02n	101:09	101:11	− 1	5.63
6 3/8	Aug 02n	102:22	102:24	− 2	5.63
6 1/4	Aug 02n	102:06	102:08	− 2	5.64
5 7/8	Sep 02n	100:27	100:29	− 1	5.63
5 3/4	Oct 02n	100:12	100:14	− 2	5.64
11 5/8	Nov 02	123:10	123:16	− 2	5.60
5 3/4	Nov 02n	100:11	100:13	− 2	5.65
5 5/8	Dec 02n	99:29	99:31	− 2	5.63
5 1/2	Jan 03n	99:15	99:17	− 2	5.61
6 1/4	Feb 03n	102:15	102:17	− 2	5.63
10 3/4	Feb 03	120:25	120:31	− 2	5.62
5 1/2	Feb 03n	99:16	99:18	− 2	5.60
5 1/2	Mar 03n	99:17	99:18	− 2	5.60
5 3/4	Apr 03n	100:21	100:22	− 2	5.59
10 3/4	May 03	121:24	121:30	− 2	5.62
	May 03wi	5:58	5:57	+ 1	....
5 3/4	Aug 03n	100:17	100:19	− 1	5.61
11 1/8	Aug 03	124:11	124:17	− 1	5.63
11 7/8	Nov 03	128:25	128:31	+ 1	5.64
5 7/8	Feb 04n	101:07	101:09	+ 2	5.61
7 1/4	May 04n	108:01	108:03	+ 3	5.63
12 3/8	May 04	133:17	133:23	+ 4	5.64
7 1/4	Aug 04n	108:09	108:11	+ 5	5.64
13 3/4	Aug 04	141:26	142:00	+ 6	5.65
7 7/8	Nov 04n	111:25	111:29	+ 6	5.65
11 5/8	Nov 04	131:25	131:31	+ 7	5.65
7 1/2	Feb 05n	110:02	110:04	+ 7	5.67
6 1/2	May 05n	104:24	104:26	+ 10	5.65
8 1/4	May 00-05	104:25	104:27	+ 3	5.41
12	May 05	135:28	136:02	+ 12	5.66
6 1/2	Aug 05n	104:25	104:27	+ 11	5.67
10 3/4	Aug 05	129:16	129:22	+ 12	5.68
5 7/8	Nov 05n	101:04	101:06	+ 12	5.68
5 5/8	Feb 06n	99:20	99:22	+ 15	5.67
9 3/8	Feb 06	122:25	122:31	+ 16	5.66
6 7/8	May 06n	107:15	107:17	+ 17	5.69
7	Jul 06n	108:10	108:12	+ 19	5.70
6 1/2	Oct 06n	105:08	105:10	+ 21	5.69
3 3/8	Jan 07i	97:00	97:01	− 1	3.78
6 1/4	Feb 07n	103:27	103:29	+ 24	5.67
7 5/8	Feb 02-07	106:09	106:11	+ 12	5.70
6 5/8	May 07n	106:15	106:17	+ 26	5.68
6 1/8	Aug 07n	103:04	103:06	+ 28	5.67
7 7/8	Nov 02-07	109:02	109:04	+ 17	5.54
3 5/8	Jan 08i	98:31	99:00	− 2	3.75
5 1/2	Feb 08n	99:10	99:11	+ 32	5.59
5 5/8	May 08n	100:12	100:13	+ 35	5.57
8 3/8	Aug 03-08	112:08	112:12	+ 22	5.60
8 3/4	Nov 03-08	113:30	114:02	+ 23	5.72
9 1/8	May 04-09	116:27	116:31	+ 24	5.72
10 3/8	Nov 04-09	124:21	124:27	+ 27	5.72
11 3/4	Feb 05-10	133:02	133:08	+ 28	5.72
10	May 05-10	124:01	124:07	+ 27	5.73
12 3/4	Nov 05-10	141:31	142:05	+ 30	5.73
13 7/8	May 06-11	151:09	151:15	+ 33	5.73
14	Nov 06-11	154:19	154:25	+ 34	5.74
10 3/8	Nov 07-12	133:01	133:07	+ 31	5.77
12	Aug 08-13	147:10	147:16	+ 34	5.78
13 1/4	May 09-14	159:24	159:30	+ 36	5.79
12 1/2	Aug 09-14	154:17	154:23	+ 35	5.80
11 3/4	Nov 09-14	149:04	149:10	+ 34	5.80
11 1/4	Feb 15	157:12	157:18	+ 45	5.82

Table 13.3 *(Continued)*

Rate	Maturity Mo/Yr	Bid	Asked	Chg.	Ask Yld.
10 5/8	Aug 15	150:30	151:04	+43	5.87
9 7/8	Nov 15	143:01	143:07	+41	5.88
9 1/4	Feb 16	136:14	136:20	+39	5.89
7 1/4	May 16	114:23	114:27	+34	5.90
7 1/2	Nov 16	117:18	117:22	+34	5.91
8 3/4	May 17	131:29	132:03	+36	5.91
8 7/8	Aug 17	133:17	133:23	+35	5.91
9 1/8	May 18	137:03	137:09	+34	5.92
9	Nov 18	136:02	136:08	+33	5.92
8 7/8	Feb 19	134:23	134:29	+33	5.93
8 1/8	Aug 19	126:04	126:10	+30	5.93
8 1/2	Feb 20	130:31	131:05	+29	5.93
8 3/4	May 20	134:07	134:13	+29	5.93
8 3/4	Aug 20	134:13	134:19	+29	5.93
7 7/8	Feb 21	123:27	124:01	+25	5.94
8 1/8	May 21	127:02	127:08	+25	5.94
8 1/8	Aug 21	127:06	127:12	+24	5.94
8	Nov 21	125:25	125:31	+24	5.93
7 1/4	Aug 22	116:22	116:26	+20	5.93
7 5/8	Nov 22	121:17	121:23	+20	5.93
7 1/8	Feb 23	115:10	115:14	+18	5.93
6 1/4	Aug 23	104:08	104:10	+16	5.92
7 1/2	Nov 24	120:27	121:01	+13	5.92
7 5/8	Feb 25	122:19	122:25	+13	5.92
6 7/8	Aug 25	112:23	112:27	+10	5.92
6	Feb 26	101:06	101:08	+9	5.91
6 3/4	Aug 26	111:10	111:14	+7	5.91
6 1/2	Nov 26	108:02	108:04	+6	5.91
6 5/8	Feb 27	109:29	109:31	+6	5.90
6 3/8	Aug 27	106:26	106:27	+4	5.88
6 1/8	Nov 27	104:04	104:05	+3	5.83
3 5/8	Apr 28i	99:14	99:15	−4	3.65

U.S. TREASURY STRIPS

Mat.	Type	Bid	Asked	Chg.	Ask Yld.
Aug 98	ci	98:29	98:29	+ 1	5.27
Aug 98	np	98:29	98:29	+ 1	5.32
Nov 98	ci	97:20	97:20	+ 1	5.35
Nov 98	np	97:18	97:19	+ 1	5.45
Feb 99	ci	96:08	96:08	+ 1	5.46
Feb 99	np	96:06	96:07	+ 1	5.54
May 99	ci	94:30	94:30	+ 1	5.52
May 99	np	94:28	94:29	+ 1	5.57
Aug 99	ci	93:19	93:20	....	5.53
Aug 99	np	93:18	93:19	+ 1	5.57
Nov 99	ci	92:11	92:12	....	5.53
Nov 99	np	92:10	92:11	....	5.56
Feb 00	ci	91:02	91:03	....	5.54
Feb 00	np	91:01	91:02	....	5.56
May 00	ci	89:28	89:29	....	5.52
May 00	np	89:26	89:27	− 1	5.57
Aug 00	ci	88:19	88:20	− 1	5.54
Aug 00	np	88:18	88:19	− 1	5.56
Nov 00	ci	87:12	87:14	− 1	5.55
Nov 00	np	87:11	87:12	− 1	5.57
Feb 01	ci	86:06	86:07	− 1	5.55
Feb 01	np	86:04	86:05	− 1	5.58
May 01	ci	84:31	85:00	− 1	5.57
May 01	np	84:29	84:31	− 1	5.59
Aug 01	ci	83:25	83:27	− 2	5.57
Aug 01	np	83:22	83:24	− 2	5.61
Nov 01	ci	82:21	82:23	− 2	5.57
Nov 01	np	82:17	82:19	− 2	5.62
Feb 02	ci	81:16	81:18	− 2	5.57
May 02	ci	80:12	80:14	− 2	5.59
May 02	np	80:08	80:10	− 2	5.62
Aug 02	ci	79:11	79:13	− 2	5.56
Aug 02	np	79:04	79:06	− 2	5.63
Nov 02	ci	78:15	78:19	− 4	5.49
Feb 03	ci	77:02	77:05	− 2	5.59
Feb 03	np	77:00	77:03	− 2	5.60
May 03	ci	75:30	76:02	− 2	5.60
Aug 03	ci	74:28	75:00	− 3	5.60
Aug 03	np	74:31	75:03	− 3	5.58
Nov 03	ci	73:28	73:31	− 3	5.60
Feb 04	ci	72:25	72:29	− 2	5.62
Feb 04	np	72:30	73:02	− 2	5.58
May 04	ci	71:23	71:27	− 2	5.63
May 04	np	71:26	71:30	− 2	5.61
Aug 04	ci	70:25	70:29	− 2	5.61
Aug 04	np	70:25	70:29	− 2	5.62
Nov 04	ci	69:21	69:25	− 3	5.65
Nov 04	bp	69:19	69:24	− 2	5.66
Nov 04	np	69:22	69:26	− 2	5.65
Feb 05	ci	68:20	68:25	− 2	5.66
Feb 05	np	68:22	68:27	− 2	5.65
May 05	ci	67:20	67:25	− 2	5.67
May 05	bp	67:17	67:21	− 3	5.70
May 05	np	67:27	67:31	− 3	5.63
Aug 05	ci	66:22	66:27	− 3	5.67
Aug 05	bp	66:18	66:22	− 3	5.70
Aug 05	np	66:24	66:28	− 3	5.66
Nov 05	ci	65:24	65:29	− 4	5.67
Nov 05	np	65:25	65:30	− 3	5.67
Feb 06	ci	64:23	64:28	− 2	5.70
Feb 06	bp	64:27	65:00	− 2	5.67
Feb 06	np	64:28	65:01	− 2	5.66
May 06	ci	63:24	63:29	− 2	5.71
Aug 06	ci	62:28	63:01	− 2	5.71
Nov 06	ci	62:00	62:04	− 2	5.71
Feb 07	ci	61:00	61:05	− 2	5.73
May 07	ci	60:04	60:09	− 3	5.74
Aug 07	ci	59:10	59:15	− 3	5.72
Nov 07	ci	58:15	58:21	− 3	5.73
Feb 08	ci	57:15	57:20	− 2	5.76
May 08	ci	56:18	56:24	− 2	5.78
Aug 08	ci	55:24	55:29	− 2	5.78
Nov 08	ci	54:26	55:00	− 2	5.80
Feb 09	ci	53:31	54:04	− 2	5.82
May 09	ci	53:04	53:10	− 2	5.83
Aug 09	ci	52:11	52:16	− 2	5.83
Nov 09	ci	51:17	51:23	− 2	5.84
Nov 09	bp	51:06	51:12	− 2	5.90
Feb 10	ci	50:23	50:28	− 2	5.86
May 10	ci	49:30	50:03	− 2	5.87
Aug 10	ci	49:05	49:11	− 2	5.87
Nov 10	ci	48:15	48:20	− 2	5.87
Feb 11	ci	47:22	47:28	....	5.88
May 11	ci	46:31	47:05	....	5.89
Aug 11	ci	46:08	46:13	....	5.90
Nov 11	ci	45:17	45:23	....	5.91
Feb 12	ci	44:26	45:00	....	5.91
May 12	ci	44:05	44:10	+ 1	5.92
Aug 12	ci	43:15	43:21	+ 1	5.92
Nov 12	ci	42:26	43:00	+ 2	5.93
Feb 13	ci	42:05	42:10	+ 1	5.93
May 13	ci	41:16	41:22	+ 1	5.94
Aug 13	ci	40:28	41:02	+ 1	5.94
Nov 13	ci	40:08	40:14	+ 1	5.95
Feb 14	ci	39:20	39:26	+ 1	5.95
May 14	ci	39:02	39:07	+ 1	5.95
Aug 14	ci	38:14	38:20	+ 1	5.95
Nov 14	ci	37:28	38:02	+ 1	5.96
Feb 15	ci	37:10	37:16	....	5.96
Feb 15	bp	38:09	38:15	....	5.80
May 15	ci	36:24	36:30	+ 1	5.96
Aug 15	ci	36:06	36:11	....	5.97
Aug 15	bp	36:18	36:24	....	5.91
Nov 15	ci	35:20	35:26	....	5.97
Nov 15	bp	35:27	36:01	+ 1	5.94
Feb 16	ci	35:02	35:08	....	5.97
Feb 16	bp	35:07	35:13	+ 1	5.95
May 16	ci	34:18	34:24	+ 1	5.98
May 16	bp	34:27	35:01	+ 1	5.93
Aug 16	ci	34:01	34:07	+ 1	5.98
Nov 16	ci	33:17	33:22	+ 1	5.98
Nov 16	bp	33:21	33:27	+ 1	5.96
Feb 17	ci	33:00	33:05	+ 2	5.99
May 17	ci	32:16	32:22	+ 1	5.99
May 17	bp	32:17	32:23	+ 1	5.98
Aug 17	ci	32:00	32:06	+ 1	5.99
Aug 17	bp	32:02	32:08	+ 1	5.98
Nov 17	ci	31:17	31:23	+ 2	5.99
Feb 18	ci	31:00	31:06	+ 1	6.00
May 18	ci	30:18	30:23	+ 1	6.00
May 18	bp	30:19	30:24	+ 1	6.00
Aug 18	ci	30:03	30:09	+ 1	6.00
Nov 18	ci	29:20	29:26	+ 1	6.01
Nov 18	bp	29:21	29:27	+ 1	6.00
Feb 19	ci	29:05	29:11	+ 2	6.01
Feb 19	bp	29:07	29:12	+ 1	6.00
May 19	ci	28:23	28:29	+ 1	6.01
Aug 19	ci	28:10	28:15	+ 1	6.01
Aug 19	bp	28:11	28:17	+ 1	6.00
Nov 19	ci	27:28	28:02	+ 1	6.01
Feb 20	ci	27:14	27:20	+ 1	6.01
Feb 20	bp	27:17	27:22	+ 1	6.00
May 20	ci	27:02	27:07	+ 1	6.01
May 20	bp	27:03	27:09	+ 1	6.01
Aug 20	ci	26:21	26:26	+ 1	6.01
Aug 20	bp	26:22	26:28	+ 1	6.01
Nov 20	ci	26:11	26:16	+ 1	6.00
Feb 21	ci	25:28	26:01	+ 1	6.01
Feb 21	bp	25:30	26:04	+ 1	6.00
May 21	ci	25:16	25:21	+ 1	6.01
May 21	bp	25:17	25:23	+ 1	6.01
Aug 21	ci	25:05	25:10	+ 2	6.01
Aug 21	bp	25:05	25:10	+ 1	6.01
Nov 21	ci	24:28	25:01	+ 1	5.99
Nov 21	bp	24:25	24:31	+ 1	6.01
Feb 22	ci	24:13	24:18	+ 1	6.01
May 22	ci	24:02	24:08	+ 1	6.01
Aug 22	ci	23:24	23:30	+ 1	6.00
Aug 22	bp	23:26	24:00	+ 1	5.99
Nov 22	ci	23:16	23:21	+ 1	5.98
Nov 22	bp	23:14	23:20	+ 1	5.99
Feb 23	ci	23:04	23:09	+ 1	5.99
Feb 23	bp	23:07	23:12	+ 1	5.97
May 23	ci	22:28	23:01	+ 1	5.97
Aug 23	ci	22:18	22:23	+ 1	5.97
Aug 23	bp	22:23	22:29	+ 1	5.93
Nov 23	ci	22:09	22:15	+ 1	5.95
Feb 24	ci	21:27	22:00	+ 1	5.97
May 24	ci	21:18	21:23	+ 1	5.97
Aug 24	ci	21:08	21:13	+ 1	5.97
Nov 24	ci	20:31	21:04	+ 1	5.96
Nov 24	bp	21:00	21:05	+ 1	5.96
Feb 25	ci	20:22	20:28	+ 1	5.95
Feb 25	bp	20:24	20:29	+ 1	5.95
May 25	ci	20:14	20:19	+ 1	5.95
Aug 25	ci	20:05	20:11	+ 1	5.94
Aug 25	bp	20:05	20:10	+ 1	5.95
Nov 25	ci	19:29	20:02	+ 1	5.93
Feb 26	ci	19:22	19:27	+ 1	5.92
Feb 26	bp	19:25	19:31	+ 1	5.90
May 26	ci	19:13	19:18	+ 1	5.92
Aug 26	ci	19:04	19:09	+ 1	5.92
Aug 26	bp	19:05	19:10	+ 1	5.91
Nov 26	ci	18:28	19:01	+ 1	5.92
Nov 26	bp	18:30	19:03	+ 1	5.91
Feb 27	ci	18:22	18:27	+ 1	5.90
Feb 27	bp	18:24	18:29	+ 1	5.89
May 27	ci	18:17	18:22	+ 1	5.88
Aug 27	ci	18:15	18:20	+ 1	5.84
Aug 27	bp	18:18	18:23	+ 1	5.82
Nov 27	ci	18:19	18:25	+ 1	5.76
Nov 27	bp	19:02	19:07	+ 1	5.68

*Source: The *Wall Street Journal*, May 29, 1998.

hypothetical data, suppose that the bid price is 129:28 and the ask price 130:00. The accrued interest, AI, is determined by the formula

$$AI \equiv C\left(\frac{\text{Number of Days Since Last Coupon Was Paid}}{\text{Total Number of Days in Current Coupon Period}}\right), \tag{13.9}$$

where C is the semiannual coupon payment. In this case, suppose that the number of days in the current coupon period is 182 and the number of days since the last coupon was paid is 85. The accrued interest is

$$AI = \left(\frac{11.625}{2}\right)\left(\frac{85}{182}\right) = 2.7145.$$

If the bond is purchased at the bid price, the total cost is

$$129(^{28}/_{32}) + 2.7145 = 132.5895.$$

If the bond is purchased at the ask price, the total cost is

$$130 + 2.7145 = 132.7145.$$

■

The final column of Table 13.3 refers to **stripped Treasuries**. A Treasury note or bond is composed of two components: coupon payments and a final payment of principal. These two components can be sold separately as synthetic zero-coupon bonds, which are the Treasury strips. The notation **np** or **bp** means it is a Treasury note or Treasury bond principal payment underlying the strip and **ci** means the instrument is a claim on coupon payments.

Floating Rate Notes

A **floating rate note** is a debt contract with specified face value, maturity, and coupon payment dates. The interest payments change over time, and they are based on the current interest rate times the principal. A floating rate note's interest payments are reset at each coupon date.

Let us consider a one-year floating rate note with semiannual interest payments and unit face value. Today, the coupon for the first six months is based on the date-0 six-month rate. Let the six-month rate at date 0 be 5.25 percent per annum; the coupon is then $c = 0.0525/2 = 0.02625$. Let us move forward six months: The first coupon has been paid and a new coupon is set. The next coupon is based on the new six-month rate. For example, suppose in six months that the new six-month rate is 5.60 percent expressed on a per annum basis; the next coupon is then

$$\begin{aligned} c &= 0.056/2 \\ &= 0.028. \end{aligned}$$

The computation of interest payments in this way results in the floating rate note always being valued at par (a dollar value) at each reset date. To see this, we start at the last payment date and work backward in time. Six months from now, the floating rate note has one coupon payment remaining plus a principal repayment. The coupon payment is \$0.028 and the principal repayment is \$1.00. The present value of these cash flows is a dollar, that is,

$$\frac{0.028 + 1}{[1 + 0.056 \times (1/2)]} = 1.$$

This occurs because the discount rate corresponds to the interest earned.

Similarly, at date 0, the value of the floating rate loan is again a dollar. The floating rate note receives the next coupon payment of 0.02625, and it can be retired at a dollar at the next reset date. The present value of these cash flows is again a dollar, that is,

$$\frac{0.02625 + 1}{[1 + 0.0525 \times (1/2)]} = 1.$$

At reset dates the floating rate note sells at its par value. In this example, the par value is a dollar.

13.3 THE TERM STRUCTURE OF DEFAULT-FREE INTEREST RATES

Now we study the term structure of interest rates, which is defined as the relationship between the yield-to-maturity on a zero-coupon bond and the bond's maturity. Figure 13.1 shows a typical term structure. In this figure, the term structure is upward sloping, which is the most common shape. Historically, however, both flat and downward sloping term structures have been observed.

Forward Rates

Here we examine forward rates. Before giving the formal definition, we explain forward rates through a simple example.

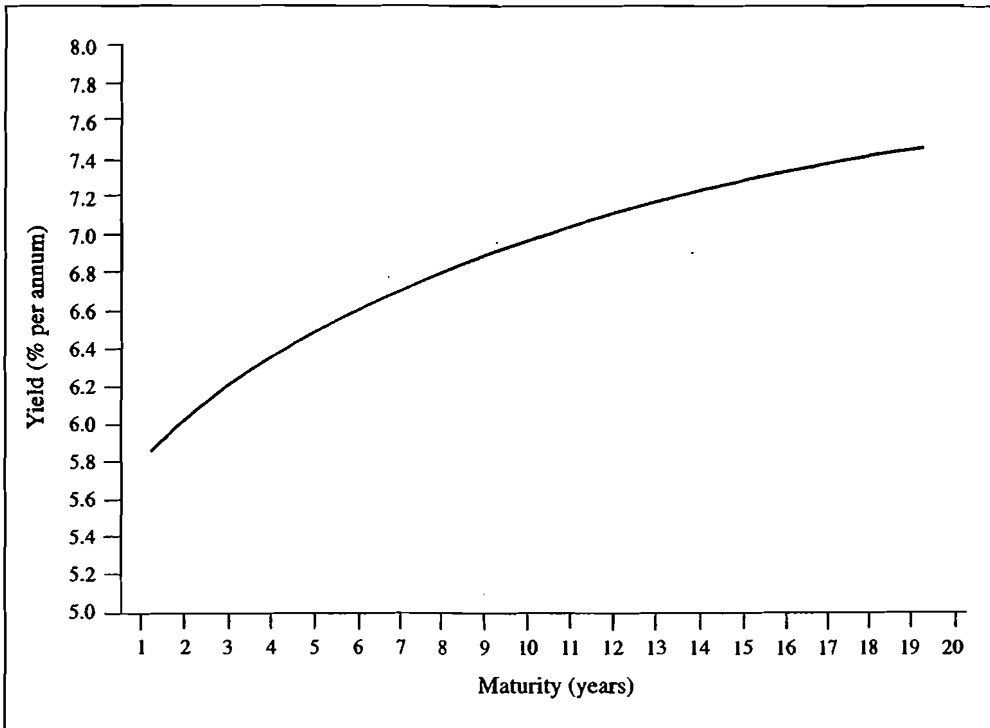
Suppose that the yield on a one-year zero-coupon bond is 5.85 percent per annum, and 6.03 percent per annum on a two-year zero-coupon bond. If we invest one dollar for two years in the two-year zero-coupon bond, we earn

$$\$(1 + 0.0603)^2$$

at the end of two years. Alternatively, we could invest one dollar in a series of one-year zero-coupon bonds, rolling over the investment at the end of the first year. At the end of the first year, the value of the investment is

$$\$(1 + 0.0585).$$

FIGURE 13.1 *A Typical Term Structure of Interest Rates*



The one-year rate of interest from year one to year two is not yet known today, given that interest rates are random. But we can always find an implied one-year "break-even" interest rate to equate the value of the two investment strategies. Above this break-even rate, the rollover strategy is better; below this break-even rate, the two-year strategy is better. By definition, the one-year break-even rate from year one to year two is

$$(1 + 0.0603)^2 \equiv (1 + 0.0585)[1 + f(0,1,2)],$$

implying that the break-even rate is

$$f(0,1,2) = 0.0621.$$

This break-even rate is called the **forward interest rate** at date zero from year one to year two. The first argument of zero in the notation $f(0,1,2)$ is used to denote the fact that the forward rate is implied by the term structure at date zero. The second argument denotes the date the forward rate starts, and the third argument denotes the date the forward rate ends. To summarize, given the term structure of interest rates at date 0, the forward rate from date 1 to date 2 is denoted by $f(0,1,2)$.

We could determine the forward rate in terms of bond prices. The value of a one-year zero-coupon bond is

$$B(0,1) = \frac{1}{1 + 0.0585}$$

and the value of a two-year zero-coupon bond is

$$B(0,2) = \frac{1}{(1 + 0.0603)^2}.$$

Therefore, the forward rate can be computed by

$$B(0,2) \equiv B(0,1)\frac{1}{1 + f(0,1,2)}.$$

This completes the simple example. The next section formalizes the above definition for arbitrary future time periods.

Formalization

Here we give the formal definition of a forward rate. The one-year forward rate from year T to year $T + 1$ implied by today's term structure is defined by

$$B(0,T + 1) \equiv B(0,T)\frac{1}{1 + f(0,T,T + 1)}. \tag{13.10}$$

Figure 13.2 shows the one-year forward rates for an upward sloping term structure. For an upward sloping term structure, observe that the forward rate is never less than the zero-coupon yield, a consequence of the mathematical definition in Expression (13.10).

In Figure 13.3, the term structure is inverted or downward sloping. In this case, the forward curve is always below the zero-coupon yield curve.

Expression (13.10) gives the definition of the one-year forward rate. In practice, it is often necessary to define forward rates for other intervals such as one month, three months, or six months. It is a straightforward exercise to generalize Expression (13.10).

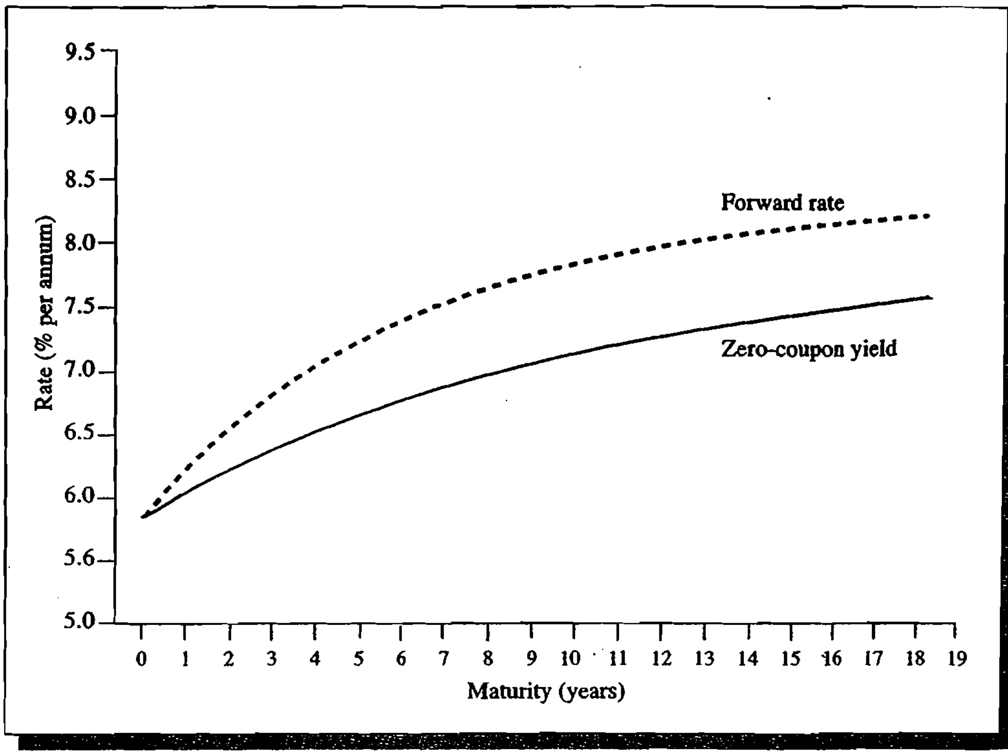
Given the term structure of interest rates at date 0, the Δ period forward rate from date T to date $T + \Delta$ is defined by

$$B(0,T + \Delta) \equiv B(0,T)\,\frac{1}{1 + f(0,T,T + \Delta)\,\Delta}, \tag{13.11}$$

where Δ is measured in units of a year.

For a six-month forward rate, $\Delta = {}^1\!/_2$; for a three-month forward rate, $\Delta = {}^1\!/_4$; and for a one-month forward rate, $\Delta = {}^1\!/_{12}$. We illustrate this definition via an example.

FIGURE 13.2 *Forward Curve Upward Sloping Term Structure*



EXAMPLE **Forward Rate Calculations.**

This example illustrates the use of Expression (13.11). We are given the following information:

Maturity	T-bill Price
12 months	91.5042
18 months	87.6083
24 months	83.9256

The one-year forward rate, from year one to year two, is given by

$$f(0,12,24) = \frac{91.5042}{83.9256} - 1 = 0.0903$$

The six-month rate, from 12 to 18 months, is given by

$$f(0,12,18) = 2\left(\frac{91.5042}{87.6083} - 1\right) = 0.0889$$

The six-month rate, from 18 to 24 months, is given by

$$f(0, 18, 24) = 2\left(\frac{87.6056}{83.9251} - 1\right) = 0.0878.$$

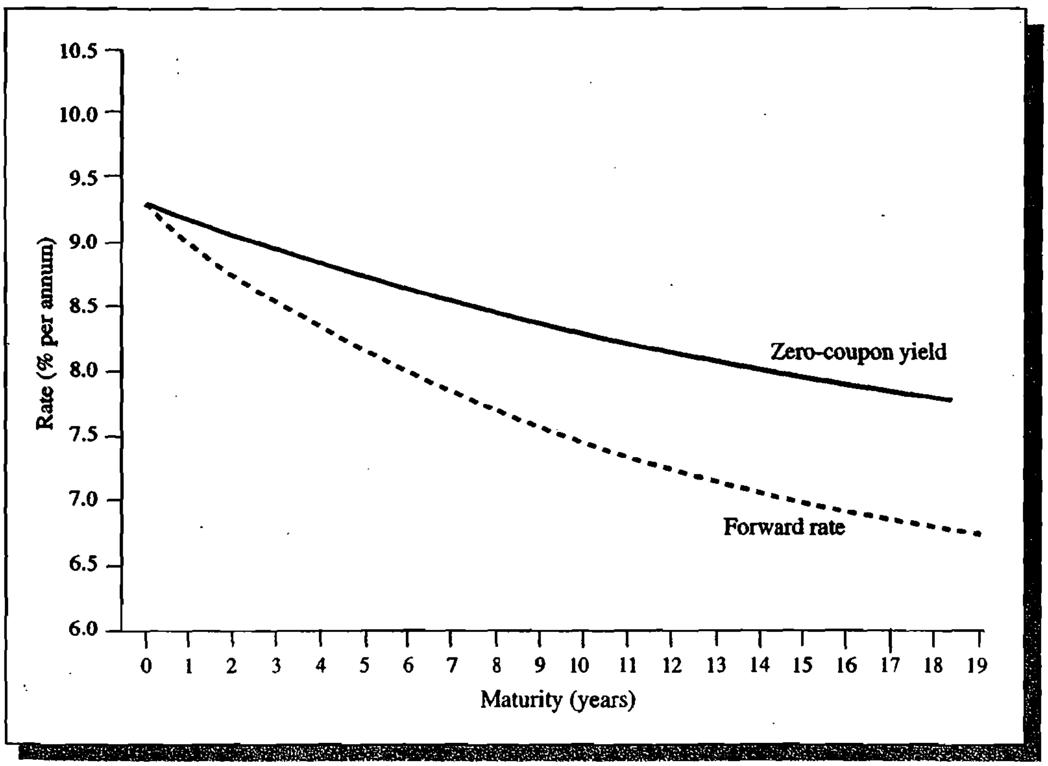
Note that the relation between the six-month rates and the one-year rate is

$$\begin{aligned}[1 + f(0, 12, 18)(1/2)][1 + f(0, 18, 24)(1/2)] &= 1.0903 \\ &= 1 + f(0, 12, 24). \quad \blacksquare\end{aligned}$$

Par Bond Yield Curve

Consider a coupon bond. Let the coupon on the bond be set such that the bond sells at par. The curve showing the relation between the bond's maturity and the coupon rate is called the **par bond yield curve**, which is usually abbreviated to **par yield curve**. We will use an example to illustrate how to calculate the par yield curve.

FIGURE 13.3 *Forward Curve Downward Sloping Term Structure*



EXAMPLE

Par Yield Curve

This example illustrates the computation of a par yield curve. We are given the following information.

Maturity (Years)	T-bill Price
1	94.94
2	89.82
3	84.70

For a bond of maturity one year with face value 100 and paying an annual coupon c, the value of the bond is

$$c \times (0.9494) + 100 \times 0.9494$$
$$= 100.$$

We want to select the coupon so that the bond sells at par. This gives the equation for the par yield:

$$c = 100(1 - 0.9494)/0.9494$$
$$= 5.33.$$

For a bond with maturity two years with face value 100 and paying an annual coupon of c, the par yield equation is

$$c \times (0.9494 + 0.8982) + 100 \times 0.8982$$
$$= 100.$$

Therefore:

$$c = 100(1 - 0.8982)/1.8476$$
$$= 5.51.$$

For a bond with maturity three years and face value 100 paying an annual coupon of c, the par yield equation is

$$c \times (0.9494 + 0.8982 + 0.8470) + 100 \times 0.8470$$
$$= 100.$$

Therefore:

$$c = 100(1 - 0.8470)/2.6946$$
$$= 5.68.$$

Generalizing this example, given a bond with maturity T years, a face value of F, and which pays an annual coupon of c dollars at dates $1, 2, \ldots, T$, the par yield equation is

$$\sum_{t=1}^{T} c \times B(0,t) + F \times B(0,T) = F. \tag{13.12}$$

Hence the par coupon yield is

$$c = F \times [1 - B(0,T)] / \sum_{t=1}^{T} B(0,t). \tag{13.13}$$

Note that to calculate the par coupon yield, it is necessary to know the entire term structure of interest rates. The discussion of the par yield curve is now complete.

Computing the Zero-Coupon Yield Curve

We now discuss some practical difficulties that arise in computing the zero-coupon yield curve using market prices. To construct the zero-coupon yield curve, we require a complete set of zero-coupon bond prices out to twenty or thirty years. Unfortunately, these zero-coupon bonds do not trade. The longest maturity for a Treasury bill is one year, implying that we must extract zero-coupon bond prices with maturities greater than one year from coupon bond prices (or use the Treasury strips). We illustrate this calculation through an example.

EXAMPLE

Implicit Zero-Coupon Bond Prices

In this example we illustrate a "bootstrapping" technique for calculating the zero-coupon yields using coupon bond prices. Suppose that we know the price of a six-month T-bill, $B(0, 6) = 0.9748$, and a twelve-month T-bill, $B(0, 12) = 0.9493$. A Treasury note with maturity 18 months and paying a semiannual coupon of 8.0 percent per annum is trading at 103.77. Given this information, we can determine the implied price of a zero-coupon bond with maturity 18 months via the equation

$$\begin{aligned} 103.77 &= 4 \times B(0,6) + 4 \times B(0,12) + (4 + 100) \times B(0,18) \\ &= 4 \times (0.9748 + 0.9493) + 104 \times B(0,18), \end{aligned}$$

implying that

$$B(0,18) = 0.9238. \qquad \blacksquare$$

This type of "bootstrapping" technique, illustrated in the previous example, can be used to determine the whole term structure of zero-coupon yields. For example, with a two-year Treasury note we can determine the value of the two-year zero-coupon bond's yield.

In practice, however, many complications arise. First, the maturity structure of existing Treasury notes and bonds is not equally spaced. For example, we may have a Treasury note with maturity of two years. The next Treasury note may have a maturity of two years and nine and a half months. This necessitates using different econometric techniques to generate the curve. Second, we may want to use only prices for the **on-the-run** bonds, that is, prices for the last auctioned bonds, which are usually the most actively traded. While we may have quotes for all the available bonds, both on-the-run and off-the-run, some of these quotes may be "old" and not reflect current conditions. Third, some bonds have special tax treatments that affect the price. These bonds should be omitted from the computation. Unfortunately, due to these three observations, constructing a term structure of interest rates is not an easy exercise.

13.4 TRADITIONAL MEASURES OF INTEREST RATE RISK

We now study the traditional measures of interest rate risk—duration and convexity—and their limitations. These traditional measures currently enjoy widespread use by commercial and investment banks. However, they will eventually be replaced by the techniques presented in Chapter 15 through Chapter 17.

Duration

Duration is often used to measure the risk of a bond. To see why, let B_c denote the current value of a coupon bond with yield-to-maturity y. Let the bond pay an annual coupon of c dollars at dates $t = 1, \ldots, T$, and have a face value of F dollars.[2] From Expression (13.5) we can write the bond's value as

$$B_c(y) = \sum_{t=1}^{T} \frac{c}{(1+y)^t} + \frac{F}{(1+y)^T}. \tag{13.14}$$

If the yield-to-maturity changes by a small amount, Δy, by how much does the bond's price change?

Let ΔB_c denote the change in the bond's price, that is,

$$\Delta B_c \equiv B_c(y + \Delta y) - B_c(y).$$

For a small change in the yield-to-maturity, the change in the bond's price can be written as

$$\Delta B_c \approx -\frac{1}{(1+y)}\left[\sum_{t=1}^{T} \frac{tc}{(1+y)^t} + \frac{TF}{(1+y)^T}\right]\Delta y. \tag{13.15}$$

[2]The analysis for a bond that pays a semiannual coupon is given in the Appendix of this chapter.

We obtain this expression by using a Taylor series expansion of the bond's price around the bond's yield.[3] Expression (13.15) shows the sensitivity of the bond's price to changes in the bond's yield. The coefficient preceding Δy is a measure of this sensitivity, that is, a measure of risk. We now show how this coefficient relates to duration.

The classical definition of duration is

$$D_c \equiv \left[\sum_{t=1}^{T} \frac{tc}{(1+y)^t} + \frac{TF}{(1+y)^T}\right] / B_c, \tag{13.16}$$

which is often referred to as Macauley's Duration.

Substituting Expression (13.16) into Expression (13.15) gives

$$\Delta B_c \simeq -B_c D_c \frac{\Delta y}{1+y}. \tag{13.17}$$

The proportional change in the bond's price is related to the duration multiplied by the change in the yield-to-maturity divided by one plus the initial yield-to-maturity. If we change the definition of duration, we can obtain a simpler relation. The second definition is referred to as **modified duration**[4]:

$$D_M = D_c/(1+y). \tag{13.18}$$

When we substitute the expression for modified duration into Expression (13.17), we find that the bond's return is directly proportional to modified duration, that is,

$$\Delta B_c \simeq -B_c \times D_M \times \Delta y. \tag{13.19}$$

For a given change in the yield-to-maturity, the change in the bond's value is the multiplicative product of the three terms on the right side of Expression (13.19). The minus sign reflects the inverse relationship between yield and price. This expression explains why modified duration is used as a measure of a bond's risk. It measures the sensitivity of the bond's return to changes in the bond's yield.

EXAMPLE

Duration and Modified Duration for a Zero-Coupon Bond

Here we show how the definitions of duration simplify for a zero-coupon bond. Consider a zero-coupon bond with a maturity of T years. The yield-to-maturity is defined by

$$B_c(y) = \frac{F}{(1+y)^T}.$$

[3]For people with a knowledge of calculus use of Taylor's series expansion implies

$$B_c(y+\Delta y) - B_c(y) = \frac{dB_c}{dy}(\Delta y) + \frac{1}{2}\frac{d^2B_c}{dy^2}(\Delta y)^2 + \dots.$$

[4]This definition is used in the CBOT publication *Understanding Duration and Convexity*.

The change in the bond's value from Expression (13.15) is

$$\Delta B_c(y) = -\frac{TF}{(1+y)^T}\frac{\Delta y}{1+y}$$
$$= -B_c(y) \times T\frac{\Delta y}{1+y}.$$

Therefore, the classical duration is

$$D_c = T$$

and the modified duration is

$$D_M = T/(1+y).$$ ■

EXAMPLE

Computation of Duration

This example illustrates the computation of a coupon bond's duration. Consider a two-year bond with an annual coupon of $8.00 and a face value of $100. The annual yield-to-maturity is 8.12 percent and the bond's price is $99.7864.

Table 13.4 shows the calculations necessary to determine the modified duration of the bond. The classical duration is 1.9258 years and the modified duration is 1.7812. This coupon bond is said to be "more risky" than a one-year zero-coupon bond and "less risky" than a two-year zero-coupon bond because a one-year zero-coupon bond has a duration of one year and a two-year zero-coupon bond has a duration of two years.

TABLE 13.4 *Calculating Modified Duration of a Bond's Annual Coupon Payments*

TIME, t (YEARS)	COUPON	PRINCIPAL	DISCOUNTED VALUE	$t \times$ DISCOUNTED VALUE
1	8	0	7.3992	7.3992
2	8	100	92.3872	184.7743
			99.7864	192.1735

$$\text{Classical Duration} = \frac{192.1735}{99.7864} = 1.9258$$

$$\text{Modified Duration} = \frac{1.9258}{1.0812} = 1.7812$$

1. We first demonstrate that for small changes in yields, Expression (13.19) works well. Suppose that the yield-to-maturity increased by one basis point to 8.13 percent. The new bond price is

$$\frac{8.00}{(1 + 0.0813)} + \frac{8}{(1 + 0.0813)^2} + \frac{100}{(1 + 0.0813)^2} = 99.7686,$$

implying that the change in the bond's price is −0.0178.

2. We can use Expression (13.19) to approximate the change in the bond's price as

$$\begin{aligned} \Delta B_c &\approx -B_c \times D_M \times \Delta y \\ &= -99.7864 \times 1.7812 \times [(8.13 - 8.12)/100] \\ &= -0.0178. \end{aligned}$$

3. The two changes are identical, showing that Expression (13.19) works well. ■

Convexity

Here we study the concept of a coupon bond's **convexity**. The use of modified duration to describe changes in the bond's price due to changes in the yield-to-maturity is accurate only for small changes in the yield-to-maturity. This can be illustrated with the help of Figure 13.4.

Figure 13.4 shows the relationship between yield and the bond's price. Modified duration only considers linear changes in the bond's price because it is obtained from a linear approximation. This is represented by the straight line on Figure 13.4. The relationship between yield and the bond's price, however, is nonlinear. In fact, the relationship between the bond's price and yield is convex. For small changes in yields, this nonlinearity is unimportant. But, for large changes in yields, it is necessary to take this nonlinear relationship into account.

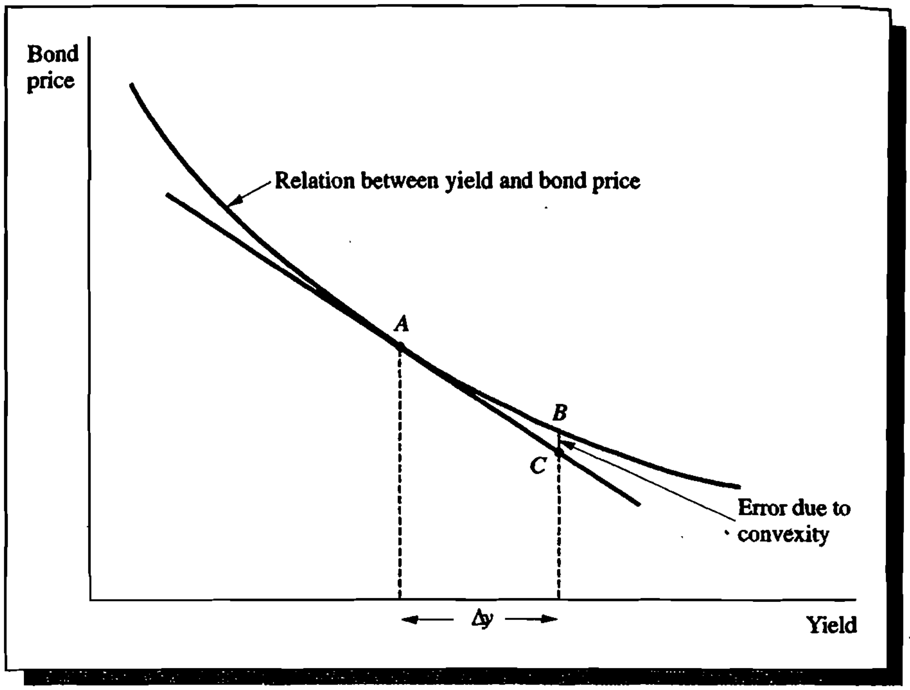
For example, suppose that the yield-to-maturity in Table 13.4 increases to 8.50 percent from 8.12 percent per annum. The new bond price[5] is 99.1144. The change in the bond's price is

$$\begin{aligned} \Delta B_c &= 99.1144 - 99.7864 \\ &= -0.6720. \end{aligned} \qquad (13.20)$$

[5]The bond price is

$$\frac{8}{1 + 0.085} + \frac{8}{(1 + 0.085)^2} + \frac{100}{(1 + 0.085)^2}$$
$$= 99.1144.$$

FIGURE 13.4 *Convexity*



From Expression (13.19), we can approximate this change as

$$\begin{aligned} -B_c \times D_M \times \Delta y &= -99.7864 \times 1.7812 \times (0.0850 - 0.0812) \\ &= -0.6754, \end{aligned} \tag{13.21}$$

which is not equal to the change in the bond's value. In fact, Expression (13.21) underestimates the value of the bond.

This value is illustrated in Figure 13.4. Given the change in yield, the bond's price moves from point A to point B. Expression (13.19) predicts the bond's price to be at point C, which is below point B. Point C will always lie below point B because the true relation between the bond's price and yield is a convex function.

The change in the bond's price, correcting for the convex relation between price and yield, is[6]

$$\Delta B_c = -B_c \times D_M \times (\Delta y) + (1/2) \times B_c \times (\text{Convexity}) \times (\Delta y)^2, \tag{13.22}$$

[6]See footnote 3.

where

$$\text{Convexity} = \frac{1}{(1+y)^2}\left[\sum_{t=1}^{T} \frac{t(t+1)c}{(1+y)^t} + \frac{T(T+1)F}{(1+y)^T}\right] / B_c.$$

The necessary steps to calculate convexity are shown in Table 13.5. The bond's convexity is 4.8155. Substituting the convexity into Expression (13.22) gives

$$\begin{aligned}\Delta B_c &= -0.6754 + \frac{1}{2} \times 99.7864 \times 4.8789 \times (0.0038)^2\\ &= -0.6754 + 0.0035\\ &= -0.6719,\end{aligned}$$

which is almost equal to the actual change of -0.6720.

Limitations of Analysis

The use of these simple forms of duration as risk measures implicitly assumes that only small and parallel shifts in the term structure can occur. To understand why this is true, note that Expressions (13.19) or (13.17) relate to changes in the bond's *own* yield to maturity. But, to measure risk *across* different coupon bonds, the change in each bond's yield must be assumed to be equal across the different bonds. This assumption is valid only if the change in the term structure of interest rates (the relation between yields on zero-coupon bonds and maturity) is a parallel shift.

In addition, using duration as a risk measure assumes that the approximation in Expression (13.19) or (13.17) provides a good estimate, which was shown to be true only for small changes in yields. Combined, these two arguments explain why duration is a valid risk measure only for small and parallel shifts in the term structure of interest rates.

Table 13.5 *Calculating the Convexity Correction Annual Coupon Payments*

Time, t (Years)	Coupon	Principal	Discounted Value	$t(t+1) \times$ Discounted Value
1	8	0	7.3992	14.7984
2	8	100	92.3872	554.3232
			99.7864	569.1216

$$\begin{aligned}\text{Convexity} &= \frac{1}{(1+0.0812)^2}[569.1216]/99.7864\\ &= 4.8789\end{aligned}$$

In practice, this never happens. Short-term rates are more volatile than long-term rates. The long and short sections of the term structure may even move in opposite directions. Furthermore, for large parallel shifts in the term structure, convexity adjustments help, but even these leave errors.

This same insight can be obtained using the techniques of Chapter 15. From a different perspective, we show that the problem with using Expression (13.19) to describe changes in the bond's price is that we are making a strong implicit assumption about the probability distribution describing bond price changes. This implicit assumption is equivalent (over infinitesimal intervals) to assuming that only small and parallel shifts in the term structure of interest rates are possible. We will expand on this issue in Chapter 15.

13.5 TREASURY BILL FUTURES

We now study the **Treasury bill futures contract**. For the Treasury bill futures contract, the underlying asset is a Treasury bill with face value of one million dollars and a maturity of 91 days. The last trading day on the T-bill futures contract is the business day immediately preceding the first delivery day. Under the terms of the contract, delivery may occur on one of three successive business days, implying that the underlying Treasury bill may have a maturity of 89, 90, or 91 days when delivered.

The price of a T-bill futures contract is quoted in terms of an index, which depends on a futures discount rate for a 90-day Treasury bill. The index value is computed as follows:

$$\text{Index Value} = 100 - \text{Futures Discount Rate (\%)}.$$

For example, if the index value is 96.81, then the futures discount rate is 3.19 percent. The futures contract price is defined by

$$1{,}000{,}000\left[1 - \frac{3.19}{100} \times \frac{90}{360}\right] = 992{,}025,$$

where by convention a 90-day maturity and a 360-day year is used.

A one-point change in the index implies a \$25 change in the T-bill futures price. For example, suppose the index increases by one point to 96.82 from 96.81, implying the futures discount rate decreased from 3.19 to 3.18. The new contract delivery price is

$$1{,}000{,}000\left[1 - \frac{3.18}{100} \times \frac{90}{360}\right] = 992{,}050.$$

The change in the T-bill futures price is \$25 (= 992,050 − 992,025).

Observe that there is a positive relationship between changes in the value of the index and the T-bill futures price, implying that the bid-quote is below the ask-quote.

TABLE 13.6 *Treasury Bill Futures Quotes, Thursday, May 28, 1998**

TREASURY BILLS (CME)-$1 mil.; pts. of 100%

	Open	High	Low	Settle	Chg	Discount Settle	Discount Chg	Open Interest
June	94.99	95.00	94.98	94.99	+ .01	5.01	− .01	2,274
Sept	94.98	94.98	94.96	94.96	− .02	5.04	+ .02	2,003

Est vol 463; vol Wed 587; open int 4,324, +118.

*Source: *The Wall Street Journal*, May 29, 1998.

Typical bid/ask quotes for T-bill futures are shown in Table 13.6. These quotes are from the T-bill futures contracts trading on the Chicago Mercantile Exchange (CME) on Thursday, May 28, 1998. For the June contract, the settlement index value is 94.99 and the futures discount rate is 5.01.

The final marking-to-market at the expiration of the futures contract sets the T-bill futures contract price equal to the value of a 90-day T-bill:

$$1{,}000{,}000\left[1 - i_d \times \frac{90}{360}\right]. \tag{13.23}$$

Thus, at delivery, the futures discount rate converges to the spot T-bill rate. Our description of the T-bill futures contract is now complete.

13.6 EURODOLLAR CONTRACTS

We study Eurodollar contracts in this section. A **Eurodollar** is a U.S. dollar-denominated deposit held by a bank outside the U.S. The rate at which a bank is willing to lend Eurodollars is referred to as the London Interbank Offer Rate (LIBOR). The rate at which a bank is willing to borrow is known as the London Interbank Bid Rate (LIBID). Eurodollar rates are generally higher than corresponding Treasury rates because Eurodollar rates are commercial lending rates containing credit risk. There are institutional factors such as reserve requirements that can also affect the spread between Eurodollar and Treasury rates.

Eurodollar Deposits

Here we study Eurodollar deposits in detail. We need to define a term structure of Eurodollar rates. This is done by first defining a term structure of "zero-coupon Eurodollar bonds." Let $L(0, T)$ denote the time-0 value of a Eurodollar deposit that pays one dollar at time T. Alternatively stated, $L(0, T)$ represents the present value of a Eurodollar paid at time T. The graph of $L(0, T)$ versus T is called the term structure of Eurodollar zero-coupon bond prices.

The T-period **LIBOR** (or Eurodollar) rate can be defined from these zero-coupon bond prices. The T-period LIBOR rate $\ell(T)$ is defined as the simple interest rate using a 360-day-year convention for the Eurodollar deposit with maturity T, that is,

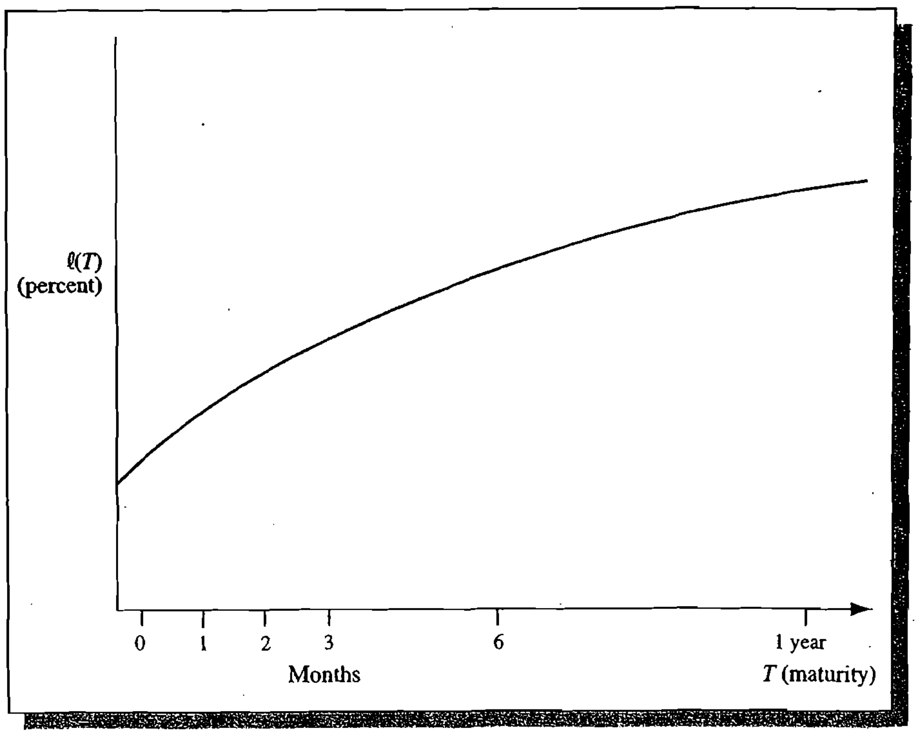
$$L(0, T) \equiv \frac{1}{1 + \ell(T)(T/360)},$$

where T is measured in days.

The graph of $\ell(T)$ versus T is the term structure of Eurodollar rates.[7] Figure 13.5 contains a typical upward sloping term structure for these Eurodollar deposits.

Table 13.7 gives LIBOR rates on Thursday, May 28, 1998, as reported in the *Wall Street Journal* on Friday, May 29. One-month rates are 5.65234 percent, three-month

FIGURE 13.5 *A Typical Term Structure of Eurodollar Rates*



[7] In fact, $\ell(T)$ should be subscripted by a zero to indicate the current date. We avoid this subscript to simplify the notation.

Table 13.7 *Money Market Rates, Thursday, May 28, 1998**

LONDON INTERBANK OFFERED RATES (LIBOR): 5.65234% one month; 5.68750% three months; 5.75000% six months; 5.87500% one year. British Bankers' Association average of interbank offered rates for dollar deposits in the London market based on quotations at 16 major banks. Effective rate for contracts entered into two days from date appearing at top of this column.

*Source: *The Wall Street Journal*, May 29, 1998.

5.6875 percent, six-month 5.75000 percent, and one-year 5.87500 percent. These rates give an increasing term structure as illustrated in Figure 13.5.

Forward Rate Agreements (FRAs)

A **forward rate agreement** (FRA) is a contract written on LIBOR that requires a cash payment at maturity based on the difference between a realized spot rate of interest and a prespecified forward rate. We first give an example, which will be generalized, and then describe some of the properties of FRAs.

Example

FRA

This example illustrates the payoff to an FRA. Consider a two-month (61-day) FRA, with principal one million dollars. The FRA is written on the three-month (91-day) LIBOR rate. Such a contract is referred to as a 2×5 FRA. The FRA rate is set today at 5.63 percent per annum. In two months' time, at the maturity of the contract, the payoff by definition is

$$\text{Principal} \times \frac{[\ell(91) - 0.0563] \times (91/360)}{1 + \ell(91) \times (91/360)},$$

where $\ell(91)$ is the spot three-month LIBOR rate at maturity.

Suppose that at maturity, $\ell(91)$ is 5.90 percent per annum. In this case, the payoff is

$$\begin{aligned}&\$1{,}000{,}000 \times \frac{(0.0590 - 0.0563) \times (91/360)}{1 + 0.0590 \times (91/360)}\\ &= \$682.50/1.0149\\ &= \$672.47.\end{aligned}$$

■

Four points should be noted from this FRA example. First, a 360-day-per-year convention is used in computing rates, in keeping with other Eurodollar contracts.

Second, the payment is discounted with the realized LIBOR rate. Third, we have assumed that there is no risk of default on the part of the writer of the FRA, implying that the payment contracted is the payment received. Fourth, the value of the contract can be positive or negative, depending on whether $\ell(91)$ is greater than or lesser than 0.0563. Thus, an FRA is a "bet" on the futures movements of the three-month LIBOR rate. Table 13.8 shows typical bid/ask quotes for forward rate agreements. FRAs typically range from 1×4 to 12×18, as illustrated in Table 13.8.

Formalization

This section formalizes the previous example.

Let us consider an FRA that matures in T months' time. The contract is written on the m month LIBOR rate. This contract is referred to as a $T \times (T + m)$ FRA. At maturity, the value of the contract is, by definition,

$$V(T) \equiv \text{Principal} \times \left[\frac{(\ell(m) - \text{FRA})\,(m/360)}{1 + \ell(m)\,(m/360)} \right], \tag{13.24}$$

where $\ell(m)$ is the m month period LIBOR rate at date T, m is months measured in days, and FRA is the $T \times (T + m)$ forward rate agreement's rate that was set when the contract was initiated.

When an FRA contract is initiated, the FRA rate is set such that the value of the contract is zero. This convention ensures that no cash is exchanged at the time the FRA contract is initiated.

We will prove that this condition implies that the FRA rate is set such that

$$\frac{1}{1 + \text{FRA}(m/360)} = \frac{L(0, T + m)}{B(0,T)}. \tag{13.25}$$

TABLE 13.8 *Forward Rate Agreement Rates*

FRAs	Bid	Offer
1–4	3.34	3.38
2–5	3.36	3.40
3–6	3.58	3.62
4–7	3.66	3.70
5–8	3.74	3.78
6–9	3.67	3.71
3–9	3.65	3.69
6–12	3.82	3.86
12–18	4.39	4.43

Before we prove this, consider Expression (13.24). This can be written in the form

$$V(T) = \frac{1 + \ell(m)(m/360) - [1 + \text{FRA}(m/360)]}{1 + \ell(m)(m/360)}$$

$$= 1 - [1 + \text{FRA}(m/360)]\frac{1}{1 + \ell(m)\,(m/360)},$$

where for simplicity we have set the principal equal to unity.

PROOF

Proof of Expression (13.25)

To prove Expression (13.25), we construct a portfolio whose payoffs match an FRA's. First, buy a Treasury bill that pays one dollar at date T. The cost of this position is $B(0,T)$.

To finance this purchase, write $1 + x \times (m/360)$ Eurodollar deposits, each deposit promising to pay one dollar at date $T + m$. The number x will be determined shortly. The price of one deposit is $L(0,T+m)$, so the price of $[1 + x \times (m/360)]$ deposits is $[1 + x \times (m/360)]L(0,T+m)$. The number x is chosen so that the initial value of the portfolio is zero, that is,

$$V_p(0) \equiv B(0,T) - [1 + x \times (m/360)] \times L(0,T+m) = 0.$$

We leave it as an exercise to determine x. We use the variable x as determined by this solution in the subsequent equations.

Now, the value of this portfolio at date T is

$$V_p(T) = 1 - [1 + x \times (m/360)] \times L(T,T+m).$$

But the value of a Eurodollar deposit that pays one dollar at date $T + m$ is (by definition) equal to

$$L(T,T+m) \equiv \frac{1}{1 + \ell(m)\,(m/360)}.$$

Therefore,

$$V_p(T) = 1 - [1 + x \times (m/360)]\frac{1}{1 + \ell(m)\,(m/360)}.$$

Notice that the value of the portfolio, $V_p(T)$, and the value of the FRA contract are identical in structure except that the portfolio has an "x" and the FRA contract has an "FRA." Since both the FRA contract and the portfolio have zero initial value, to avoid arbitrage, $x = \text{FRA}$ must be true. Expression (13.25) then follows from $V_p(0) = 0$ and the fact that $x = \text{FRA}$. ■

Futures Contracts

The Eurodollar futures contract is written on a one-million dollar, three-month Eurodollar deposit. Settlement is in cash. Eurodollar futures prices are shown in Table 13.9. These quotes are from Eurodollar futures contracts trading on the Chicago Mercantile Exchange. Delivery dates for the Eurodollar futures contracts range from three months to ten years. The largest open interest occurs for the shortest maturity contracts (three months to three years).

The Eurodollar futures prices are quoted in the form of an index as follows:

$$\text{Index Value} = 100 - \text{Futures Deposit Rate (\%)}.$$

TABLE 13.9 *Eurodollar Futures Contracts, Thursday, May 28, 1998**

EURODOLLAR (CME)-$1 million; pts of 100%

	Open	High	Low	Settle	Chg	Yield Settle	Yield Chg	Open Interest
June	94.32	94.32	94.31	94.31	– .01	5.69	+ .01	432,277
July	94.31	94.31	94.27	94.31	– .01	5.69	+ .01	11,391
Aug	94.30	94.30	94.30	94.30	– .01	5.70	+ .01	4,268
Sept	94.30	94.30	94.28	94.29	– .02	5.71	+ .02	467,451
Oct	....	....	.....	94.25	– .01	5.75	+ .01	1,440
Dec	94.23	94.24	94.21	94.22	– .02	5.78	+ .02	390,001
Mr99	94.26	94.26	94.23	94.25	– .02	5.75	+ .02	295,502
June	94.22	94.23	94.20	94.21	– .02	5.79	+ .02	227,488
Sept	94.20	94.20	94.16	94.18	– .01	5.82	+ .01	207,276
Dec	94.08	94.08	94.06	94.07	– .02	5.93	+ .02	179,281
Mr00	94.15	94.15	94.12	94.14	– .01	5.86	+ .01	142,431
June	94.11	94.12	94.10	94.11	– .02	5.89	+ .02	125,166
Sept	94.09	94.10	94.08	94.09	– .02	5.91	+ .02	86,942
Dec	94.03	94.03	94.01	94.02	– .02	5.98	+ .02	71,681
Mr01	94.07	94.07	94.05	94.06	– .02	5.94	+ .02	54,897
June	94.05	94.05	94.03	94.04	– .02	5.96	+ .02	55,097
Sept	94.03	94.03	94.01	94.02	– .02	5.98	+ .02	46,142
Dec	93.96	93.96	93.94	93.95	– .02	6.05	+ .02	40,950
Mr02	93.99	94.00	93.98	93.99	– .02	6.01	+ .02	45,494
June	93.98	93.98	93.96	93.97	– .02	6.03	+ .02	46,519
Sept	93.96	93.96	93.95	93.95	– .02	6.05	+ .02	40,486
Dec	93.89	93.90	93.87	93.88	– .02	6.12	+ .02	29,986
Mr03	93.92	93.93	93.91	93.91	– .02	6.09	+ .02	24,958
June	93.89	93.90	93.88	93.89	– .02	6.11	+ .02	13,336
Sept	93.87	93.88	93.87	93.87	– .02	6.13	+ .02	15,532
Dec	93.80	93.81	93.80	93.80	– .02	6.20	+ .02	8,186
Mr04	93.83	93.84	93.83	93.83	– .02	6.17	+ .02	7,964
June	93.80	93.81	93.80	93.80	– .02	6.20	+ .02	8,151
Sept	93.78	93.79	93.78	93.78	– .02	6.22	+ .02	6,965
Dec	93.71	93.72	93.71	93.71	– .02	6.29	+ .02	6,838
Mr05	93.74	93.75	93.74	93.74	– .02	6.26	+ .02	4,893
June	93.71	93.72	93.71	93.71	– .02	6.29	+ .02	4,130
Sept	93.69	93.70	93.69	93.69	– .02	6.31	+ .02	4,074
Dec	....	....	....	93.62	– .02	6.38	+ .02	3,747
Mr06	93.65	93.66	93.65	93.65	– .02	6.35	+ .02	5,813
June	93.62	93.63	93.62	93.62	– .02	6.38	+ .02	3,440
Sept	93.60	93.61	93.60	93.60	– .02	6.40	+ .02	3,890
Dec	93.53	93.54	93.53	93.53	– .02	6.47	+ .02	4,320
Mr07	....	....	....	93.56	– .02	6.44	+ .02	3,538
June	....	....	....	93.53	– .02	6.47	+ .02	3,515
Sept	....	....	....	93.51	– .02	6.49	+ .02	3,923
Dec	....	....	....	93.44	– .02	6.56	+ .02	2,931
Mr08	....	....	....	93.47	– .02	6.53	+ .02	2,777

Est vol 294,288; vol Wed 665,333; open int 3,145,587, +12,050.

*Source: *The Wall Street Journal*, May 29, 1998.

For example, if the index value is 94.12, the Eurodollar futures deposit rate is 5.88 percent per annum. The contract price, by definition, is

$$1{,}000{,}000\left[1 - \frac{5.88}{100} \times \frac{90}{360}\right]$$
$$= 985{,}300.$$

A one-basis-point change in the index value causes a \$25 change in the Eurodollar futures price. For example, if the index value increases by one basis point to 94.13 from 94.12, the futures deposit rate decreases by one point to 5.87, and the new Eurodollar futures contract price is

$$1{,}000{,}000\left[1 - \frac{5.87}{100} \times \frac{90}{360}\right]$$
$$= 985{,}325.$$

The change in the value of the Eurodollar futures contract is therefore \$25 per one-point change in the index.

The final marking-to-market at the expiration of the Eurodollar futures contract sets the contract price equal to

$$1{,}000{,}000\left[1 - \ell(90) \times \frac{90}{360}\right], \tag{13.26}$$

where $\ell(90)$ is the 90-day LIBOR rate[8] at the expiration of the contract. The Eurodollar futures deposit rate converges to the Eurodollar spot rate at the delivery date of the contract.

Expression (13.26) looks similar in form to Expression (13.23) for the T-bill futures contract, but there is an important difference. In Expression (13.23) for Treasury bill futures, i_d is a discount rate. In contrast, in Expression (13.26), $\ell(90)$ is a simple interest rate.

Recall that there is a simple linear relationship between the value of a Treasury bill and the discount rate:

$$\text{Value of Treasury Bill} = 1{,}000{,}000\left[1 - i_d \times \frac{90}{360}\right].$$

An inverse relationship exists between the value of a Eurodollar deposit and the LIBOR rate (a simple interest rate):

$$\text{Value of Eurodollar Deposit} = 1{,}000{,}000\left[\frac{1}{1 + \ell(90) \times \frac{90}{360}}\right].$$

[8]The rate used is an average rate. For contracts traded on the London Exchange, the settlement price is computed by using the quotes between 9:30 a.m. and 11:00 a.m. on the last trading day stated by a random sample of 16 from a list of designated banks. The three highest and three lowest quotes are disregarded. The settlement price will be 100 minus the average of the remaining 10 rates.

For contracts traded on the CME, a slightly different averaging procedure is used.

This difference between using a discount rate versus a simple interest rate has important ramifications for the pricing of these contracts. Our description of Eurodollar futures contracts is now complete.

13.7 TREASURY BOND AND NOTE FUTURES

Here we examine Treasury bond and Treasury note futures contracts. These futures contracts have complicated delivery features that are explained below. The delivery features make the T-bond and T-note futures contracts difficult to price and to hedge.

Treasury bond futures

The basic trading unit is a Treasury bond with a face value of $100,000. The futures contract is written on any U.S. Treasury bond that, if callable, is not callable for at least 15 years from the first day of the delivery month or, if not callable, has a maturity of at least 15 years from the first business day of the delivery month.

Because any one of many Treasury bonds can be used to satisfy the delivery requirements of the futures contract, the Chicago Board of Trade has developed a procedure for adjusting the price of the deliverable bond so that it is equivalent to a bond trading with a nominal coupon of 8 percent. We will explain this in more detail later.

Treasury bond futures contract delivery months are March, June, September, and December. The **last trading day** in the futures contract is the seventh business day preceding the last business day of the delivery month. The **last delivery day** is the last business day of the delivery month.

Price quotes are in terms of dollars and 32nds of a dollar for a $100 par value. For example, a quote of 114-26 means $\$114^{26}/_{32}$ or $114.8125 per $100 par value. See Table 13.10 for price quotes on T-bond futures contracts on the Chicago Board of Trade (CBT). These are listed in the first five rows of this table. The settlement price for the September contract trading on the CBT is 121-09. The open interest for this contract is 394,564 contracts.

10-Year U.S. Treasury note futures

The basic trading unit is a U.S. Treasury note with a face value of $100,000. The futures contract is written on any U.S. Treasury note maturing in at least $6^1/_2$ years, but not more than 10 years from the first business day of the delivery month. Details about the contract delivery month, the last trading day, and last delivery day are similar to those of Treasury bond futures. Table 13.10 contains price quotes on these Treasury note futures. For example, the settlement price for the June contract is 112-29.

5-Year U.S. Treasury note futures

The basic trading unit is a U.S. Treasury note with a face value of $100,000. The futures contract is written on any of the four most recently auctioned 5-year U.S.

TABLE 13.10 *Treasury Bond and Notes Features, Thursday, May 28, 1998**

TREASURY BONDS (CBT)-$100,000; pts. 32nds of 100%

	Open	High	Low	Settle	Change	Lifetime High	Lifetime Low	Open Interest
June	121-15	121-21	121-08	121-15	– 3	123-28	104-03	474,040
Sept	121-08	121-15	121-02	121-09	– 2	123-10	103-22	394,564
Dec	120-30	121-04	120-27	121-00	+ 2	122-30	103-13	61,254
Mr99	....	....	....	120-24	– 2	121-26	103-04	51

Est vol 500,000; vol Wed 662,618; open int 929,947, –2,140.

TREASURY BONDS (MCE)-$50,000; pts. 32nds of 100%

	Open	High	Low	Settle	Change	Lifetime High	Lifetime Low	Open Interest
June	121-17	121-22	121-09	121-18	+ 6	123-27	114-08	8,252

Est vol 6,000; vol Wed 9,487; open int 8,770, –842.

TREASURY NOTES (CBT)-$100,000; pts. 32nds of 100%

	Open	High	Low	Settle	Change	Lifetime High	Lifetime Low	Open Interest
June	112-29	113-01	112-25	112-29	– 2	115-06	106-26	329,439
Sept	113-00	113-06	112-28	113-02	– 2	114-22	110-25	206,805
Dec	112-29	112-30	112-26	112-30	– 2	114-21	111-11	9,462

Est vol 201,000; vol Wed 316,572; open int 545,707, +11,802.

5 YR TREAS NOTES (CBT)-$100,000; pts. 32nds of 100%

	Open	High	Low	Settle	Change	Lifetime High	Lifetime Low	Open Interest
June	09-055	109-07	09-025	09-045	– 2.5	111-00	107-28	187,753
Sept	09-075	09-085	109-04	09-055	– 3.0	109-175	108-09	95,596

Est vol 92,212; vol Wed 140,243; open int 283,434, –6,150.

2 YR TREAS NOTES (CBT)-$200,000, pts. 32nds of 100%

	Open	High	Low	Settle	Change	Lifetime High	Lifetime Low	Open Interest
June	04-017	04-045	04-005	104-01	– 1.2	104-175	103-21	27,643
Sept	104-01	04-025	104-01	04-012	– 1.7	104-042	03-225	24,910

Est vol 7,500; vol Wed 21,155; open int 52,553, –2,322.

*Source: *The Wall Street Journal*, May 29, 1998.

Treasury notes that have an original maturity of not more than 5 years and 3 months and remaining maturity of not less than 4 years and 3 months, as of the first business day of the delivery month.

The remaining details, with the exception of price quotes, are similar to those of Treasury bond futures. Prices are quoted in terms of dollars and one half of 1/32 of a dollar. For example, a quote of 111-205 means $111(20.5/32) or $111.640625 per $100 par value. Table 13.10 contains price quotes on the 5-year Treasury note futures. The June contract has a settlement price of 109-045.

2-Year U.S. Treasury note futures

For the 2-year Treasury note futures, the basic trading unit is a U.S. Treasury note with a face value of $200,000. The futures contract is written on any U.S. Treasury note with an original maturity of not more than 5 years and 3 months and a remaining maturity of not less than 1 year and 9 months from the first day of the delivery month but not more than 2 years from the last day of the delivery month.

The remaining details, with the exception of the price quotes, are similar to those of Treasury bond futures. Prices are quoted in terms of dollars and one quarter of 1/32 of a dollar; for example,

106-230 means 106 (23/32) = 106.718750
106-232 means 106 (23.25/32) = 106.726563

106-235 means 106 (23.50/32) = 106.734375
106-237 means 106 (23.75/32) = 106.742188.

Table 13.10 contains price quotes for the 2-year Treasury note futures. The June contract has a settlement price of 104-01.

The Delivery Process

We now discuss the delivery process of Treasury bond and Treasury note futures contracts. This is a particular example of the process described in Chapter 1. Delivery for these futures contracts is a three-day process. This provides the three parties—the buyer or long, the seller or short, and the clearing corporation—time to make the necessary notifications, delivery, and payment arrangements.

The short can initiate the three-day sequence any time during a period that begins two business days prior to the first business day of the delivery month and ends two business days before the last business day of the month. Trading in the deliverable futures contract stops on the seventh business day preceding the last business day of the delivery month.

The three-day delivery sequence begins when the short notifies the clearing corporation of the intention to deliver. This day is called the **Position Day**. On the second day of the delivery sequence, the clearing corporation matches the oldest long to the delivering short. The corporation notifies both parties to the trade.

TABLE 13.11 *The Delivery Process in Futures Markets*

DAY 1: POSITION DAY
The short informs the clearing corporation of the intent to make delivery.
DAY 2: NOTICE OF INTENTION DAY
The clearing corporation matches the oldest long to the delivering short and then notifies both parties.
The short invoices the long.
DAY 3: DELIVERY DAY
The short delivers the financial instruments to the long.
The long makes payment to the short.
TITLE PASSES
The long assumes all ownership rights and responsibilities.

After receiving this information, the short invoices the long. The second day is called the **Notice of Intention Day**. On the third day, the long takes possession of the instrument and the short receives the invoice amount. The third day is called the **Delivery Day**. For future reference, we summarize the delivery process in Table 13.11.

13.8 TREASURY BOND FUTURES

Now we return to the Treasury bond futures contract to discuss its delivery features in more detail. Recall that the Treasury bond futures contract is a commitment to deliver a nominal 8 percent, $100,000 face value U.S. Treasury bond with at least fifteen years to maturity or to the first call date, whichever comes first. The seller of the futures contract has a choice of eligible bonds that can be used for delivery. In an attempt to neutralize the effects of bonds with different coupons, the Chicago Board of Trade (CBT) designed a system of conversion factors, which we now explain.

Conversion Factors

Basically, conversion factors provide a means of equating bonds with different coupons and maturities so as to allow the seller of the futures contract a choice of eligible bonds to use for delivery, thus minimizing the possibility of price distortions, illiquidities, and manipulation.

The CBT system adjusts the T-bond futures price based on an 8 percent, fifteen-year Treasury bond with a semiannual coupon. The amount the short invoices the long is computed as follows:

$$\begin{array}{l}\text{Cash Received}\\ \text{by the Short}\end{array} \equiv \text{Quoted Futures Price} \times \begin{array}{l}\text{Conversion Factor for the}\\ \text{Bond Delivered}\end{array}$$

$$+ \text{ Accrued Interest on the Bond Delivered.}$$

For example, suppose the quoted T-bond futures price is 95-19. Remember that this is quoted in units of 32nds, so the actual price is 95 (19/32) or 95.59375 per $100 face value. Suppose that for the particular Treasury bond the short picks to deliver, the conversion factor is 1.0514, and the accrued interest is $2.85 per $100 face value. Each T-bond futures contract is for delivery of $100,000 face value of bonds. Therefore, the total dollar amount that the short invoices the long is

$$\left(\frac{100{,}000}{100}\right)(95.59375 \times 1.0514 + 2.85)$$
$$= 103{,}357.27.$$

The conversion factor for the Treasury bond delivered by the short is given by a formula, explained in the following text, as described by the Chicago Board of Trade. The maturity of the bond is rounded down to the nearest three months for the purpose of the calculation. If the maturity of the bond is an exact number of half years after the rounding, the first coupon is assumed to be paid in six months. If the maturity of the bond after rounding is not an exact number of half years—there is an extra three months—the first coupon is assumed to be paid after three months. We will use two examples to show how to calculate the conversion factor.

EXAMPLE

Example One (Conversion Factor Calculation)

Consider an $8\frac{1}{2}$% Treasury bond with a maturity of 22 years and 2 months. For the purposes of calculating the conversion factor, the bond is assumed to have a maturity of 22 years. The first coupon payment of 4.25 is assumed to be paid after six months. The value of the bond is defined to be

$$\sum_{t=1}^{44} \frac{4.25}{(1+.04)^t} + \frac{100}{(1+0.4)^{44}} = 105.1372,$$

assuming semiannual coupon payments and a face of 100. The conversion factor is defined to be 1.0514, using four-decimal-place accuracy. ■

EXAMPLE

Example Two (Conversion Factor Calculation)

This example illustrates the conversion factor calculation for a more difficult situation. Consider an $8\frac{1}{2}$% Treasury bond with a maturity of 22 years and 11 months. The semiannual coupon payment is 4.25. For the purpose of calculating the conversion factor, the bond is assumed to have a maturity of 22 years and 9 months. The value of this bond in three months' time, the maturity being 22 years and 6 months, is defined to be

$$4.25 + \sum_{t=1}^{45} \frac{4.25}{(1+.04)^t} + \frac{100}{(1+.04)^{45}} = 109.4300.$$

Recall that the coupon is assumed to be paid in three months' time. This value is discounted back for the three-month period. The discount rate is $\sqrt{1.04} - 1 = 0.0198$. The bond's value today is

$$\frac{1}{1.0198}\, 109.4300 = 107.3049.$$

For purposes of consistency, the accrued interest, 2.125, is subtracted from the price of the bond to give 105.1799. The conversion factor is 1.0518. ■

The Chicago Board of Trade publishes tables of conversion factors (see Table 13.12). Check the conversion factors calculated in the above two examples to see that they match those in Table 13.12.

Cheapest to Deliver

This section discusses an embedded option within the Treasury bond futures contract known as the **cheapest to deliver option**, or the **quality option**. For contracts traded on the Exchange of the Chicago Board of Trade, the investor who is short at any given time has a choice of bonds that satisfy the conditions of the futures contract. The short invoices the long for the amount:

(Quoted Futures Price × Conversion Factor) + Accrued Interest.

Table 13.12 *Conversion Factor to Yield 8.000 Percent**

	Coupon Rate							
Yrs-Mos:	8%	$8\frac{1}{8}$%	$8\frac{1}{4}$%	$8\frac{3}{8}$%	$8\frac{1}{2}$%	$8\frac{5}{8}$%	$8\frac{3}{4}$%	$8\frac{7}{8}$%
22-0	1.0000	1.0128	1.0257	1.0385	1.0514	1.0642	1.0771	1.0899
22-3	.9998	1.0127	1.0256	1.0385	1.0514	1.0643	1.0772	1.0901
22-6	1.0000	1.0130	1.0259	1.0389	1.0518	1.0648	1.0777	1.0907
22-9	.9998	1.0128	1.0258	1.0388	1.0518	1.0648	1.0778	1.0908
23-0	1.0000	1.0131	1.0261	1.0392	1.0522	1.0753	1.0783	1.0914
23-3	.9998	1.0129	1.0260	1.0391	1.0522	1.0653	1.0784	1.0915
23-6	1.0000	1.0132	1.0263	1.0395	1.0526	1.0658	1.0789	1.0921
23-9	.9998	1.0130	1.0262	1.0394	1.0526	1.0658	1.0790	1.0922
24-0	1.0000	1.0132	1.0265	1.0397	1.0530	1.0662	1.0795	1.0927
24-3	.9998	1.0131	1.0264	1.0397	1.0530	1.0663	1.0795	1.0928
24-6	1.0000	1.0133	1.0267	1.0400	1.0534	1.0667	1.0800	1.0934
27-9	.9998	1.0132	1.0266	1.0399	1.0533	1.0667	1.0801	1.0935
25-0	1.0000	1.0134	1.0269	1.0403	1.0537	1.0671	1.0806	1.0940
25-3	.9998	1.0133	1.0267	1.0402	1.0537	1.0671	1.0806	1.0941
25-6	1.0000	1.0135	1.0270	1.0405	1.0540	1.0676	1.0811	1.0946
25-9	.9998	1.0134	1.0269	1.0405	1.0540	1.0675	1.0811	1.0946
26-0	1.0000	1.0136	1.0272	1.0408	1.0544	1.0680	1.0816	1.0951
26-3	.9998	1.0134	1.0271	1.0407	1.0543	1.0679	1.0816	1.0952
26-6	1.0000	1.0137	1.0273	1.0410	1.0547	1.0684	1.0820	1.0957
26-9	.9998	1.0135	1.0272	1.0409	1.0546	1.0683	1.0820	1.0957
27-0	1.0000	1.0137	1.0275	1.0412	1.0550	1.0687	1.0825	1.0962
27-3	.9998	1.0136	1.0274	1.0411	1.0549	1.0687	1.0825	1.0963
27-6	1.0000	1.0138	1.0276	1.0415	1.0553	1.0691	1.0829	1.0967
27-9	.9998	1.0137	1.0275	1.0414	1.0552	1.0691	1.0829	1.0968

*Source: Chicago Board of Trade, *Treasury Futures for Institutional Investors*, Table 2.1.

The cost to the short of purchasing the bond to deliver is

Quoted Bond Price + Accrued Interest.

Therefore, the net return to the short is

Quoted Futures Price × Conversion Factor − Quoted Bond Price.

It is in the interest of the short to pick the bond that maximizes this difference.

For example, suppose that the quoted futures price is 94-2, or 94.0625. There are three bonds with the following prices and conversion factors:

Bond	Price	Conversion Factor
1	94.25	1.0820
2	126.00	1.4245
3	142.125	1.5938.

The net return to the short is

Bond 1 $94.0625 \times 1.0820 - 94.25 = 7.53$
Bond 2 $94.0625 \times 1.4245 - 126.00 = 7.99$
Bond 3 $94.0625 \times 1.5938 - 142.125 = 7.79$.

It would be in the interest of the short to deliver Bond 2, as it is the cheapest bond to deliver.

Over time, the identity of the cheapest-to-deliver Treasury bond changes. The conversion factor system tends to favor the delivery of relatively low-coupon, long-maturity bonds when yields are in excess of 8 percent. When yields are less than 8 percent, the conversion factor system favors high-coupon, short-maturity bonds. The shape of the term structure affects the outcome. For an upward sloping curve, a positive relationship exists between maturity and yield, implying a negative relationship between maturity and price. This causes a tendency for long maturity bonds to be the cheapest to deliver.

Wild Card Option

There is another embedded option within the Treasury Bond futures option known as the **wild card option**. The Chicago Board of Trade interest rate futures markets stop trading at 2:00 P.M. (C.S.T.), while the cash market for Treasury bonds continues to trade past this time. The deadline for notifying the clearing corporation of an intent to deliver is 8:00 P.M. (C.S.T.). This difference of six hours creates a window each day within the delivery period during which the short may potentially take advantage of a decline in the cash market prices. This window generates the "wild card option." The futures price reflects the value of this option: the greater the value of the option, the lower the futures price.

Timing Option

The third embedded option within the Treasury bond futures contract is known as the **timing option**. The deliverable contract will stop trading on the seventh business day preceding the last business day of the delivery month. During this seven-business-day period, all open positions must be settled by delivery. The short has the flexibility of determining when to deliver during this period and can take advantage of any decline in the cash market. This timing option is valuable to the short, and the greater the value, the lower the Treasury bond futures price will be prior to the time the future ceases to trade.

13.9 SUMMARY

In this chapter, we discussed the basic instruments underlying the term structure of interest rates: Treasury bills, Treasury notes, and Treasury bonds. We described how Treasury bills are quoted in terms of discount rates. We also gave definitions for simple interest rates, money market rates, and continuously compounded rates. We described market yields on Treasury bonds and notes, and the valuation of floating rate notes.

Duration has a long history as a tool for hedging interest rate risk. We introduced duration for bonds and explained some of its properties. As a hedging tool, duration assumes that the term structure of interest rates shifts by small parallel amounts. In practice, this rarely happens. We will discuss in Chapter 15 a better method for measuring risk that avoids these limitations.

We also examined additional instruments related to the term structure of interest rates, including Eurodollar deposits, FRAs, Eurodollar futures contracts, and Treasury bill, note, and bond futures contracts.

In Treasury bond and Treasury note futures contracts, the writer of the futures contract has three embedded options:

1. quality option—the choice of the cheapest-to-deliver bond;
2. wild card option—on any day during the delivery month, deadline for the notice of intention to deliver is 8:00 P.M., six hours after the futures market stops trading; and
3. timing option—option to deliver during the last seven business days of the delivery month at a price based on the last settlement price at the end of trading.

These options all tend to reduce the Treasury futures price and make futures contract pricing difficult.

REFERENCES

Burghardt, G., T. Belton, M. Lane, G. Luce, and R. McVey, 1991. *Eurodollar Futures and Options*. Chicago: Probus Publishing Company.

Chicago Board of Trade Publications:
Treasury Futures for Institutional Investors
Two Year Treasury Note Futures
Five Year Treasury Note Futures
The Delivery Process in Brief: Treasury Board and Treasury Note Futures
Understanding Duration and Convexity

The London International Financial Futures Exchange, *Eurodollar Interest Rate Futures and Options*.

Stigum, M., 1981. *Money Market Calculations*. Homewood, IL: Dow Jones-Irwin.

QUESTIONS

Question 1

The current discount rate is 3 percent on a 90-day T-bill, face value $1 million. What is the price of the T-bill? If the discount rate increases by one basis point to 3.01 percent, what is the *change* in the price of the T-bill?

Question 2

The continuously compounded yield on a deposit that pays one million dollars in 365 days' time is 4.5 percent. What is the current value of the deposit? What is the yield expressed as a simple interest rate, assuming a 365-day year?

Question 3

The current discount rate on a 91-day T-bill is 3.65 percent, assuming a 360-day year. What is the simple interest rate, assuming a 365-day year?

Question 4

A bond pays interest on an annual basis. The maturity of the bond is 4 years, the coupon is 6.25 percent, the face value of the bond is 100, and its market value is 104.33. Determine the discretely compounded yield-to-maturity and the continuously compounded yield-to-maturity.

Question 5

A bond pays interest semiannually. The coupon is 5.50 percent per annum, implying that the amount paid every six months is 2.75 dollars per 100 dollar face value. The maturity of the bond is 3 years and its market value is 100.05. Determine the effective yield-to-maturity and the continuously compounded yield-to-maturity.

Question 6

Let the quote on the 15 July 1999 $6^3/_8$% Treasury note be Bid 105:26 Ask 105:28. If you purchased this bond on May 25, 1994 at the ask price, what would be the total cost? The total number of days in the current coupon period is 182.

Question 7

Consider a $12^1/_2$% Treasury bond with a maturity of 20 years. Show that the conversion factor is 1.4452.

Question 8

The quoted futures price is 114-26. Which of the following three bonds is cheapest to deliver?

Bond	Price	Conversion Factor
1	162:20	1.3987
2	138:31	1.2870
3	131.02	1.273

Question 9

Consider a Treasury note with a maturity of 5 years and a coupon of 10 percent per annum. The coupon is paid semiannually. You are given the following information about the term structure of interest rates:

Maturity	0.5	1.0	1.5	2.0	2.5
T-bill Price	0.9748	0.9494	0.9238	0.8982	0.8725
Maturity	3.0	3.5	4.0	4.5	5.0
T-bill Price	0.8470	0.8216	0.7965	0.7717	0.7472

a) What is the market price of the Treasury note?
b) What is the annual yield-to-maturity?
c) What is the modified duration of the Treasury note?

Question 10

Use the information in Question 9 to answer the following questions.

a) Determine the continuously compounded yield-to-maturity for the T-bills.
b) Determine the continuously compounded six-month forward rates.
c) Plot the term structure of zero-coupon yields and the term structure of forward rates.

APPENDIX: DURATION AND CONVEXITY CORRECTION FOR A SEMIANNUAL COUPON BOND

Let y denote the semiannual yield-to-maturity defined by Expression (13.6). The definition of modified duration is

$$D_M \equiv \frac{(1/2)}{(1 + y/2)}\left[\sum_{t=1}^{2T} \frac{tc}{(1 + y/2)^t} + \frac{2TF}{(1 + y/2)^{2T}}\right]/B_c.$$

EXAMPLE

Modified Duration

Consider a $2^1/_2$-year bond with a semiannual coupon of $4.00. The semiannual yield-to-maturity is 8.12 percent and the bond price is $99.6733. Table A1 shows the calculations necessary to determine the modified duration of the bond. Using the figures in Table A1 and substituting into Expression (13.16) gives

$$\begin{aligned}\Delta B_c &= -99.6733 \times 2.2251 \times \Delta y \\ &= -221.78\Delta y.\end{aligned}$$

If the semiannual yield-to-maturity increases by one basis point to 8.13 percent, the bond price is

$$\sum_{t=1}^{5} \frac{4.00}{(1 + 0.04065)^t} + \frac{100}{(1 + 0.04065)^5}$$
$$= 99.6500,$$

TABLE A1 *Calculating Modified Duration of a Bond with Semiannual Coupon Payments*

TIME, t (YEARS)	COUPON	PRINCIPAL	DISCOUNTED VALUE*	t × DISCOUNTED VALUE × 0.5
1	4.00	0	3.8439	1.9220
2	4.00	0	3.6340	3.6340
3	4.00	0	3.5498	5.3248
4	4.00	0	3.4113	6.8227
5	4.00	100	85.2343	213.0858
			99.6733	230.7893

$$\text{Classical Duration} = \frac{230.7893}{99.6733} = 2.3155$$

$$\text{Modified Duration} = \frac{2.35155}{1.0406} = 2.2251$$

*The semiannual yield-to-maturity is 4.06 percent.

implying that the bond price has changed by −0.02. Alternatively, we could compute an approximation to this change using modified duration to give

$$
\begin{aligned}
B_c &= -221.78 \times [(8.13 - 8.12)/100] \\
&= -0.02.
\end{aligned}
$$

For this small change in yield, the two values are identical. ■

Convexity correction

The convexity correction is defined by

$$
\text{Convexity} \equiv \frac{(1/4)}{(1+y/2)^2}\left[\sum_{t=1}^{2T}\frac{t(t+1)c}{(1+y/2)^t} + \frac{2T(2T+1)F}{(1+y/2)^{2T}}\right]/B_c.
$$

EXAMPLE

(Continued)

Suppose that the semiannual yield-to-maturity increased to 8.50 percent. The new bond price is

$$
\begin{aligned}
&\sum_{t=1}^{5}\frac{4}{(1+0.0425)^t} + \frac{100}{(1+0.0425)^5} \\
&= 98.8948,
\end{aligned}
$$

TABLE A2 *Calculating the Convexity Correction for a Semiannual Coupon Bond*

TIME, t (YEARS)	COUPON	PRINCIPAL	PRESENT VALUE*	$t(t+1) \times$ PRESENT VALUE
1	4.00	0	3.8439	7.6878
2	4.00	0	3.6340	21.8040
3	4.00	0	3.5498	42.5976
4	4.00	0	3.4113	68.2260
5	4.00	100	85.2343	2,557.0290
			99.6733	2,697.3444

$$
\begin{aligned}
\text{Convexity} &= \left[\frac{(1/4)}{(1+0.0406)^2}\,2{,}697.3444\right]/99.6733 \\
&= 6.2478 \\
\text{Price Correction} &= (1/2)\,B_c\,\text{Convexity}\,(\Delta y)^2 \\
&= (1/2) \times 99.6733 \times 6.2478(0.0850 - 0.0812)^2 \\
&= 0.0045.
\end{aligned}
$$

*The semiannual yield-to-maturity is 4.06 percent.

implying that the change in the bond price is

$$\begin{aligned}\Delta B_c &= 98.8948 - 99.6733 \\ &= -0.7785.\end{aligned}$$

The steps to calculate the convexity term are shown in Table A2. Using Expression (13.19), the change in the bond price is

$$\begin{aligned}&-99.6733 \times 2.2251 \times (0.0038) + {}^1\!/_2 \times 99.6733 \times 6.2478 \times (0.0038)^2 \\ &= -0.8428 + 0.0045 \\ &= -0.8383.\end{aligned}$$

■