Appendix to "Fiscal Shocks and Fiscal Risk Management" by Huw Lloyd-Ellis and Xiaodong Zhu

1 Data

All of the data and the code used in this paper can be downloaded from the internet at http://qed.econ.queensu.ca/pub/faculty/lloyd-ellis/papers/fiscrisk.html.

Fiscal Variables

The quarterly primary surplus was calculated as the difference between total federal revenues and expenditures less interest payments on the debt, as published by Statistics Canada. For institutional reasons, this data exhibits considerable seasonal variation. Specifically, annual crown corporation cash flows are attributed only to the second quarter yielding a large "spike". We therefore used seasonally adjusted data. The surplus data does not include charges and subsidies relating to the Petroleum Compensation fund. Quarterly public debt figures are taken from IMF International Financial Statistics. The effective interest rate was calculated as the ratio of actual interest payments on the debt to value of the debt.

Asset Returns

VWR is the index of value–weighted returns on the NYSE taken from the CRSP tape. DIV is the dividend yield on the NYSE from the CRSP tape. LONGR is the nominal interest rate on 10 year US. government bonds. TBILL is the nominal 3–month US. treasury bill rate. TBMA is a one–year fixed–weight moving average of TBILL. All of these returns were converted into Canadian dollars using the spot U.S.–Canadian exchange rate taken from CITIBASE. Note that these returns should therefore be interpreted as the return in Canadian dollars on each U.S. dollar invested.

Data used to Compute the Epstein–Zin Stochastic Discount Factor

Real per capita US consumption was calculated using data from CITIBASE. The real rate of return on the market portfolio was taken to be equal to VWR divided by the US CPI.

2 Sensitivity of Results to Empirical Specification

2.1 Endogeneity of the Exchange Rate

In our benchmark regressions, we specify all of our variables in Canadian dollars. We convert US asset returns by multiplying them by the Canada/US exchange rate. This implies that payments made by the Canadian government to investors are denominated in US dollars and so the Canadian government cannot affect their value by inflating, for example. However, by converting in this way it is not so clear that these variables are independent of Canadian fiscal policy and truly represent exogenous shocks. To address this issue, we allow for the possibility of endogeneity by estimating the regression equation using the US dollar–denominated asset returns as instruments.. The results are reported in Table A1. As can be seen, the results are fairly insensitive to the alternative specification, suggesting that the endogeneity is of little importance empirically. This is confirmed by the Hausman tests reported at the bottom of each column. These indicate that the hypothesis that the coefficients are significantly different when estimated by two–stage least squares is be rejected at the 10% level.

Table A1

	Shocks Only		Shocks + Shift	
Variable	OLS	2SLS	OLS	2SLS
Constant	.0134 (18.71)	.0130 (17.65)	.0126 (23.15)	.0124 (21.93)
VWR	.0052 (0.57)	.0021 (0.66)	-.0010 (0.55)	-.0031 (0.42)
DIV	-.3190 (12.97)	-.3140 (12.32)	-.2722 (14.18)	-.2600 (13.48)
LONGR	-.0561 (3.57)	-.0548 (3.45)	-.1272 (9.24)	-.1300 (9.00)
TBILL	.0469 (3.34)	.0500 (3.36)	.0874 (7.71)	.0880 (7.64)
T3MA	.0954 (5.92)	.0935 (5.50)	.1036 (8.46)	.0985 (7.96)
DUM	—	—	.0039 (10.30)	.0039 (10.19)
R^2	0.69	0.69	0.82	0.82
$\overline{R}^2$	0.68	0.68	0.82	0.81
D–W	0.69	0.69	1.20	1.15
Hausman	$\chi^2(5) = 5.3097$ [0.38]		$\chi^2(5) = 7.2839$ [0.20]	

Notes:

(1) t-statistics are in parenthesis.

(2) P-values are in square brackets

(3) Instruments are the US denominated asset returns and the dummy variable.

(4) The X-variables are demeaned.

2.2 Stability

To test the stability of the parameters on the shocks and the lagged surplus across the two regimes, we regress the residuals from our preferred regression on the explanatory variables within each regime. Table A2 documents the results. FTEST indicates that we cannot reject the joint hypothesis that these parameters are constant across regimes. In other words, the change in policy stance is largely consistent with an increase in the permanent components of the surplus after the first quarter of 1985 rather than a change in the marginal responsiveness of the surplus to the exogenous shocks.

Table A2: Forecasting the Primary Surplus

	Variable	In–Sample	Forecast
X	VWR	0.004	0.004
		(0.04)	
	DIV	-0.70	-0.70
		(5.05)	
	LONGR	-0.30	-0.30
		(2.55)	
	TBILL	0.22	0.22
		(2.47)	
	TBMA	0.28	0.28
		(2.47)	
	Constant	0.0319	0.0276
		(5.39)	(7.24)
	SLAG	0.24	0.24
		(2.26)	
	DUM	—	0.0108
			(10.72)
	NOBS	76	
	R^2	0.75	
	D–W	1.96	
	F-TEST	2.09	
	[P–value]	[0.08]	

Notes:

(1) t–statistics are given in parenthesis.

(2) In the out of sample regression we restrict the coefficients on the X–variables to be the same as in the in–sample regression. FTEST tests this restriction.

2.3 Inclusion of Other Variables

Our baseline regression includes only financial asset returns. However, it may be the case that the inclusion of other cyclical variables will affect the results. In their multi–country panel data analysis, Roubini and Sachs (1989) find the effects of changes in the unemployment rate and the growth rate to be significant. Table A3 reports regression results for the effects of changes in real GNP growth, DG, and the unemployment rate, DU, in our sample. Although the unemployment variable is significant, it does not add much to the explanatory power of the model. Moreover the coefficients on the shock variables are robust to the inclusion of these alternative cyclical variables.

Table A3

	Variable	Basic Model	Extended Model
	VWR	0.00	0.00
		(0.43)	(0.43)
	DIV	-0.70	-0.62
		(7.09)	(6.40)
X	LONGR	-0.33	-0.32
		(5.68)	(5.54)
	TBILL	0.27	0.23
		(6.34)	(4.97)
	TBMA	0.22	0.24
		(3.97)	(4.41)
	DG	—	-0.07
			(1.85)
	DU	—	-0.37
			(2.38)
	Constant	0.0317	0.0287
		(3.70)	(4.13)
	SLAG	0.38	0.43
		(5.86)	(6.47)
	DUM	0.0108	0.0097
		(6.70)	(5.93)
	$\overline{R}^2$	0.85	0.86
	D–W	2.04	2.14

Notes:

(1) t–statistics are given in parenthesis.

(2) Unemployment data is take from OECD, Main Economic Indicators, various issues.

2.4 VAR Estimates

Table A4

	VWR	DIV	TBILL	LONGR	μ	r
VWR(-1)	-0.10 (1.18)	-0.01 (3.99)	0.01 (1.25)	0.01 (0.70)	-0.08 (1.57)	-0.002 (1.82)
DIV(-1)	3.82 (4.45)	0.96 (37.06)	0.05 (0.58)	0.18 (1.27)	-2.00 (3.12)	-0.01 (0.81)
TBILL(-1)	-0.25 (0.47)	0.004 (0.03)	0.90 (17.43)	0.05 (0.54)	-0.44 (1.36)	-0.005 (0.49)
LONGR(-1)	-1.01 (2.16)	0.02 (1.62)	0.06 (1.28)	0.81 (10.33)	0.14 (0.49)	0.02 (2.98)
$r(-1)$	—	—	—	—	—	0.91 (32.3)
Constant	-0.10 (0.31)	-0.01 (0.03)	0.01 (0.79)	0.01 (0.32)	-0.02 (4.09)	0.22 (3.18)

3 Details of Calibration

3.1 Calibration of the Real Output Process

Given our specifications for the real stochastic discount factor and productivity, we have

$$y_t = \left(\psi_0 e^{\mu_\psi t + \mathsf{b}'_\psi \tilde{\mathsf{X}}_{t-1} + \nu_t}\right) \left(\frac{\alpha(1-\tau) e^{\mu_\psi t + \mathsf{b}'_\psi \tilde{\mathsf{X}}_{t-1} + \frac{1}{2}\sigma_\nu^2 - \mu_m - \mathsf{b}'_m \tilde{\mathsf{X}}_{t-1} + \frac{1}{2}\sigma_m^2}}{1-(1-\delta(1-\tau)) e^{-\mu_m - \mathsf{b}'_m \tilde{\mathsf{X}}_{t-1} + \frac{1}{2}\sigma_m^2}}\right)^{\frac{\alpha}{1-\alpha}}. \tag{1}$$

Taking logs on both sides yields

$$\begin{aligned} \ln y_t \;=\; & \frac{1}{1-\alpha}\left(\ln \psi_0 + \frac{1}{2}\alpha\sigma_v^2 + \alpha \ln\left(\alpha(1-\tau)\right)\right) + \frac{\mu_\psi}{1-\alpha} t + \frac{\mathbf{b}'_\psi \tilde{\mathbf{X}}_{t-1}}{1-\alpha} + \nu_t \\ & - \frac{\alpha}{1-\alpha} \ln\left(e^{\mu_m - \frac{1}{2}\sigma_m^2 + \mathsf{b}'_m \tilde{\mathsf{X}}_{t-1}} - (1-\delta(1-\tau))\right) \end{aligned} \tag{2}$$

Now taking a first–order linear approximation around $\mathbf{b}'_m \tilde{\mathbf{X}}_{t-1} = 0$ for the last term, we can write

$$\ln y_t \approx q_0(\tau) + \mu_y t + \mathbf{q}(\tau)' \tilde{\mathbf{X}}_{t-1} + \nu_t \tag{3}$$

where

$$q_0(\tau) = \frac{1}{1-\alpha}\left(\ln \psi_0 + \frac{1}{2}\alpha\sigma_v^2 + \alpha \ln\left(\alpha(1-\tau)\right)\right) - \frac{\alpha}{1-\alpha} \ln\left(e^{\mu_m - \frac{1}{2}\sigma_m^2} - (1-\delta(1-\tau))\right), \tag{4}$$

$$\mathbf{q}(\tau) = \frac{1}{1-\alpha}\left(\mathbf{b}_\psi - \frac{\alpha e^{\mu_m - \frac{1}{2}\sigma_m^2}}{e^{\mu_m - \frac{1}{2}\sigma_m^2} - (1-\delta(1-\tau))} \mathbf{b}_m\right), \tag{5}$$

and

$$\mu_y = \overline{\mu}_y - \mu_p - \frac{1}{2}\sigma_p^2. \tag{6}$$

Regressing $\ln y_t - \mu_y t$ on the state variables over a period during which the effective tax rate is deemed constant yields the coefficients q_0, and $\mathbf{q}$, plus the variance of ν_t, σ_ν^2. The stochastic process followed by the productivity parameter can then be backed out using the method of undetermined coefficients.[1]

3.2 Calibrating the Two–factor Asset Pricing Model

The moment condition for the risk premium be written as follows:

$$\exp\left(E_t\left[R_{t+1}^{m,n}\right] - r_t^n + \frac{1}{2}\sigma_{1,u}^2 - \rho_1\sigma_{1,u}^2 - \rho_4\sigma_{14,u}\right) = 1 \tag{7}$$

or

$$E_t\left[R_{t+1}^{m,n}\right] - r_t^n + \frac{1}{2}\sigma_{1,u}^2 - \rho_1\sigma_{1,u}^2 - \rho_4\sigma_{14,u} = 0. \tag{8}$$

Taking unconditional expectations of the left hand side of the equation yields

$$E\left[R_t^{m,n} - r_t^n\right] + \frac{1}{2}\sigma_{1,u}^2 - \rho_1\sigma_{1,u}^2 - \rho_4\sigma_{14,u} = 0. \tag{9}$$

Replacing the theoretical moments with sample moments, we have

$$\frac{1}{T}\sum_{t=1}^{T}\left(R_t^{m,n} - r_t^n\right) + \frac{1}{2}\sigma_{1,u}^2 - \rho_1\sigma_{1,u}^2 - \rho_4\sigma_{14,u} = 0. \tag{10}$$

Then we have

$$E_t\left[\frac{M_{t+2j}^n}{M_t^n}\right] = \exp\left(-\left(E[r_t^n] + \frac{1}{2}\sigma_n^2\right)2j - m_z^n(t,2j) + \frac{1}{2}V_{zz}^n(t,2j)\right). \tag{11}$$

where $m_z^n(t,2j) = E_t\left[-(\ln M_{t+2j}^n - \ln M_t^n) - 2\mu_n j\right]$ and
$V_{zz}^n(t,2j) = E_t\left[\left(-(\ln M_{t+2j}^n - \ln M_t^n) - 2\mu_n j - m_z(t,2j)\right)^2\right]$. So, the moment condition for the term premium can be written as

$$\begin{aligned} 1 \quad = \quad & \frac{1}{2}\left[\sum_{j=1}^{20}\exp\left(-\left(E[r_t^n] + \frac{1}{2}\sigma_n^2\right)2j - m_z^n(t,2j) + \frac{1}{2}V_{zz}^n(t,2j)\right)\right] r_t^{n,L} \\ & + \exp\left(-\left(E[r_t^n] + \frac{1}{2}\sigma_n^2\right)40 - m_z^n(t,40) + \frac{1}{2}V_{zz}^n(t,40)\right). \end{aligned} \tag{12}$$

Taking the sample average of the right hand side of this equation yields

$$\begin{aligned} 1 \quad = \quad & \frac{1}{T}\sum_{t=1}^{T}\left\{\frac{1}{2}\left[\sum_{j=1}^{20}\exp\left(-\left(E[r_t^n] + \frac{1}{2}\sigma_n^2\right)2j - m_z^n(t,2j) + \frac{1}{2}V_{zz}^n(t,2j)\right)\right] r_t^{n,L}\right\} \\ & + \frac{1}{T}\sum_{t=1}^{T}\exp\left(-\left(E[r_t^n] + \frac{1}{2}\sigma_n^2\right)40 - m_z^n(t,40) + \frac{1}{2}V_{zz}^n(t,40)\right). \end{aligned} \tag{13}$$

[1] Note also that $\mu_\psi = (1-\alpha)\mu_y$.

We choose the values of ρ_1 and ρ_4 so that they are the solutions to the equations (10) and (13). We do so by first using (10) to express ρ_1 as a linear function of ρ_4 and substituting it into equation (13). We then numerically look for the value of ρ_4 that solves equation (13).

4 Details Relating to Monte Carlo Simulation

To compute the probability of policy shifts and the associated welfare impacts, we conducted a Monte Carlo simulation. For each set of parameters, we estimated the underlying VAR to determine the parameters of the system and the associated joint distribution of the errors. We used this to generate N paths of T periods for the entire system. For each path and at each date, we computed the implied debt level, D_t, and the present value of future forecasted primary surpluses under the current policy, $V_t(\tau_t)$. We did this for both the hedged and unhedged government cash flow processes. We then computed the associated net debt and used it to determine the tax rate to be set in the next period according to the policy rule described in the paper. This generated a joint numerical distribution over the tax rate and the discount factor which we used to compute welfare. Since changes in the tax rate occur infrequently (i.e. only when the bounds on the net debt are hit), a large number of paths and time periods were required before our estimated welfare gain converged. Specifically, $N = 5,000,000$ and $T = 500$ were sufficient for convergence of the welfare gain estimate up to the second decimal place.

4.1 Nominal Present Value Calculations

We need to calculate present value of nominal cash–flows of the following form:

$$\Lambda(t,j) = \frac{1}{M_t} E_t[M^n_{t+j}\left(\eta_0 + \boldsymbol{\eta}_1'\tilde{\mathbf{X}}_{t+j} + \boldsymbol{\eta}_2' X_{5,t+j}\right)\overline{Y}_{t+j}] \tag{14}$$

Let $Z^n(t,j) = -\ln M^n_{t+j} - \mu_n j$. Then

$$Z^n(t,j) = Z^n(t,j-1) + \mathbf{b}_n'\tilde{\mathbf{X}}_{t+j-1} + \omega_{n,t+j}, \tag{15}$$

where $Z^n(t,0) = -\ln M^n_t$. It follows that

$$\begin{aligned}\Lambda(t,j) &= \overline{Y}_t e^{(\overline{\mu}_y - \mu_n)j} E_t[e^{-Z^n(t,j)}\left(\eta_0 + \boldsymbol{\eta}_1'\tilde{\mathbf{X}}_{t+j} + \boldsymbol{\eta}_2' X_{5,t+j}\right)] &(16)\\ &\equiv \overline{Y}_t e^{(\overline{\mu}_y - \mu_n)j}\left[\eta_0 E_t[e^{-Z^n(t,j)}] + \boldsymbol{\eta}_1' E_t[e^{-Z^n(t,j)}\tilde{\mathbf{X}}_{t+j}] + \boldsymbol{\eta}_2' E_t[e^{-Z^n(t,j)} X_{5,t+j}]\right]. &(17)\end{aligned}$$

Let $\mathbf{m}_x(t,j) = E_t\left[\tilde{\mathbf{X}}_{t+j}\right]$, $m^n_z(t,j) = E_t\left[Z^n(t,j)\right]$, $\mathbf{V}_{\mathsf{xx}}(j) = E_t\left[\left(\tilde{\mathbf{X}}_{t+j} - \mathbf{m}_x(t,j)\right)\left(\tilde{\mathbf{X}}_{t+j} - \mathbf{m}_x(t,j)\right)'\right]$, $\mathbf{V}^{\mathsf{n}}_{\mathsf{xz}}(j) = E_t\left[\left(\tilde{\mathbf{X}}_{t+j} - \mathbf{m}_x(t,j)\right)(Z^n(t,j) - m_z(t,j))\right]$, and $V^n_{zz}(j) = E_t\left[(Z^n(t,j) - m_z(t,j))^2\right]$.

These moments can be calculated recursively as follows:

$$\begin{aligned}
\mathbf{m}_x(t,j) &= \mathbf{A}\mathbf{m}_{\mathsf{x}}(t,j-1), && (18)\\
m_z^n(t,j) &= m_z^n(t,j-1) + \mathbf{b}_n'\mathbf{m}_{\mathsf{x}}(t,j-1), && (19)\\
\mathbf{V}_{\mathsf{xx}}(j) &= \mathbf{A}\mathbf{V}_{\mathsf{xx}}(j-1)\mathbf{A}' + \mathbf{\Sigma}, && (20)\\
\mathbf{V}_{\mathsf{xz}}^{\mathsf{n}}(j) &= \mathbf{A}\mathbf{V}_{\mathsf{xz}}^{\mathsf{n}}(j-1) + \mathbf{A}\mathbf{V}_{\mathsf{xx}}(j-1)\mathbf{b}_n + \mathbf{v}_n, && (21)\\
V_{zz}^n(j) &= V_{zz}^n(j-1) + 2\mathbf{b}_n'\mathbf{V}_{\mathsf{xz}}^{\mathsf{n}}(j-1) + \mathbf{b}_n'\mathbf{V}_{\mathsf{xx}}(j-1)\mathbf{b}_n + \sigma_n^2 && (22)
\end{aligned}$$

where $m_z^n(t,1) = \mathbf{b}_n'\widetilde{\mathbf{X}}_t - \ln M_t^n$, $V_{zz}^n(1) = \sigma_n^2$, $\mathbf{m}_{\mathsf{x}}(t,1) = \mathbf{A}\widetilde{\mathbf{X}}_t$, $\mathbf{V}_{\mathsf{xx}}(1) = \mathbf{\Sigma}$, and $\mathbf{V}_{\mathsf{xz}}^{\mathsf{n}}(1) = \mathbf{v}_n$. Given these moments, one can calculate the present values as follows:

$$E_t[e^{-Z^n(t,j)}] = e^{-m_z^n(t,j)+\frac{1}{2}V_{zz}^n(t,j)}, \tag{23}$$

$$E_t[e^{-Z^n(t,j)}\widetilde{\mathbf{X}}_{t+j}] = e^{-m_z^n(t,j)+\frac{1}{2}V_{zz}^n(t,j)}\left(\mathbf{m}_x(t,j) - \mathbf{V}_{\mathsf{xz}}^{\mathsf{n}}(t,j)\right), \tag{24}$$

and

$$E_t[e^{-Z^n(t,j)}X_{5,t+j}] = e^{-m_z^n(t,j)+\frac{1}{2}V_{zz}^n(t,j)}\left(m_{5,x}^n(t,j) - V_{5,xz}^n(j)\right), \tag{25}$$

where

$$m_{5,x}(t,j) = \frac{1}{4}\mathbf{1}_4'\left[\mathbf{m}_x(t,j-1) + \mathbf{m}_x(t,j-2) + \mathbf{m}_x(t,j-3) + \mathbf{m}_x(t,j-4)\right], \tag{26}$$

$$\mathbf{V}_{5,xx}(j) = \frac{1}{4}\Big[\mathbf{V}_{xx}(j-1) + \mathbf{A}\mathbf{V}_{xx}(j-2) + \mathbf{A}^2\mathbf{V}_{xx}(j-3) + \mathbf{A}^3\mathbf{V}_{xx}(j-4)\Big]\mathbf{1}_4, \tag{27}$$

$$\begin{aligned}
V_{5,xz}^n(j) \;=\; & \frac{1}{4}\mathbf{1}_4'[\mathbf{V}_{xz}^n(j-1) + \mathbf{V}_{xz}^n(j-2) + \mathbf{V}_{xz}^n(j-3) + \mathbf{V}_{xz}^n(j-4) + \mathbf{V}_{xx}(j-1)\mathbf{b}_n\\
& + (\mathbf{I}+\mathbf{A})\mathbf{V}_{xx}(j-2)\mathbf{b}_n + (\mathbf{I}+\mathbf{A}+\mathbf{A}^2)\mathbf{V}_{xx}(j-3)\mathbf{b}_n + (\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3)\mathbf{V}_{xx}(j-4)\mathbf{b}_n] \qquad (28)
\end{aligned}$$

Here, $\mathbf{1}_4$ denotes the vector $(0,0,0,1)'$.

4.2 Real Present Value Calculation

We need also to calculate present value of real cash–flows of the following form:

$$\Lambda(t,j) = \frac{1}{M_t}E_t[M_{t+j}y_{t+j}\left(\eta_0 + \boldsymbol{\eta}_1'\widetilde{\mathbf{X}}_{t+j} + \boldsymbol{\eta}_2'X_{5,t+j}\right)] \tag{29}$$

Let $Z(t,j) = -\ln M_{t+j} - \mu_m j$. Then

$$Z(t,j) = Z(t,j-1) + \mathbf{b}_m'\tilde{\mathbf{X}}_{t+j-1} + \omega_{m,t+j}, \tag{30}$$

where $Z(t,0) = -\ln M_t$ and let

$$Q(t,j) = -Z(t,j) + \mathbf{q}'\tilde{\mathbf{X}}_{t+j-1} \tag{31}$$

It follows that

$$\Lambda(t,j) = e^{q_0+\mu_y t}e^{(\mu_y-\mu_m)j}E_t[\exp(Q(t,j))\left(\eta_0+\boldsymbol{\eta}_1'\widetilde{\mathbf{X}}_{t+j}+\boldsymbol{\eta}_2'X_{5,t+j}\right)]. \tag{32}$$

$$\equiv e^{q_0+\mu_y t}e^{(\mu_y-\mu_m)j}\left[\eta_0 E_t[e^{Q(t,j)}]+\boldsymbol{\eta}_1'E_t[e^{Q(t,j)}\widetilde{\mathbf{X}}_{t+j}]+\boldsymbol{\eta}_2'E_t[e^{Q(t,j)}X_{5,t+j}]\right] \tag{33}$$

Let $m_z(t,j)$, $\mathbf{V}_{\mathsf{xz}}(t,j)$ and $V_{zz}(t,j)$ denote the real counterparts of $m_z^n(t,j)$, $\mathbf{V}^{\mathsf{n}}_{\mathsf{xz}}(t,j)$ and $V_{zz}^n(t,j)$ and let $m_Q(t,j) = E_t\left[Q(t,j)\right]$, $\mathbf{V}_{\mathsf{xQ}}(j) = E_t\left[\left(\widetilde{\mathbf{X}}_{t+j}-\mathbf{m}_x(t,j)\right)(Q(t,j)-m_Q(t,j))\right]$, and $V_{QQ}(j) = E_t\left[(Q(t,j)-m_Q(t,j))^2\right]$. These moments can be calculated recursively as follows:

$$m_z(t,j) = m_z(t,j-1)+\mathbf{b}_m'\mathbf{m}_{\mathsf{x}}(t,j-1), \tag{34}$$

$$m_Q(t,j) = -m_z(t,j)+\mathbf{q}'\mathbf{m}_{\mathsf{x}}(t,j-1) \tag{35}$$

$$\mathbf{V}_{\mathsf{xz}}(t,j) = \mathbf{A}\mathbf{V}_{\mathsf{xz}}(t,j-1)+\mathbf{A}\mathbf{V}_{\mathsf{xx}}(t,j-1)\mathbf{b}_m+\mathbf{v}_m, \tag{36}$$

$$V_{zz}(t,j) = V_{zz}(t,j-1)+2\mathbf{b}_m'\mathbf{V}_{\mathsf{xz}}(t,j-1)+\mathbf{b}_m'\mathbf{V}_{\mathsf{xx}}(t,j-1)\mathbf{b}_m+\sigma_m^2 \tag{37}$$

$$V_{QQ}(t,j) = V_{zz}(t,j-1)-2(\mathbf{q}-\mathbf{b}_m)'\mathbf{V}_{\mathsf{xz}}(t,j-1)+(\mathbf{q}-\mathbf{b}_m)'\mathbf{V}_{\mathsf{xx}}(t,j-1)(\mathbf{q}-\mathbf{b}_m)+\sigma_\nu^2 \tag{38}$$

$$\mathbf{V}_{\mathsf{xQ}}(t,j) = -\mathbf{V}_{\mathsf{xz}}(t,j)+\mathbf{A}\mathbf{V}_{\mathsf{xx}}(t,j-1)\mathbf{q} \tag{39}$$

where $m_z(t,1) = \mathbf{b}_m'\widetilde{\mathbf{X}}_t - \ln M_t$, $m_Q(t,1) = (\mathbf{b}_m-\mathbf{q})'\widetilde{\mathbf{X}}_t - \ln M_t$, $V_{zz}(t,1) = \sigma_m^2$, $\mathbf{m}_{\mathsf{x}}(t,1) = \mathbf{A}\widetilde{\mathbf{X}}_t$, $V_{QQ}(t,1) = \sigma_\nu^2$, and $\mathbf{V}_{\mathsf{xQ}}(t,1) = -\mathbf{v}_m$. Given these moments, we can calculate the present values as follows:

$$E_t[e^{Q(t,j)}] = e^{m_Q(t,j)+\frac{1}{2}V_{QQ}(t,j)}. \tag{40}$$

$$E_t[e^{Q(t,j)}\widetilde{\mathbf{X}}_{t+j}] = e^{m_Q(t,j)+\frac{1}{2}V_{QQ}(t,j)}\left(\mathbf{m}_x(t,j)+\mathbf{V}_{\mathsf{xQ}}(t,j)\right), \tag{41}$$

and

$$E_t[e^{Q(t,j)}\mathbf{X}_{5,t+j}] = e^{m_Q(t,j)+\frac{1}{2}V_{QQ}(t,j)}\left(m_{5,x}(t,j)-V_{5,xz}(j)+\mathbf{q}'\mathbf{V}_{5,xx}(j-1)\right), \tag{42}$$

where $m_{5,x}(t,j)$, $V_{5,xz}(j)$, and $\mathbf{V}_{5,xx}(j)$ are defined as above.

4.3 Second–Order Approximation of Present Values

For computational speed it was necessary to take a second–order approximation in computing these present values. Using the same notation as above, we can write

$$\mathbf{m}_{\mathsf{x}}(t,j) = \mathbf{A}^j\mathbf{X}_t \tag{43}$$

and for the nominal discount factor

$$m_z^n(t,j) = \mathbf{b}_n'\sum_{i=1}^{j}\mathbf{A}^{i-1}\mathbf{X}_t - \ln M_t^n \tag{44}$$

$$= \mathbf{b}_n'(\mathbf{I}-\mathbf{A})^{-1}\left(\mathbf{I}-\mathbf{A}^j\right)\mathbf{X}_t - \ln M_t^n \tag{45}$$

Let

$$\mathbf{B}'(j) = -\mathbf{b}_n' \left(\mathbf{I} - \mathbf{A}\right)^{-1} \left(\mathbf{I} - \mathbf{A}^j\right) \tag{46}$$

and let $\mathbf{B}_i(j)$ denote the ith row of $\mathbf{B}(j)$ and let $[\mathbf{A}^j]_i$ denote the ith row of $\mathbf{A}^j$.

The present value of an asset with return stream $\left\{\overline{Y}_{t+j} X_{i,t+j}\right\}_{j=1}^{\infty}$ is

$$\begin{aligned}
\Pi_i(t) &= \frac{\overline{Y}_t}{M_t^n} \sum_{j=1}^{\infty} \exp[(\overline{\mu}_y - \mu_n)j - m_z^n(t,j) + \frac{1}{2}V_{zz}^n(j)]\left(m_i(t,j) + V_{iz}^n(j)\right) \\
&= \overline{Y}_t \sum_{j=1}^{\infty} \exp[(\overline{\mu}_y - \mu_n)j + \mathbf{B}'(\tau)\mathbf{X}_t + \frac{1}{2}V_{zz}^n(\tau)] \left([\mathbf{A}^j]_i \mathbf{X}_t + V_{iz}^n(j)\right) \\
&= \overline{Y}_t \sum_{j=1}^{\infty} \exp\left((\overline{\mu}_y - \mu_n)j + \frac{1}{2}V_{zz}^n(j)\right) \left[e^{\mathsf{B}(j)'\mathsf{X}_t}[\mathbf{A}^j]_i \mathbf{X}_t + e^{\mathsf{B}(j)'\mathsf{X}_t} V_{iz}^n(j)\right] \qquad (47)
\end{aligned}$$

Taking second–order Taylor series expansions around $\mathbf{X}_t = \mathbf{0}$ yields

$$e^{\mathsf{B}'\mathsf{X}_t} \simeq 1 + \mathbf{B}'\mathbf{X}_t + \frac{1}{2}\mathbf{X}_t'\mathbf{B}\mathbf{B}'\mathbf{X}_t \tag{48}$$

and

$$e^{\mathsf{B}'\mathsf{X}_t}[\mathbf{A}^j]_i \mathbf{X}_t \simeq [\mathbf{A}^j]_i \mathbf{X}_t + \frac{1}{2}\mathbf{X}_t' \left(\mathbf{B}[\mathbf{A}^j]_i + \mathbf{B}'[\mathbf{A}^j]_i'\right)\mathbf{X}_t \tag{49}$$

Substituting and collecting terms, it follows that the present value can be approximated by

$$\begin{aligned}
\Pi_i(t) &= \overline{Y}_t \sum_{j=1}^{\infty} \exp\left((\overline{\mu}_y - \mu_n)j + \frac{1}{2}V_{zz}^n(j)\right) V_{iz}^n(j) \qquad (50) \\
&+ \overline{Y}_t \sum_{j=1}^{\infty} \exp\left((\overline{\mu}_y - \mu_n)j + \frac{1}{2}V_{zz}^n(j)\right) \left[[\mathbf{A}^j]_i + V_{iz}^n(j)\mathbf{B}'(j)\right]\mathbf{X}_t \\
&+ \mathbf{X}_t' \left[\frac{\overline{Y}_t}{2} \sum_{j=1}^{\infty} \exp\left((\overline{\mu}_y - \mu_n)j + \frac{1}{2}V_{zz}^n(j)\right) \left(\mathbf{B}(\tau)[\mathbf{A}^j]_i + \mathbf{B}'(j)[\mathbf{A}^j]_i' + V_{iz}^n(j)\mathbf{B}(j)\mathbf{B}'(j)\right)\right]\mathbf{X}_t.
\end{aligned}$$

Note also that the present value of the asset with return stream $\{\overline{Y}_{t+j}\}_{j=1}^{\infty}$ is

$$\begin{aligned}
\Pi_0(t) &= \overline{Y}_t \sum_{j=1}^{\infty} \exp\left((\overline{\mu}_y - \mu_n)j + \frac{1}{2}V_{zz}^n(j)\right)\left(1 + \mathbf{B}'(j)\mathbf{X}_t + \frac{1}{2}\mathbf{X}_t'\mathbf{B}(j)\mathbf{B}'(j)\mathbf{X}_t\right) \qquad (51) \\
\Pi_0(t) &= \overline{Y}_t \sum_{j=1}^{\infty} \exp\left((\overline{\mu}_y - \mu_n)j + \frac{1}{2}V_{zz}^n(j)\right) \qquad (52) \\
&+ \left[\overline{Y}_t \sum_{j=1}^{\infty} \exp\left((\overline{\mu}_y - \mu_n)j + \frac{1}{2}V_{zz}^n(j)\right)\mathbf{B}'(j)\right]\mathbf{X}_t \\
&+ \mathbf{X}_t' \left[\frac{\overline{Y}_t}{2} \sum_{j=1}^{\infty} \exp\left((\overline{\mu}_y - \mu_n)j + \frac{1}{2}V_{zz}^n(j)\right)\mathbf{B}(j)\mathbf{B}'(j)\right]\mathbf{X}_t
\end{aligned}$$

Analogous second–order approximations were also made for real cash flows.

4.4 Decomposition of Welfare Gains

The welfare function can be written in the following form:

$$\hat{W}_0 = (1-\delta(1-\tau_0))k_0 + (1-\alpha)\frac{1}{M_0}\sum_{t=0}^{\infty} E_0\left[M_t f(\tau_t, z_t)\right], \tag{53}$$

where z_t is a random variable that is independent of fiscal policy. If the tax rate follows a three point process, we can express the tax rate as

$$\tau_t = \tau_h I_{A_t} + \tau_l I_{B_t} + \tau_0 I_{\Omega - A_t \cup B_t}, \tag{54}$$

where I_A is the indicator function for set A, A_t is the collection of states in which the tax rate will be high, B_t the collection of states in which the tax will be low. A generic term of the welfare function can then be written as

$$\begin{aligned}
& E_0 M_t[f(\tau_h, z_t)I_{A_t} + f(\tau_l, z_t)I_{B_t} + f(\tau_0, z_t)I_{\Omega - A_t \cup B_t}] \\
= \ & E_0[M_t f(\tau_0, z_t)] + E_0[M_t(f(\tau_h, z_t) - f(\tau_0, z_t))I_{A_t}] + E_0[M_t(f(\tau_l, z_t) - f(\tau_0, z_t))I_{B_t}] \quad (55) \\
= \ & E_0[M_t f(\tau_0, z_t)] + E_0[M_t(f(\tau_h, z_t) - f(\tau_0, z_t))]P_{tH} + Cov[M_t(f(\tau_h, z_t) - f(\tau_0, z_t)), I_{A_t}] \\
& + E_0[M_t(f(\tau_l, z_t) - f(\tau_0, z_t))]P_{tL} + Cov[M_t(f(\tau_l, z_t) - f(\tau_0, z_t)), I_{B_t}] \quad (56) \\
= \ & E_0[M_t f(\tau_0, z_t)] - E_0[M_t(2f(\tau_0, z_t) - f(\tau_h, z_t) - f(\tau_l, z_t)]P_{tH} + \\
& E_0[M_t(f(\tau_l, z_t) - f(\tau_0, z_t)](P_{tL} - P_{tH}) + \\
& -Cov[M_t(f(\tau_0, z_t) - f(\tau_h, z_t)), I_{A_t}] + Cov[M_t(f(\tau_l, z_t) - f(\tau_0, z_t)), I_{B_t}] \quad (57) \\
= \ & W_{t0} + W_{t1} + W_{t2} + W_{t3} \quad (58)
\end{aligned}$$

Note that $M_t(f(\tau_0, z_t) - f(\tau_h, z_t))$ and $M_t(f(\tau_l, z_t) - f(\tau_0, z_t))$ are respectively the marginal cost of a tax hike and marginal benefit of a tax cut, both of which are unaffected by hedging. The welfare gains of hedging can come from three potential sources:

(1) Hedging reduces both P_{tH} and P_{tL} equally and does not change either of the covariances. In this case, W_{2t} and W_{3t} do not change with hedging. But W_{1t} increases because f is a concave function. This gain is purely due to the tax smoothing effect.

(2) The reduction in P_{tH} is more than the reduction in P_{tL}. This will increase W_{2t}.

(3) Hedging increases the covariance between the marginal benefit of tax cut and the even of tax cut or reduces the covariance between the marginal cost of tax hike and the event of tax hike. That is, hedging may cause the tax hike to occur in the states when the marginal cost of tax hike is low or the tax cut to occur in the states when the marginal benefits of tax cut is high. This gain is due to diversification.

4.5 An Alternative Asset Pricing Model

Here we detail the specification and calibration of the real stochastic discount factor implied by Epstein-Zin's consumption CAPM:

$$M_t = \left(\beta^t C_t^{-\frac{1}{\sigma}}\right)^{\theta} \left(\Pi_{s=0}^{t} \frac{1}{R_s^m}\right)^{1-\theta}. \tag{59}$$

Here, C_τ denotes real US per capita consumption, R_t^m denotes the gross real return on the market portfolio (measured by the real value–weighted stock return index on the NYSE), σ is the elasticity of intertemporal substitution, and θ is defined, following Campbell (1993), as $\theta = (1-\gamma)/(1-(1/\sigma))$, where γ is the coefficient of relative risk aversion.

By the definition of R_t^m, we have $X_{1,t} = (\ln R_t^m - E[\ln R_t^m]) + \ln(P_t/P_{t-1}) - \mu_p$, which implies

$$\ln R_t^m = E[\ln R_t^m] + (\mathbf{a}'_{(1)} - \mathbf{b}'_{\mathsf{p}})\tilde{\mathbf{X}}_{t-1} + u_{1,t} - \omega_{p,t}, \tag{60}$$

where $\mathbf{a}'_{(1)}$ is the first row vector of the coefficient matrix $\mathbf{A}$ from the VAR. If we assume that the real US consumption growth rate evolves according to:

$$\ln\left(\frac{C_t}{C_{t-1}}\right) = \mu_c + \mathbf{b}'_c \tilde{\mathbf{X}}_{t-1} + \omega_{c,t}, \tag{61}$$

where $\omega_{c,t}$ is i.i.d., $\omega_{c,t} \sim N(0, \sigma_c^2)$, $E[\omega_{c,t}\mathbf{u}_t] = \mathbf{v}_c$ and $E[\omega_{c,t}\omega_{p,t}] = \sigma_{pc}$, then, we can write the growth rate of the real stochastic discount factor as

$$-\ln\left(\frac{M_t}{M_{t-1}}\right) = -\theta\ln\beta + \frac{\theta}{\sigma}\ln\left(\frac{C_t}{C_{t-1}}\right) + (1-\theta)\ln R_t^m \tag{62}$$

$$\equiv \mu_m + \mathbf{b}'_{\mathsf{m}}\tilde{\mathbf{X}}_{t-1} + \omega_{m,t}, \tag{63}$$

where $\mu_m = -\theta\ln\beta + \frac{\theta}{\sigma}\mu_c + (1-\theta)E[\ln R_t^m]$, $\mathbf{b}_m = \frac{\theta}{\sigma}\mathbf{b}_c + (1-\theta)(\mathbf{a}_{(1)} - \mathbf{b}_{\mathsf{p}})$, $\omega_{m,t} = \frac{\theta}{\sigma}\omega_{c,t} + (1-\theta)(u_{1,t} - \omega_{p,t})$.

Calibration. The moment conditions in this case can be written as follows:

$$1 = \exp\left(r_t^n - (\mu_n - \frac{1}{2}\sigma_n^2) - \mathbf{b}'_{\mathsf{n}}\tilde{\mathbf{X}}_t\right), \tag{64}$$

$$1 = \exp\left(E_t[R_{t+1}^{m,n}] - (\mu_n - \frac{1}{2}\sigma_n^2) - \mathbf{b}'_{\mathsf{n}}\tilde{\mathbf{X}}_t + \frac{1}{2}\sigma_{1,u}^2 - \mathbf{1}'_1\mathbf{v}_n\right); \tag{65}$$

or

$$0 = r_t^n - (\mu_n - \frac{1}{2}\sigma_n^2) - \mathbf{b}'_{\mathsf{n}}\tilde{\mathbf{X}}_t, \tag{66}$$

$$0 = E_t[R_{t+1}^{m,n}] - (\mu_n - \frac{1}{2}\sigma_n^2) - \mathbf{b}'_{\mathsf{n}}\tilde{\mathbf{X}}_t + \frac{1}{2}\sigma_{1,u}^2 - \mathbf{1}'_1\mathbf{v}_n; \tag{67}$$

where

$$\mu_n = -\theta\ln\beta + \frac{\theta}{\sigma}\mu_c + (1-\theta)(E[\ln R_t^m] - \mu_p) + \mu_p, \tag{68}$$

$$\mathbf{b}_n = \frac{\theta}{\sigma}\mathbf{b}_c + (1-\theta)(\mathbf{a}_{(1)} - \mathbf{b}_p) + \mathbf{b}_p, \tag{69}$$

$$\mathbf{v}_n = \frac{\theta}{\sigma}\mathbf{v}_c + (1-\theta)(\mathbf{\Sigma 1}_1 - \mathbf{v}_p) + \mathbf{v}_p, \tag{70}$$

$$\begin{aligned} \sigma_n^2 &= \left(\frac{\theta}{\sigma}\right)^2 \sigma_c^2 + (1-\theta)^2 \sigma_{1,u}^2 + \theta^2 \sigma_p^2 \\ &\quad +2\left[\frac{\theta(1-\theta)}{\sigma}\mathbf{1}_1'\mathbf{v}_c + \frac{\theta^2}{\sigma}\sigma_{cp} + \frac{\theta(1-\theta)}{100}\mathbf{1}_1'\mathbf{v}_p\right]. \end{aligned} \tag{71}$$

Taking unconditional expectations yields

$$0 = E[r_t^n] - (\mu_n - \frac{1}{2}\sigma_n^2) - \mathbf{b}_{\mathsf{n}}' E[\tilde{\mathbf{X}}_t], \tag{72}$$

$$0 = E[R_t^{m,n}] - (\mu_n - \frac{1}{2}\sigma_n^2) - \mathbf{b}_{\mathsf{n}}' E[\tilde{\mathbf{X}}_t] + \frac{1}{2}\sigma_{1,u}^2 - \mathbf{1}_1'\mathbf{v}_n; \tag{73}$$

or

$$\mathbf{b}_{\mathsf{n}}' E[\tilde{\mathbf{X}}_t] - E[r_t^n] = \mu_n - \frac{1}{2}\sigma_n^2, \tag{74}$$

$$E[R_t^{m,n} - r_t^n] = -\frac{1}{2}\sigma_{1,u}^2 + \mathbf{1}_1'\mathbf{v}_n; \tag{75}$$

Replacing the theoretical moments with sample moments, we have

$$\frac{1}{T}\sum_{t=1}^{T}\left(\mathbf{b}_{\mathsf{n}}'\tilde{\mathbf{X}}_t - r_t^n\right) = \mu_n - \frac{1}{2}\sigma_n^2, \tag{76}$$

$$\frac{1}{T}\sum_{t=1}^{T}(R_t^{m,n} - r_t^n) = -\frac{1}{2}\sigma_{1,u}^2 + \mathbf{1}_1'\mathbf{v}_n. \tag{77}$$

Using the above notation, we can write

$$E_t\left[\frac{M_{t+2j}^n}{M_t^n}\right] = \exp\left(-\left(\mu_n - \frac{1}{2}\sigma_n^2\right)2j\right)\exp\left(-m_z^n(t,2j) + \frac{1}{2}\widetilde{V}_{zz}^n(t,2j)\right), \tag{78}$$

where

$$\widetilde{V}_{zz}^n(t,2j) = V_{zz}^n(t,2j) - \sigma_n^2 2j. \tag{79}$$

Also from above, we have $m_z^n(t,1) = \mathbf{b}_n'\widetilde{\mathbf{X}}_t$, $\widetilde{V}_{zz}^n(t,1) = 0$, and, for any $i > 1$,

$$m_z^n(t,i) - m_z^n(t,i-1) = \mathbf{b}_n'\mathbf{m}_{\mathsf{x}}(t,i-1), \tag{80}$$

$$\widetilde{V}_{zz}^n(t,i) - \widetilde{V}_{zz}^n(t,i-1) = 2\mathbf{b}_n'\mathbf{V}_{xz}^n(t,j-1) + \mathbf{b}_n'\mathbf{V}_{\mathsf{xx}}(t,j-1)\mathbf{b}_n, \tag{81}$$

$$\mathbf{V}_{xz}^n(t,j) = \mathbf{A}\mathbf{V}_{xz}^n(t,j-1) + \mathbf{A}\mathbf{V}_{xx}(t,j-1)\mathbf{b}_n + \mathbf{v}_n. \tag{82}$$

Thus, both $m_z^n(t,2j)$ and $\widetilde{V}_{zz}^n(t,2j)$ are independent of μ_n and σ_n^2, and are functions of $\mathbf{b}_n$ and $\mathbf{v}_n$ only. Let $\eta_1 = \frac{\theta}{\sigma}$ and $\eta_2 = 1-\theta$. Then, both $\mathbf{b}_n$ and $\mathbf{v}_n$ are linear functions of η_1 and η_2. Thus, (78) can be written as

$$E_t\left[\frac{M_{t+2j}^n}{M_t^n}\right] = \exp\left(-\left(\mu_n - \frac{1}{2}\sigma_n^2\right)2j\right) H\left(t,2j,\eta_1,\eta_2\right), \tag{83}$$

where $H\left(t,2j,\eta_1,\eta_2\right) = \exp\left(-m_z^n(t,2j) + \frac{1}{2}\widetilde{V}_{zz}^n(t,2j)\right)$. From (76), the moment condition can be written as

$$\begin{aligned} 1 \quad = \quad & \frac{1}{T}\sum_{t=1}^{T}\left\{\frac{1}{2}\left[\sum_{j=1}^{20}\exp\left(-\frac{1}{T}\sum_{t=1}^{T}\left(\mathbf{b}_\mathsf{n}'\tilde{\mathbf{X}}_t - r_t^n\right)2j\right)H\left(t,2j,\eta_1,\eta_2\right)\right] r_t^{n,L}\right\} \\ & + \frac{1}{T}\sum_{t=1}^{T}\exp\left(-\frac{1}{T}\sum_{t=1}^{T}\left(\mathbf{b}_\mathsf{n}'\tilde{\mathbf{X}}_t - r_t^n\right)40\right)H\left(t,40,\eta_1,\eta_2\right). \end{aligned} \tag{84}$$

Furthermore, equation (74) can be written as follows:

$$\frac{1}{T}\sum_{t=1}^{T}(R_t^{m,n} - r_t^n) = -\frac{1}{2}\sigma_{1,u}^2 + \mathbf{1}_1'\mathbf{v}_p + \eta_1\mathbf{1}_1'\mathbf{v}_c + \eta_2(\frac{1}{100}\mathbf{1}_1'\mathbf{\Sigma}\mathbf{1}_1 - \mathbf{1}_1'\mathbf{v}_p). \tag{85}$$

We choose the values of η_1 and η_2 that solve equation (84) and (85). Given η_1 and η_2, we can determine the values of θ and σ as follows:

$$\theta \quad = \quad 1-\eta_2, \tag{86}$$

$$\sigma \quad = \quad \frac{1-\eta_2}{\eta_1}. \tag{87}$$

Finally, given the solutions to θ and σ, we choose the parameter β so that it solves equation (68).

4.6 Convergence

The one-period nominal interest rate is

$$r_t^n = \mu_n + \mathbf{b}_n'\widetilde{\mathbf{X}}_t - \frac{1}{2}\sigma_n^2, \tag{88}$$

and, more generally, the j-period nominal interest rate is given by

$$r^n(t,j) = \mu_n + \left(m_z^n(t,j) - \frac{1}{2}V_{zz}^n(t,j)\right)j^{-1}. \tag{89}$$

Furthermore, the one-period forward rate between $t+j$ and $t+j+1$ is

$$f^n(t,j) = (j+1)r^n(t,j+1) - jr^n(t,j) = \mu_n + \mathbf{b}_n'\mathbf{m}_\mathsf{x}(t,j) - \mathbf{b}_n'\mathbf{V}_{\mathsf{xz}}^{\mathsf{n}}(t,j) - \frac{1}{2}\mathbf{b}_n'\mathbf{V}_{\mathsf{xx}}(t,j)\mathbf{b}_n - \frac{1}{2}\sigma_n^2. \tag{90}$$

Since the eigenvalues of the matrix $\mathbf{A}$ are all within the unit circle, $\mathbf{m}_x(t,j)$ should converge to zero and both $\mathbf{V}_{\mathsf{xx}}(t,j)$ and $\mathbf{V}^{\mathsf{n}}_{\mathsf{xz}}(t,j)$ should converge to a constant matrix or vector as j increases. Let $\overline{\mathbf{V}}_{\mathsf{xx}}$ and $\overline{\mathbf{V}}^{\mathsf{n}}_{\mathsf{xz}}$ denote the limit of $\mathbf{V}_{\mathsf{xx}}(t,j)$ and $\mathbf{V}^{\mathsf{n}}_{\mathsf{xz}}(t,j)$, respectively, then, we have

$$\begin{aligned} \overline{\mathbf{V}}_{\mathsf{xx}} &= \mathbf{A}\overline{\mathbf{V}}_{\mathsf{xx}}\mathbf{A}' + \mathbf{\Sigma}, &(91)\\ \overline{\mathbf{V}}^{\mathsf{n}}_{\mathsf{xz}} &= \mathbf{A}\overline{\mathbf{V}}^{\mathsf{n}}_{\mathsf{xz}} + \mathbf{A}\overline{\mathbf{V}}_{\mathsf{xx}}\mathbf{b}_n + \mathbf{v}_n. &(92)\end{aligned}$$

Note that $\overline{\mathbf{V}}_{\mathsf{xx}}$ is simply the unconditional variance-covariance matrix of $\widetilde{\mathbf{X}}_t$, which can be solved as

$$\operatorname{vec}\left(\overline{\mathbf{V}}_{\mathsf{xx}}\right) = \left(\mathbf{I} - \mathbf{A}\otimes\mathbf{A}\right)^{-1}\operatorname{vec}\left(\mathbf{\Sigma}\right). \tag{93}$$

Given $\overline{\mathbf{V}}_{\mathsf{xx}}$, $\overline{\mathbf{V}}^{\mathsf{n}}_{\mathsf{xz}}$ is given by the following formula

$$\overline{\mathbf{V}}^{\mathsf{n}}_{\mathsf{xz}} = \left(\mathbf{I} - \mathbf{A}\right)^{-1}\left(\mathbf{A}\overline{\mathbf{V}}_{\mathsf{xx}}\mathbf{b}_n + \mathbf{v}_n\right). \tag{94}$$

Thus, the limiting one-period forward rate is given by

$$\overline{f}^n = \mu_n - \mathbf{b}_n'\overline{\mathbf{V}}^{\mathsf{n}}_{\mathsf{xz}} - \frac{1}{2}\mathbf{b}_n'\overline{\mathbf{V}}_{\mathsf{xx}}\mathbf{b}_n - \frac{1}{2}\sigma_n^2. \tag{95}$$

The real interest rates can be determined in the same way as that used for the nominal interest rates. In particular, the one-period real interest rate is

$$r_t = \mu_m + \mathbf{b}_m'\widetilde{\mathbf{X}}_t - \frac{1}{2}\sigma_m^2, \tag{96}$$

and the limiting one-period forward rate is

$$\overline{f} = \mu_m - \mathbf{b}_m'\overline{\mathbf{V}}_{\mathsf{xz}} - \frac{1}{2}\mathbf{b}_m'\overline{\mathbf{V}}_{\mathsf{xx}}\mathbf{b}_m - \frac{1}{2}\sigma_m^2, \tag{97}$$

where $\overline{\mathbf{V}}_{\mathsf{xz}} = \left(\mathbf{I} - \mathbf{A}\right)^{-1}\left(\mathbf{A}\overline{\mathbf{V}}_{\mathsf{xx}}\mathbf{b}_m + \mathbf{v}_m\right)$.

The present value of nominal surplus converges if and only if the nominal growth rate is less than the limiting nominal forward rate, i.e.,

$$\mu_{\overline{y}} < \overline{f}^n = \mu_n - \mathbf{b}_n'\overline{\mathbf{V}}^{\mathsf{n}}_{\mathsf{xz}} - \frac{1}{2}\mathbf{b}_n'\overline{\mathbf{V}}_{\mathsf{xx}}\mathbf{b}_n - \frac{1}{2}\sigma_n^2, \tag{98}$$

and the present value of real surplus converges if and only if the real growth rate is less than the limiting real forward rate, i.e.

$$\mu_y < \overline{f} = \mu_m - \mathbf{b}_m'\overline{\mathbf{V}}_{\mathsf{xz}} - \frac{1}{2}\mathbf{b}_m'\overline{\mathbf{V}}_{\mathsf{xx}}\mathbf{b}_m - \frac{1}{2}\sigma_m^2 \tag{99}$$