Appendix

Proof of Proposition 1: From the household's Euler equation we have

$$\sigma g(t) + \rho = r(t). \tag{58}$$

Differentiating (18) yields $\frac{\dot{V}(t)}{V(t)} = \frac{\dot{w}(t)}{w(t)} = g(t)$. Substituting into (17) and rearranging gives

$$r(t) = \frac{\delta(1 - e^{-\gamma})(1 - X(t))}{e^{-\gamma}} + \delta\gamma X(t) - \delta X(t) \tag{59}$$

Equating (58) and (59) and solving for the stationary allocation of labor to entrepreneurship thus yields

$$X(t) = X^* = \frac{\delta(1 - e^{-\gamma}) - \rho e^{-\gamma}}{\delta - (1 - \sigma)e^{-\gamma}\delta\gamma}. \tag{60}$$

Substituting into (16) gives (20). Note that with $\sigma < 1$, the existence of a positive growth path requires that $\delta(1-e^{-\gamma}) > \rho e^{-\gamma}$ which rearranges to the second inequality in (19). Also for utility to bounded and the transversality condition to hold requires that $r(t) > g(t)$. Using (20) and (58) a sufficient condition given by the first inequality in (19).

Proof of Lemma 1 We show: (1) that if a signal of success from a potential entrepreneur is credible, other entrepreneurs stop innovation in that sector; (2) given (1) entrepreneurs have no incentive to falsely claim success.

Part (1): If entrepreneur i's signal of success is credible then all other entrepreneurs believe that i has a productivity advantage which is e^{γ} times better than the existing incumbent. If continuing to innovate in that sector, another entrepreneur will, with positive probability, also develop a productive advantage of e^{γ}. Such an innovation yields expected profit of 0, since, in developing their improvement, they do not observe the non-implemented improvements of others, so that both firms Bertrand compete with the same technology. Returns to attempting innovation in another sector where there has been no signal of success, or from simply working in production, $w\,(t) > 0$, are thus strictly higher, .

Part (2): If success signals are credible, entrepreneurs know that upon success, further innovation in their sector will cease from Part (1) by their sending of a costless signal. They are thus indifferent between falsely signalling success when it has not arrived, and sending no signal. Thus, there exists a signalling equilibrium in which only successful entrepreneurs send a signal of success.

Proof of Lemma 2: From the production function we have $\ln y(t) = \int_0^1 \ln \frac{y(t)}{p_i(t)} di$. Substituting for $p_i(t)$ using (8) yields $0 = \int_0^1 \ln \frac{w(t)e^{\gamma}}{A_i(T_{v-1})} di$ 0 which re-arranges to (27).

Proof of Lemma 3: Note that in any preceding no-entrepreneurship phase, $r\,(t) = \rho$. Thus, since, in a cycling equilibrium, the date of the next implementation is fixed at T_v, the expected value of entrepreneurship, δV^D, also grows at the rate $\rho > 0$. Thus, if under $X(T_v^E) = 0$, $\delta V^D(T_v^E) > w_v$, then the same inequality is also true the instant before, i.e. at $t \to T_v^E$, since w_v is constant within the cycle. But this violates the assertion that entrepreneurship commences at T_v^E. Thus necessarily, $\delta V^D(T_v^E) = w_v$ at $X\left(T_v^E\right) = 0$.

Proof of Proposition 2: From (28), long–run productivity growth is given by

$$\Gamma_{v+1} = (1 - P\left(T_v\right))\gamma \tag{61}$$

Integrating (37) over the entrepreneurship phase and substituting for $X(\cdot)$ using (34) yields

$$1 - P(T_v) = \delta \int_{T_{v+1}^E}^{T_{v+1}} \left(1 - e^{-\frac{\rho}{\sigma}[t - T_{v+1}^E]}\right) d\tau. \tag{62}$$

Substitution into (61) and integrating gives (39).

Proof of Proposition 3: The increase in output from the beginning of one cycle to the beginning of the next reflects only the improvement in productivity $y_0(T_v) = e^{\Gamma_v} y_0(T_{v-1})$. Moreover, since all output is consumed it follows that $c_0(T_v) = e^{\Gamma_v} c_0(T_{v-1})$. This implies that the long run discount factor is given by $\beta(t) = \sigma\Gamma_v + \rho\left(T_{v+1} - T_v\right) - \int_{T_v}^{t} r(s)ds$. In particular, since $r(t) = 0$ during the downturn, $\beta(t) = \sigma\Gamma_v + \rho\Delta_v^E \quad \forall\ t \in (T_v^E, T_v)$. Combining this with (43) and (44) yields (45).

Proof of Proposition 4: The discounted monopoly profits from owning an innovation at time T_v is given by $V_0^I(T_v) = (1 - e^{-\gamma}) \int_{T_v}^{T_{v+1}} e^{-\int_{T_v}^{\tau} r(s)ds} y(\tau)d\tau + P(T_v)e^{-\beta(T_v)} V_0^I(T_{v+1})$. Substituting for $V_0^I(T_{v+1})$ using (48), and integrating yields

$$V_0^I(T_v) = \left(\frac{(1 - e^{-\gamma})y_0(T_v)}{1 - P(T_v)e^{\Gamma - \beta(T_v)}}\right)\left[\frac{1 - e^{-\rho(T_{v+1}^E - T_v)}}{\rho} + e^{-\rho(T_{v+1}^E - T_v)}\left(\frac{1 - e^{-\frac{\rho}{\sigma}\Delta^E}}{\rho/\sigma}\right)\right]. \tag{63}$$

Asset market clearing during the boom and the fact that $X(T_v) = 0$ implies (using (11)) that $\delta V_0^I(T_v) = w_{v+1} = e^{-\gamma} y_0(T_v)$. Substituting into (63) we have

$$(1 - e^{-\gamma})\delta\left[\frac{1 - e^{-\rho(T_{v+1}^E - T_v)}}{\rho} + e^{-\rho(T_{v+1}^E - T_v)}\left(\frac{1 - e^{-\frac{\rho}{\sigma}\Delta^E}}{\rho/\sigma}\right)\right] = e^{-\gamma}\left(1 - P(T_v)e^{\Gamma - \beta(T_v)}\right). \tag{64}$$

But $\beta(T_v) = \rho(T_{v+1}^E - T_v) + \Gamma$, so that multiplying through by $e^{\rho(T_{v+1}^E - T_v)}$ and collecting terms yields

$$\left(\frac{(1 - e^{-\gamma})\delta}{\rho} - e^{-\gamma}\right)\left(e^{\rho(T_{v+1}^E - T_v)} - 1\right) = e^{-\gamma}\left(1 - P(T_v)\right) - (1 - e^{-\gamma})\delta\left(\frac{1 - e^{-\frac{\rho}{\sigma}\Delta^E}}{\rho/\sigma}\right). \tag{65}$$

Since $\left[1 - P(T_v)\right]\gamma = \Gamma_v$ from (45) we have that $1 - P(T_v) = \frac{\rho\Delta^E}{\gamma(1-\sigma)}$, substituting this into the above, using (46) to substitute out the second term on the right hand side, and rearranging yields

$$e^{\rho(T_{v+1}^E - T_v)} = 1 + \mu\Delta^E, \tag{66}$$

where μ is defined in (50). Taking logs and noting that $T_{v+1}^E - T_v = T_{v+1} - T_v - \Delta^E = \Delta_v - \Delta^E$ yields (49).

Proof of Proposition 5: It is easily verified that under (51) there does exist a unique triple $(\Delta^E, \Delta, \Gamma) > 0$ which solves (39), (45) and (49). The remainder of the proof shows that each of the conditions (E1) through (E4) under (51):

(E1): Since $V_0^I(T_{v+1}) = e^{\Gamma} V_0^I(T_v)$, we can write $V_0^D(T_v) = e^{-\beta(T_v)+\Gamma} V_0^I(T_v)$. From the proof of Proposition 3, $\beta(T_v) = \rho\Delta + \sigma\Gamma$, so that condition **(E1)** requires that $\rho\Delta > (1-\sigma)\Gamma$, which must be true for the consumer's optimization problem to be bounded. Using (45), this condition simply requires that $\Delta > \Delta^E$, which, from (49) and the definition of μ in (50) is true as long as $\frac{\rho}{\delta} > \gamma(1-\sigma)\left(1 - e^{-\gamma}\right)$. This holds if the left–hand inequality in (51) is satisfied.

(E2): This inequality can be written as

$$V^I(t) = \int_t^{T_v} e^{-\int_t^{\tau} r(s)ds} \pi(\tau) d\tau + \frac{P(T_v)}{P(t)} V^D(t) < V^D(t) \qquad \forall\, t \in (T_v^E, T_v) \tag{67}$$

During the downturn we know that $V^D(t) = w_{v-1}/\delta = e^{-\gamma} y_0/\delta$ and $r(t) = 0$. Substituting these and rearranging yields

$$e^{-\gamma}\left(1 - \frac{P(T_v)}{P(t)}\right) - (1 - e^{-\gamma})\delta \left(\frac{1 - e^{-\frac{\rho}{\sigma}(T_v - t)}}{\rho/\sigma}\right) > 0, \tag{68}$$

where $P(t) = 1 - \int_{T_v^E}^t \delta \left(1 - e^{-\frac{\rho}{\sigma}[\tau - T_v^E]}\right) d\tau = 1 - \delta[t - T_v^E] + \delta \left(\frac{1 - e^{-\frac{\rho}{\sigma}[t - T_v^E]}}{\rho/\sigma}\right)$. When $t = T_v$, this becomes $P(T_v) = 1 - \delta\Delta^E + \delta \left(\frac{1 - e^{-\frac{\rho}{\sigma}\Delta^E}}{\rho/\sigma}\right)$. It is easily verified that $\ln P(t)$ is decreasing and convex in t. It follows that

$$-\frac{\ln P(T_v) - \ln P(t)}{T_v - t} \geq \left.\frac{d \ln P(t)}{dt}\right|_{t=T_v} \tag{69}$$

Let

$$q = -\frac{\sigma}{\rho} \left.\frac{d \ln P(t)}{dt}\right|_{t=T_v} = \frac{\delta \left(\frac{1 - e^{-\frac{\rho}{\sigma}\Delta^E}}{\rho/\sigma}\right)}{1 - \delta\Delta^E + \delta \left(\frac{1 - e^{-\frac{\rho}{\sigma}\Delta^E}}{\rho/\sigma}\right)}. \tag{70}$$

Now note that condition (51) implies that $q > 1$. To see this, note that it follows from (70) that $q > 1$, if and only if $\delta\Delta^E > 1$. So we first demonstrate that $\delta\Delta^E > 1$. In Figure 2, at the positive intersection of (39) and (45), the former (linear function) must be steeper than the latter (concave function). Differentiating these two curves, this implies that Δ^E must satisfy

$$1 - e^{-\frac{\rho}{\sigma}\Delta^E} > \frac{\rho}{\delta\gamma(1-\sigma)}. \tag{71}$$

Substituting using (46) this implies

$$\delta\Delta^E > \frac{\sigma}{\gamma(1-\sigma) - \rho/\delta}. \tag{72}$$

So that a sufficient condition for $\delta\Delta^E > 1$ is that $\frac{\rho}{\delta} > \gamma(1-\sigma) - \sigma$, which must be true under (51).

We now have that $q > 1$, and it follows that

$$\log P(T_v) - \log P(t) \leq -q\frac{\rho}{\sigma}(T_v - t) \tag{73}$$

Rearranging gives $1 - \frac{P(T_v)}{P(t)} \geq 1 - e^{-q\frac{\rho}{\sigma}(T_v - t)}$. It follows that a sufficient condition for (68) is that

$$e^{-\gamma}\left(1 - e^{-q\frac{\rho}{\sigma}(T_v-t)}\right) - \left(1 - e^{-\gamma}\right)\delta\left(\frac{1 - e^{-\frac{\rho}{\sigma}(T_v-t)}}{\rho/\sigma}\right) \geq 0 \tag{74}$$

We know that (68), and hence (74), holds with equality at $t = T_v$, thus a sufficient condition is that the left hand side of (74) declines monotonically with $t < T_v$. That is

$$-e^{-\gamma}q\frac{\rho}{\sigma}e^{-q\frac{\rho}{\sigma}(T_v-t)} + \left(1 - e^{-\gamma}\right)\delta e^{-\frac{\rho}{\sigma}(T_v-t)} < 0 \;\forall\; t \in [T_v^E, T_v]. \tag{75}$$

Since $q > 1$, $e^{-q\frac{\rho}{\sigma}(T_v-t)} \leq e^{-\frac{\rho}{\sigma}(T_v-t)}$, and so a sufficient condition is $q > \frac{\sigma(1-e^{-\gamma})\delta}{\rho e^{-\gamma}}$. From (70), this inequality holds if

$$\delta\left(\frac{1 - e^{-\frac{\rho}{\sigma}\Delta^E}}{\rho/\sigma}\right) > \frac{\sigma(e^{\gamma}-1)\delta}{\rho}\left[1 - \delta\Delta^E + \delta\left(\frac{1 - e^{-\frac{\rho}{\sigma}\Delta^E}}{\rho/\sigma}\right)\right] \tag{76}$$

Since $\delta\Delta^E > 1$, **(E2)** will hold if $\frac{\rho}{\delta} > \sigma(e^{\gamma} - 1)$. If, instead, however, $\frac{\rho}{\delta} < \sigma(e^{\gamma} - 1)$, **(E2)** can still hold, so long as (51) is satisfied. To see this note that from (46)

$$\Delta^E = \frac{1 - e^{-\frac{\rho}{\sigma}\Delta^E}}{\rho/\sigma\left(1 - \frac{\rho}{\delta\gamma(1-\sigma)}\right)} \tag{77}$$

Substituting into (76) and rearranging yields:

$$\left(\frac{\rho}{\delta\gamma(1-\sigma)}\right)^2 e^{-\gamma}\left(1 - \frac{\sigma(e^{\gamma}-1)\delta}{\rho}\right) < \frac{\rho}{\delta\gamma(1-\sigma)} - \left(1 - e^{-\gamma}\right) \tag{78}$$

Since the left hand side is negative when $\frac{\rho}{\delta} < \sigma(e^{\gamma} - 1)$, it is sufficient that $\frac{\rho}{\delta} > (1 - e^{-\gamma})\gamma(1-\sigma)$, which is true under the left–hand inequality in (51).

(E3): Long–run market clearing implies that $\delta V^I(T_{v-1}) = w_v$. It follows that a sufficient condition for **(E3)** is $\frac{dV^I(t)}{dt} < 0, \forall\, t \in (0, T_v^E)$. Since during this phase $r(t) = \rho$ and $g = 0$, the value of immediate implementation can be expressed as

$$V^I(t) = (1 - e^{-\gamma})\left(\frac{1 - e^{-\rho(T_v^E - t)}}{\rho}\right)y_0(T_{v-1}) + e^{-\rho(T_v^E - t)}V^I(T_v^E). \tag{79}$$

Differentiating w.r.t. to t yields

$$\frac{dV^I(t)}{dt} = -(1 - e^{-\gamma})e^{-\rho(T_v^E - t)}y_0(T_{v-1}) + \rho e^{-\rho(T_v^E - t)}V^I(T_v^E). \tag{80}$$

If (51) holds then from **(E2)**, we have that $V^I(T_v^E) < w_v/\delta = e^{-\gamma}y_0(T_{v-1})/\delta$, and so

$$\frac{dV^I(t)}{dt} < -\frac{e^{-\rho(T_v^E - t)}y_0}{\delta}\left[(1-e^{-\gamma})\delta - \rho e^{-\gamma}\right] < 0. \tag{81}$$

This requires that $\frac{\rho}{\delta} < e^{\gamma} - 1$. Since $e^{\gamma} - 1 > \gamma > \gamma(1-\sigma)$, this follows from the right–hand inequality of (51).

(E4): This is equivalent to $\Gamma < \gamma$. Substituting for Δ^E in (46) using (45) and rearranging slightly yields

$$\frac{\left(1 - e^{-\frac{1-\sigma}{\sigma}\Gamma}\right)}{\left(\frac{1-\sigma}{\sigma}\right)\Gamma} = 1 - \frac{\rho}{\delta\gamma(1-\sigma)} \tag{82}$$

The left hand side of this equation is monotonically *decreasing* in Γ (to see this note that $1-e^{-x}$ is an increasing, concave function of x and its slope, e^{-x}, is just equal to 1 at $x = 0$, and then declines with x). It follows that $\Gamma < \gamma$ requires that

$$\frac{\left(1 - e^{-\frac{1-\sigma}{\sigma}\gamma}\right)}{\left(\frac{1-\sigma}{\sigma}\right)\gamma} < \frac{\left(1 - e^{-\frac{1-\sigma}{\sigma}\Gamma}\right)}{\left(\frac{1-\sigma}{\sigma}\right)\Gamma} = 1 - \frac{\rho}{\delta\gamma(1-\sigma)} \tag{83}$$

So a necessary and sufficient condition for **(E4)** is $\frac{\rho}{\delta} < \gamma(1-\sigma) - \sigma\left(1 - e^{-\frac{1-\sigma}{\sigma}\gamma}\right)$, which holds under the right hand inequality in (51).

Proof of Proposition 6: Growth in the acyclical economy is given by g^a in (20). In the cyclical economy, from (45) the average long run growth rate can be expressed as $g^c = \frac{\Gamma}{\Delta} = \frac{\rho}{1-\sigma}\frac{\Delta^E}{\Delta}$. Using (49) and the fact that for any $x > 0$, $\ln(1+x) < x$ we have $\Delta < \Delta^E + \frac{\mu}{\rho}\Delta^E$. It follows that

$$g^c > \frac{\rho}{1-\sigma}\left(\frac{\Delta^E}{\Delta^E + \frac{\mu}{\rho}\Delta^E}\right) = \frac{[\delta(1-e^{-\gamma}) - \rho e^{-\gamma}]\gamma}{1-(1-\sigma)\,\gamma e^{-\gamma}} = g^a. \tag{84}$$

Proof of Proposition 7: Suppose that the economies start at T_0 with identical distributions of implemented innovations and no unimplemented innovations. Hence the maximum level of output, $\overline{y}(T_0)$, that could be produced if all labor were being used in manufacturing is the same in both equilibria. In the acyclical equilibrium, household welfare is given by

$$W^A(T_0) = \frac{c(T_0)^{1-\sigma}}{1-\sigma}\left(\frac{1}{\rho-(1-\sigma)g^a}\right) = \frac{\overline{y}(T_0)^{1-\sigma}}{1-\sigma}\left(\frac{(1-X^*)^{1-\sigma}}{\rho-(1-\sigma)g^a}\right) \tag{85}$$

where X^* is the fraction of labor effort in entrepreneurship given by (60). In the cyclical equilibrium, household welfare at the beginning of the first cycle is

$$\begin{aligned} W^C(T_0) &= \frac{c(T_0)^{1-\sigma}}{1-\sigma}\sum_{v=0} e^{-\rho(\Delta-\Delta^E)v}\left\{\frac{1-e^{-\rho(\Delta-\Delta^E)}}{\rho} + e^{-\rho(\Delta-\Delta^E)}\left(\frac{1-e^{-\frac{\rho}{\sigma}\Delta^E}}{\rho/\sigma}\right)\right\} \\ &= \frac{\overline{y}(T_0)^{1-\sigma}}{1-\sigma}\left\{\frac{1}{\rho} + \frac{1}{\mu}\left(1-\frac{\rho}{\delta\gamma(1-\sigma)}\right)\right\} \end{aligned} \tag{86}$$

Observe that rearranging (84) yields $\frac{1}{\mu} = \left(\frac{1-\sigma}{\rho}\right) \frac{g_a}{\rho-(1-\sigma)g_a}$. Substituting into (86) and noting that $\frac{1}{\rho} \equiv \frac{1}{\rho-(1-\sigma)g_a}\left(1 - \frac{(1-\sigma)g_a}{\rho}\right)$ yields

$$W^C(T_0) = \frac{\overline{y}(T_0)^{1-\sigma}}{1-\sigma}\left(1 - \frac{g_a}{\delta\gamma}\right)\frac{1}{\rho-(1-\sigma)g_a}. \tag{87}$$

But since $g_a = \delta\gamma X^*$, it follows that the ratio of welfare in the cyclical economy to that in the acyclical one is given by $\frac{W^C(T_0)}{W^A(T_0)} = (1 - X^*)^\sigma < 1$.