ECON435/835: Topics in Development Economics

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Assignment #3 – Answer Key

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1. Consider an economy, like that considered by Murphy, Shleifer and Vishny (1989), consisting of three (representative) households. Each household supplies 1 unit of labour which could be used either in manufacturing or agriculture. Household 1 owns no shares, household 2 owns a fraction $\hat{\gamma} < \frac{1}{2}$ of the claims to both manufacturing firm profits and agricultural rents, and household 3 owns the remaining $1 - \hat{\gamma} > \frac{1}{2}$.

(a) The agricultural sector produces output using only labour according to

$$Y_A = \theta L_A^{\frac{1}{2}}.$$

Derive the agricultural profit function, $\pi_A(L_A)$, and wage function, $w(L_A)$ implied by profit maximization.

Landowners maximize

$$\pi_A = \theta L_A^{\frac{1}{2}} - w L_A$$

The necessary condition for a maximum is

$$\frac{d\pi_A}{dL_A} = \frac{\theta}{2} L_A^{-\frac{1}{2}} - w = 0.$$

It follows that

$$w(L_A) = \frac{\theta}{2} L_A^{-\frac{1}{2}},$$

and

$$\pi_A(L_A) = \theta L_A^{\frac{1}{2}} - \frac{\theta}{2} L_A^{-\frac{1}{2}} L_A = \frac{\theta}{2} L_A^{\frac{1}{2}}.$$

(b) Let $z = 1$ be the maximum food requirement of all households. Assuming that $w < z$, derive an expression for the equilibrium amount of labour effort in agriculture as a function of θ. For what range of values of θ is it true that $w < z$.

Equilibrium in agriculture occurs when total agricultural income equals total spending on food:

$$\begin{aligned} w(L_A)L_A + \pi_A(L_A) &= zN + w(L_A)(L - N) \\ \theta L_A^{\frac{1}{2}} &= 2 + \frac{\theta}{2} L_A^{-\frac{1}{2}} \end{aligned}$$

Multiplying through by $2L_A^{\frac{1}{2}}$ yields

$$2\theta L_A - 4L_A^{\frac{1}{2}} - \theta = 0$$

Let $x = L_A^{\frac{1}{2}}$, then this is a quadratic equation in x:

$$2\theta x^2 - 4x - \theta = 0$$

Applying the quadratic formula yields

$$L_A^{\frac{1}{2}} = x = \frac{4 + \sqrt{16 + 8\theta^2}}{4\theta} = \frac{2 + \sqrt{4 + 2\theta^2}}{2\theta}$$

(note that the other root would be negative). The wage is given by

$$w(L_A) = \frac{\theta}{2} L_A^{-\frac{1}{2}} = \frac{\theta^2}{2 + \sqrt{4 + 2\theta^2}}$$

for $w(L_A) < z$ we require that θ satisfies

$$\begin{aligned} \theta^2 &< 2 + \sqrt{4 + 2\theta^2} \\ \left(\theta^2 - 2\right)^2 &< 4 + 2\theta^2 \\ \theta^4 - 4\theta^2 + 4 &< 4 + 2\theta^2 \\ \theta^2 &< 6 \\ \theta &< \sqrt{6} = 2.45 \end{aligned}$$

(c) Manufacturing production is the same as in Murphy, Shleifer and Vishny. The fixed labour requirement for the modern technology is $C = 0.5$ and the traditional technology is 50% less productive than the modern technology: $\alpha = 1.5$. Show that if all sectors up to Q industrialize, only household 3 will demand output from the Qth sector.

Since they limit price traditional firms out of the market, so that $p = \alpha w$, the profits of modern firms are given by

$$\pi = \alpha w x - w x - w C = w[(\alpha - 1)x - C],$$

where x is the demand for their product. If all sectors up to Q industrialize, this means that firms just earn zero profits in sector Q, and the total number of households that buy this good is given by

$$N^* = \frac{C}{\alpha - 1} = \frac{0.5}{1.5 - 1} = 1.$$

It follows that the demand for the Qth good comes from the richest household 3.

(d) Derive the equilibrium level of profits in the manufacturing sector as a function of θ, $\hat{\gamma}$ and L_A. How do the equilibrium profits depend on these parameters?

Since household 1 buys only food and household 3's expenditures covers the fixed costs of all industrial firms, the expenditure of household 2 generates pure profits for modern firms. Since household 2 buys 1 of each good up to Q, these pure profits are aggregate to

$$\pi = \alpha w Q - wQ.$$

Since the total expenditure of household 2 must equal its income, Q is determined by

$$\begin{aligned} z + pQ &= w + \hat{\gamma}(\pi + \pi_A) \\ Q &= \frac{w + \hat{\gamma}(\pi + \pi_A) - z}{\alpha w} \end{aligned}$$

Substituting for Q yields

$$\begin{aligned} \pi &= \left(\frac{\alpha - 1}{\alpha}\right)[w + \hat{\gamma}(\pi + \pi_A) - z] \\ \pi &= \frac{1}{3}\left[\hat{\gamma}(\pi + \frac{\theta}{2}L_A^{\frac{1}{2}}) - (1 - \frac{\theta}{2}L_A^{-\frac{1}{2}})\right] \end{aligned}$$

Solving for π yields the MM-Curve

$$\pi = \frac{\hat{\gamma}\frac{\theta}{2}L_A^{\frac{1}{2}} - (1 - \frac{\theta}{2}L_A^{-\frac{1}{2}})}{3 - \hat{\gamma}}.$$

Clearly, π is increasing with $\hat{\gamma}$. The bigger the share of profits going to the middle class (which is household 2 as long as $\hat{\gamma} < \frac{1}{2}$), the bigger the demand for industrial goods. This in turn yields greater profits which results in even greater demand. An increase in θ, the productivity of agriculture also increases π, if L_A is held fixed: it raises the incomes of all households which induces them to demand more industrial goods. The impact of an increase in L_A, holding the other parameters fixed is given by

$$\frac{d\pi}{dL_A} = \frac{\theta/4}{3 - \hat{\gamma}}\left(\hat{\gamma}L_A^{-\frac{1}{2}} - L_A^{-\frac{3}{2}}\right) = \frac{(\theta/4)\, L_A^{-\frac{3}{2}}}{3 - \hat{\gamma}}(\hat{\gamma}L_A - 1)$$

In general, the effect of an increase in the agricultural labour force has an ambiguous impact on π. If $L_A > 1/\hat{\gamma}$, the effect is positive.

(e) Using your answers to (b) and (d) characterize the impact of an increase in θ on the equilibrium value of L_A.

In this example, from (b) the AA-curve is given by

$$L_A^{\frac{1}{2}} = \frac{2 + \left(4 + 2\theta^2\right)^{\frac{1}{2}}}{2\theta}$$

and from (d) the MM-Curve is given by

$$\pi = \frac{\hat{\gamma}\frac{\theta}{2}L_A^{\frac{1}{2}} - (1 - \frac{\theta}{2}L_A^{-\frac{1}{2}})}{3 - \hat{\gamma}}$$

These are illustrated in Figure 1. Here the MM-curve is drawn as an upward sloping curve ($L_A > 1/\hat{\gamma}$), but if it were downward sloping the implications are similar. An increase in θ causes the AA curve to shift to the left:

$$\begin{aligned} \frac{dL_A^{\frac{1}{2}}}{d\theta} &= \frac{2\theta.4\theta.\frac{1}{2}\left(4 + 2\theta^2\right)^{-\frac{1}{2}} - 2\left(2 + \left(4 + 2\theta^2\right)^{\frac{1}{2}}\right)}{4\theta^2} \\ &= \frac{-2\left(4 + 2\theta^2\right)^{\frac{1}{2}} - 4}{2\theta^2\left(4 + 2\theta^2\right)^{\frac{1}{2}}} < 0 \end{aligned}$$

An increase in θ causes the MM curve to shift up (see part (d)). It follows that, in equilibrium, L_A declines and π increases as we move from E_1 to E_2.

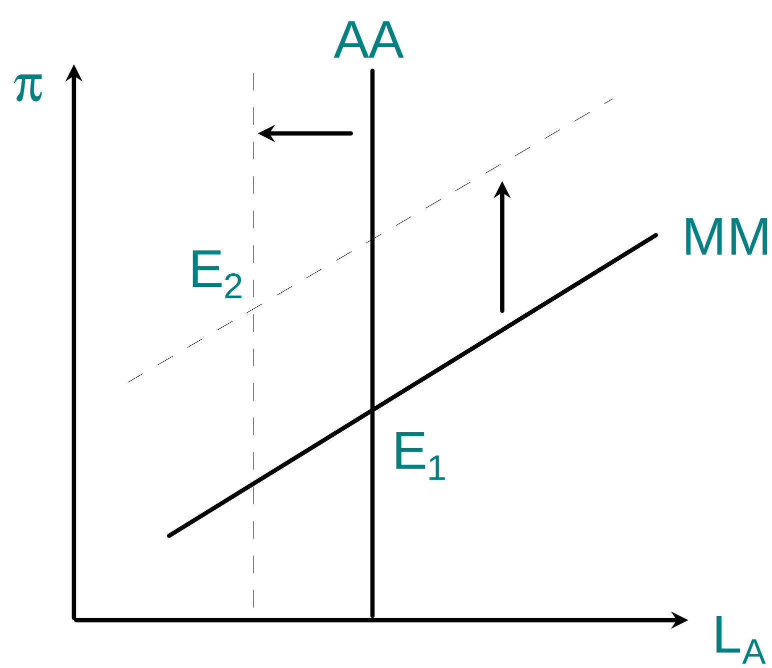

Figure 1: Impact of Increased Agricultural Productivity

(f) Ceteris paribus the increase in θ raises the marginal product of, and hence the demand for, labour in agriculture. Explain intuitively how your answer in part (e) comes about despite this.

The main reason is that the demand for agricultural products from the richest 2 households is fixed by assumption. When productivity rises it is not necessary to employ as much labour to produce this amount of food. Although, the demand for food from the poorest household does increase it is not enough to generate a higher demand for labour.

2. Consider the following credit market. The output of a risk–neutral farmer depends stochastically on her effort level, x. Specifically, suppose her output is Q with probability $p(x) = 1 - e^{-ax}$ and is 0 otherwise. The utility cost to the farmer of providing this effort is x. In order to undertake any production at all, the farmer requires fixed capital K. The farmer has wealth W which she could either invest in her own production or place in a development bank and earn interest i.

(a) Suppose $W > K$, so that the farmer can self–finance. What would the farmer's optimal choice of effort, x^*, be?

In this case the farmer solves

$$\max_x \; p(x)Q - x - (1+i)K$$

The FOC is

$$p'(x)Q - 1 = 0$$

Since $p'(x) = ae^{-ax}$, we have

$$\begin{aligned} ae^{-ax}Q &= 1 \\ e^{ax} &= aQ \\ x^* &= \frac{1}{a}\ln aQ. \end{aligned}$$

(b) Now suppose the farmer only has wealth $W < K$ and must borrow the remainder $K - W$. The repayment to the lender is $R = (1+r)(K-W)$, where $r \geq i$ is the lending rate. What would the farmer's utility-maximizing level of effort be? How does it depend on r?

In this case the farmer solves

$$\max_x \; p(x)\left[Q - (1+r)(K-W)\right] - x - (1+i)W$$

The FOC is

$$\begin{aligned} ae^{-ax}\left[Q-(1+r)(K-W)\right] &= 1 \\ e^{-ax} &= \frac{1}{a\left[Q-(1+r)(K-W)\right]} \end{aligned} \tag{1}$$

and so the choice of effort (assuming the implied payoff is positive) would be

$$\hat{x} = .\frac{1}{a}\ln a\left[Q-(1+r)(K-W)\right] \tag{IC}$$

Clearly $\hat{x}$ is decreasing in r — having to pay a higher repayment in the good state of the world reduces the farmer's incentives to provide effort.

(c) If lenders are competitive and face the marginal cost of funds i, what must be the relationship be between r and x ? Explain.

The lender's net expected profit is

$$\pi = p(x)R - (1+i)(K-W)$$

Under competitive conditions profits would be driven to zero and so

$$\begin{aligned} \left(1-e^{-ax}\right)(1+r)(K-W) &= (1+i)(K-W) \\ \left(1-e^{-ax}\right)(1+r) &= (1+i) \end{aligned}$$

We can express the lender's zero-profit condition as

$$e^{-ax} = 1 - \frac{1+i}{1+r} \tag{2}$$

Since the left hand side is decreasing with x and the right hand side is increasing with r this equation represents a negative relationship between the two variables. We can express the condition as

$$x = -\frac{1}{a}\ln\left(1-\frac{1+i}{1+r}\right) \tag{ZP}$$

(d) Characterize the interest rate $\hat{r}$ and the effort level $\hat{x}$ that obtains in an information–constrained Pareto efficient equilibrium. (Hint: use a diagram)

Figure 2 illustrates the two curves (IC) and (ZP). As can be seen, there are potentially two intersections. Substituting for e^{-ax} in (1) using (2), we get

$$a\left(1-\frac{1+i}{1+\hat{r}}\right)\left[Q-(1+\hat{r})(K-W)\right] = 1$$

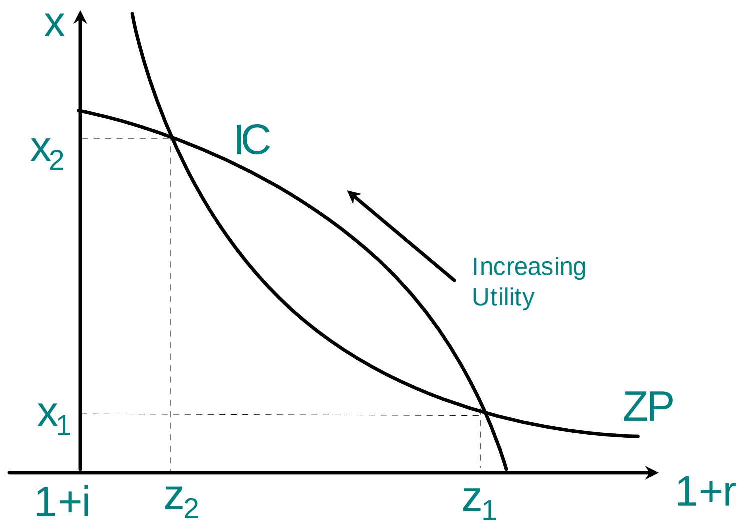

Figure 2: Question 2 (d)

Letting $z = 1 + r$ and multiply through by z yields

$$\begin{aligned} a(z - (1+i))\left[Q - z(K - W)\right] &= z \\ azQ - az^2(K - W) - a(1+i)Q + az(K - W) &= z \\ a(K - W)z^2 - (aQ + a(K - W) - 1)z + a(1+i)Q &= 0 \end{aligned}$$

The solution for z solves this quadratic equation. Given this we can compute $\hat{r}$. Then we can substitute this into (2) to solve for x. Since this procedure will yield two pairs of solutions, we must determine which pair maximizes the utility of the farmer.

(e) Suppose $a = \frac{1}{2}$, $Q = 30$, $K = 20$, $W = 10$ and $i = 0$. What are the constrained–efficient values of $\hat{r}$ and $\hat{x}$? How does the latter compare with x^* computed for these parameters?

After substituting in these parameters, the quadratic equation above becomes

$$5z^2 - 19z + 15 = 0$$

The solution is

$$z = \frac{19 \pm \sqrt{361 - 300}}{10} = \frac{19 \pm 7.81}{10}$$

Therefore we have two potential solutions: $(z_1,\ z_2) = (2.68,\ 1.12)$. Substituting these into (2) we have

$$\begin{aligned} e^{-\frac{1}{2}x} &= \frac{z-1}{z} \\ \hat{x} &= 2\ln\frac{z}{z-1} \end{aligned}$$

and so we have two corresponding solutions for x given by $(x_1,\ x_2) = (0.934,\ 4.467)$.

Under these parameter values, the expected utility of the borrower is

$$EU(z,x) = \left(1 - e^{-\frac{1}{2}x}\right)[30 - 10z)] - x - 10$$

Substituting in these two sets of potential solutions yields

$$\begin{aligned} EU(z_1, x_1) &= -9.74 \\ EU(z_2, x_2) &= 2.32 \end{aligned}$$

The first pair of solutions does not satisfy the borrower's participation constraint, since it yields negative utility. It follows that the constrained efficient solution is $(\hat{r}, \hat{x}) = (0.12,\ 4.467)$. Under these parameter values, the unconstrained efficient level of effort is

$$x^* = 2\ln 15 = 5.416.$$

Clearly, this exceeds the effort in the information–constrained situation. Consequently expected output must be higher too.

3. A risk–neutral farmer produces output, Y, using capital, K, according to

$$Y = K^{\frac{1}{2}}$$

The capital is completely used up in production each period and the opportunity cost of each dollar invested is $1 + r$, where $r = 0.1$.

(a) If the farmer were to finance the investment out of her owning savings, what would the optimal level of investment be in each period? What is the farmer's income?

In this case the farmer solves

$$\max_K \ K^{\frac{1}{2}} - (1 + r)K$$

The first-order condition for a maximum is

$$\begin{aligned} \frac{1}{2}K^{-\frac{1}{2}} &= 1 + r \\ K &= \left(\frac{1}{2.2}\right)^2 = 0.21 \end{aligned}$$

The implied income of the farmer is

$$\pi = (0.21)^{\frac{1}{2}} - (1.1)0.21 = 0.22$$

Now suppose that the farmer has no savings and must finance the investment in each period by borrowing from a risk–neutral lender. The loan contract specifies that in return for the initial loan of K, the farmer should pay the lender R. The marginal cost of funds to the lender is $1+r$. There is no uncertainty, but the lender has no means to enforce repayment directly. Assume that if the farmer defaults in any period she cannot access credit in the future. In this "autarky" case the farmer moves to an urban area where she can earn a wage v per period. It is prohibitively costly for the lender to track her down. The farmer discounts the future at rate $\delta = 0.6$.

(b) Write down the farmer's incentive constraint and the lender's participation constraint.

The farmer's incentive constraint is

$$\begin{aligned} \frac{K^{\frac{1}{2}} - R}{1-\delta} &\geq K^{\frac{1}{2}} + \frac{\delta v}{1-\delta} \\ K^{\frac{1}{2}} - R &\geq (1-\delta) K^{\frac{1}{2}} + \delta v \\ R &\leq \delta \left(K^{\frac{1}{2}} - v \right) \end{aligned}$$

The lender's participation constraint is

$$R \geq (1+r)K$$

(c) Assuming that $v = 0$, illustrate these constraints on a diagram with R on the vertical and K on the horizontal axis. Derive the value of K at all intersections of the two constraints, if any.

The constraints are illustrated in Figure 3 and intersect when

$$(1+r)K = \delta \left(K^{\frac{1}{2}} - v \right)$$

Substituting in the values assumed we get

$$1.1K = 0.6K^{\frac{1}{2}}$$

Thus there are intersections at $K = 0$ and at

$$\bar{K} = \left(\frac{0.6}{1.1} \right)^2 = 0.3$$

(d) If the debt contract is designed so as to maximize the farmer's income, is the incentive constraint binding? Explain your answer using the diagram in part (c). What is the implied level of investment?

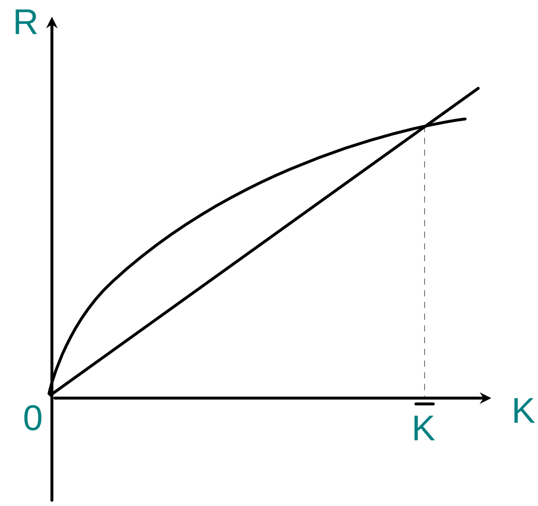

Figure 3: Constraints

The farmer's income is

$$\pi = K^{\frac{1}{2}} - R$$

For a given value of π, the iso-profit curve of the farmer can be expressed as

$$R = K^{\frac{1}{2}} - \pi$$

The slope of this isoprofit line is

$$\left.\frac{dR}{dK}\right|_{\pi} = \frac{1}{2}K^{-\frac{1}{2}}$$

At a point of tangency between this isoprofit line and the lender's participation constraint (as illustrated in Figure 4), we have

$$\begin{aligned}\frac{1}{2}K^{-\frac{1}{2}} &= 1.1 \\ K^* &= 0.21\end{aligned}$$

Since K^* satisfies both constraints it is feasible. Note that in this case, the incentive constraint is not binding and the outcome is efficient.

(e) Now assume that opportunities in the urban sector improve (e.g. due to globalization), so that $v = 0.1$. Does the incentive constraint bind now? What is the implied level of investment?

In this case, the points of intersection are determined by

$$\begin{aligned}1.1K &= 0.6\left(K^{\frac{1}{2}} - 0.1\right) \\ 1.1K - 0.6K^{\frac{1}{2}} + 0.06 &= 0\end{aligned}$$

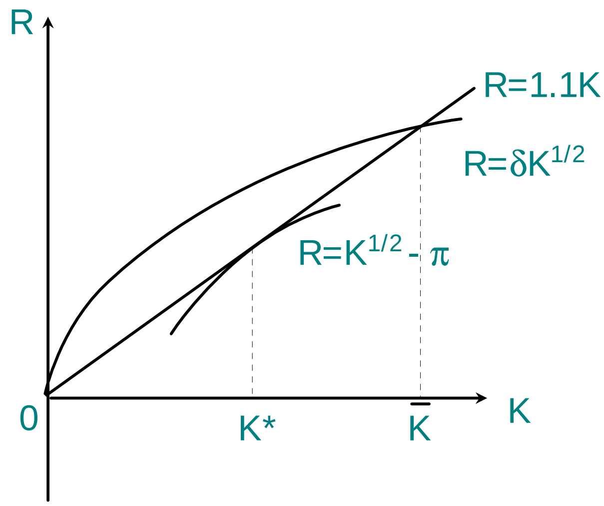

Figure 4: Unconstrained Outcome

Letting $x = K^{\frac{1}{2}}$, this is a quadratic equation with roots

$$K^{\frac{1}{2}} = \frac{0.6 \pm \sqrt{0.36 - 0.264}}{2.2} = 0.273 \pm 0.14$$

Therefore the points of intersection are

$$(K_1, K_2) = (0.02, 0.17)$$

Since the point of tangency, $K^* = 0.21$, lies outside this range, the maximum feasible income for the farmer must occur at the highest point of intersection of the two constraints (see Figure 5):

$$K^{**} = 0.17$$

In this case, there is investment, but its level is constrained by the incentive constraint.

(f) What happens if $v = 0.2$?

Now points of intersection must satisfy

$$\begin{aligned} 1.1K &= 0.6\left(K^{\frac{1}{2}} - 0.2\right) \\ 1.1K - 0.6K^{\frac{1}{2}} + 0.12 &= 0 \end{aligned}$$

Letting $x = K^{\frac{1}{2}}$, this is a quadratic equation with roots

$$K^{\frac{1}{2}} = \frac{0.6 \pm \sqrt{0.36 - 0.526}}{2.2}$$

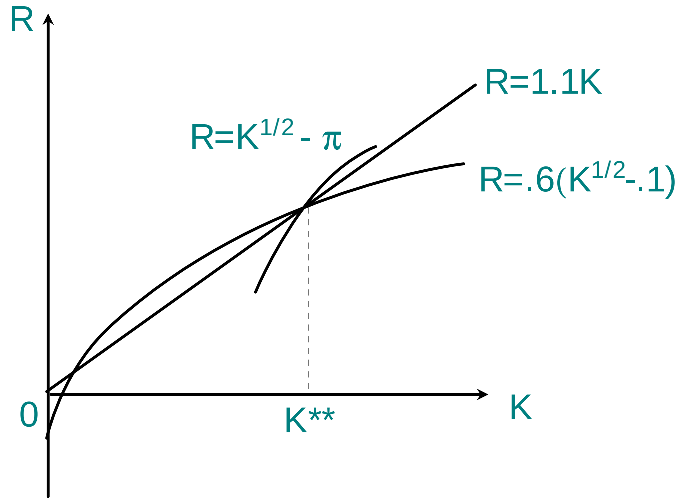

Figure 5: Constrained Equilibrium Lending

Since this involves the square root of a negative number, it implies that there are no roots (i.e. no points of intersection) for real (i.e. non–imaginary) values of K. This situation is illustrated in Figure 6.

Intuitively, this represents a situation of complete credit rationing (market shut down), in which there is no level of K that lenders can offer and which will lead to repayment that borrowers will find profitable. The reason is that the outside option is so good, the borrower's always prefer to disappear into the urban area even if this drives them in to autarky in rural credit markets.

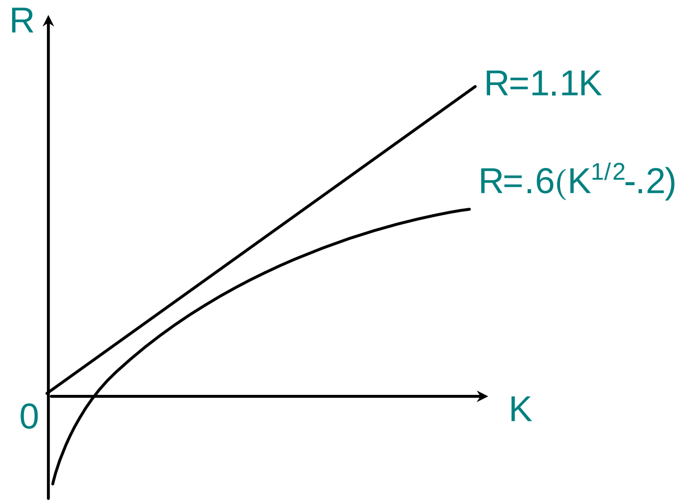

Figure 6: Complete Credit-Rationing