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ECON435/835: Development Economics
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Assignment #1 — Answer Guide
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1. 1. Suppose the economy consists of a large number of households with identical preferences represented by the indirect utility function

$$u(y_h) = Ay_h^{\alpha},$$

where y_h is the income of household h and α and A are constants.
(a) For any general distribution of household income, is the growth in average household utility proportional to the growth in average household income ? Explain why.
No. Since the marginal utility of income varies with household income.

Now suppose also that income is log–normally distributed across households in the economy. This means that the natural log of y, $\ln y$, is distributed normally with mean μ and variance, Var$(\ln y) = \sigma^2$. This kind of distribution is actually a pretty good first approximation to the actual distribution of income in a typical OECD economy — skewed to the left with a long, thin upper tail.[1] **A key property is that the mean value of income is given by**

$$E[y] = e^{\mu + \frac{1}{2}\sigma^2}.$$

You should also know that since $\ln y$ is normally distributed with mean μ and variance σ^2, then any linear transformation of $\ln y$, $a + b\ln y$, is also normally distributed with mean $a + b\mu$ and variance $b^2\sigma^2$.
(b) What is the average utility of households in terms of α, μ and σ.
First note that a household's utility can be expressed as

$$u(y_h) = Ae^{\alpha \ln y_h}.$$

[1] You can find out more about this distribution at http://en.wikipedia.org/wiki/Log-normal_distribution.

Since $\ln y$ is normally distributed with mean μ_y and variance σ_y^2, the variable $\alpha \ln y$ must also be normally distributed with mean $\alpha\mu_y$ and variance $\alpha^2\sigma_y^2$. Consequently, household utility must be log–normally distributed and its average value is

$$U = E\left[u(y_h)\right] = Ae^{\alpha\mu_y + \frac{\alpha^2}{2}\sigma_y^2}.$$

(c) Suppose we (incorrectly) assume that the economy consists of a hypothetical single representative household whose income is equal to the average income in the actual economy. Write down the utility of that hypothetical household in terms of α, μ and σ.

Because income is log–normally distributed, average income is given by

$$Y = E\left[y_h\right] = e^{\mu + \frac{1}{2}\sigma^2}.$$

The utility of this hypothetical representative household is then

$$u\left(Y\right) = A\left(Y\right)^{\alpha} = A\left(e^{\mu+\frac{1}{2}\sigma^2}\right)^{\alpha} = Ae^{\alpha\mu + \frac{\alpha}{2}\sigma^2}.$$

(d) Income inequality changes very slowly over time in Canada. What does this imply for your answer to part (a) under the log-normal distribution case ? What about the comparison of per capita income across countries that have very different degrees of inequality ?

The growth in a variable is the change in the log of that variable. The growth rate of the true average utility is given by

$$\Delta \ln U = \Delta \ln Ae^{\alpha\mu + \frac{\alpha^2}{2}\sigma^2} = \alpha\Delta\mu + \frac{\alpha^2}{2}\Delta\sigma^2,$$

where Δ denotes the change over time. The growth in average income is

$$\Delta \ln Y = \Delta \ln e^{\mu + \frac{1}{2}\sigma^2} = \Delta\mu + \frac{1}{2}\Delta\sigma^2$$

If that inequality (as measured by the variance of log income) changes very slowly over time, so that $\Delta\sigma^2 \simeq 0$, this suggests that that the two variables are proportional to one another in this case. That is

$$\Delta \ln U = \alpha\Delta \ln Y.$$

In such a case per capita income growth may provide a reasonable index for average utility growth.

If we interpret Δ as the difference across countries and $\Delta\sigma^2 \neq 0$, then

$$\begin{aligned}\Delta \ln U - \alpha \Delta \ln Y &= \frac{\alpha^2}{2}\Delta\sigma^2 - \frac{\alpha}{2}\Delta\sigma^2 \\ \Delta \ln U &= \alpha\Delta \ln Y - \frac{\alpha(1-\alpha)}{2}\Delta\sigma^2.\end{aligned}$$

In this case the difference in the log of average utility is not proportional to the difference in log of average income. In particular (assuming $\alpha < 1$), a country with higher inequality would have lower average utility for the same average income.

2. Consider an economy in which the representative household has utility over 2 goods given by

$$u(x_1, x_2) = \beta \ln x_1 + (1-\beta) \ln x_2.$$

with price p_1 and p_2. The household spends all its income, m, on these two goods. The two goods are produced by separate industries which have production functions given by

$$x_1 = \theta_1 k_1^{\gamma} l_1^{1-\gamma}$$

and

$$x_2 = \theta_2 k_2^{\delta} l_2^{1-\delta},$$

where $\theta_1 \neq \theta_2$ are measures of TFP for each industry and $\gamma \neq \delta$.

(a) Derive a valid index of real GDP growth in terms of the growth in the final outputs of the two goods.

With Cobb–Douglas utility and homogeneity of degree one, we know that the share of expenditure on each commodity will equal the exponent in the utility function. Hence, a measure of GDP growth is

$$\hat{y} = \beta\hat{x}_1 + (1-\beta)\hat{x}_2,$$

where $\hat{y} = \dot{y}/y$, etc.

(b) Derive a measure of real GDP growth for this economy in terms of the growth of factor inputs and TFP, assuming perfect competition.

Under perfect competition, the optimal factor shares will equal the exponents on each factor. The growth in the output of good 1 is

$$\hat{x}_1 = \hat{\theta}_1 + \gamma\hat{k}_1 + (1-\gamma)\hat{l}_1,$$

and that of good 2 is

$$\hat{x}_2 = \hat{\theta}_2 + \delta\hat{k}_2 + (1-\delta)\hat{l}_2.$$

It follows that the growth of real GDP is

$$\begin{aligned}
\hat{y} &= \beta\left(\hat{\theta}_1 + \gamma\hat{k}_1 + (1-\gamma)\hat{l}_1\right)\hat{x}_1 + (1-\beta)\left(\hat{\theta}_2 + \delta\hat{k}_2 + (1-\delta)\hat{l}_2\right) \\
&= \left[\beta\hat{\theta}_1 + (1-\beta)\hat{\theta}_2\right] + \left[\beta\gamma\hat{k}_1 + (1-\beta)\delta\hat{k}_2\right] + \left[\beta(1-\gamma)\hat{l}_1 + (1-\beta)(1-\delta)\hat{l}_2\right].
\end{aligned}$$

Now consider an economy in which households have preferences over a single composite consumption good index, C, given by

$$u(C) = \ln C$$

and that good is produced by a single production sector according to

$$Y = AK^{\alpha}L^{1-\alpha},$$

where A, K and L represent composite indices of TFP, real capital and labour services, respectively.

(c) Derive a measure of real GDP growth for this economy in terms of the growth in A, K and L.

As before GDP growth is given by

$$\hat{y} = \hat{A} + \alpha\hat{K} + (1-\alpha)\hat{L}$$

(d) Show that it is possible for the single–sector economy to represent the two–sector economy above. What would the composite indices C, A, K and L and the parameter α have to be?

For GDP growth in the two economies to be the same, we would need the following to be true:

$$\begin{aligned}
\hat{A} &= \beta\hat{\theta}_1 + (1-\beta)\hat{\theta}_2 \\
\alpha\hat{K} &= \beta\gamma\hat{k}_1 + (1-\beta)\delta\hat{k}_2 \\
(1-\alpha)\hat{L} &= \beta(1-\gamma)\hat{l}_1 + (1-\beta)(1-\delta)\hat{l}_2
\end{aligned}$$

In growth accounting, the weights are shares and so must add up to one. Hence,

$$\alpha = \beta\gamma + (1-\beta)\delta$$

Thus, the required composite indices are

$$
\begin{aligned}
A &= \theta_1^{\beta}\theta_2^{1-\beta} \\
K &= k_1^{\frac{\beta\gamma}{\beta\gamma+(1-\beta)\delta}} k_2^{\frac{(1-\beta)\delta}{\beta\gamma+(1-\beta)\delta}} \\
L &= l_1^{\frac{\beta(1-\gamma)}{1-\beta\gamma-(1-\beta)\delta}} l_2^{\frac{(1-\beta)(1-\delta)}{1-\beta\gamma-(1-\beta)\delta}}
\end{aligned}
$$

The index for consumption is

$$C = x_1^{\beta}x_2^{1-\beta}.$$

In other words, provided we compute the aggregate indices for capital, labour and TFP correctly (and the assumptions of constant returns and competition hold), it can make sense to represent the economy using a single aggregate production function.

3. An economy produces final output using capital, K, and labour, L, according to the technology

$$Y = K^{\alpha}(AL)^{1-\alpha},$$

where A denotes the effectiveness of labour. Total output is growing at the rate of 5% per year. The rental rate per unit of capital is equal to 0.1 units of final output. The physical capital–output ratio is 3:1. The stocks of capital and population are growing at the rate of 3 and 2% respectively. Assume that everybody works.

(a) Under the assumption that all output is paid in wages and rent, calculate the implied shares of capital and labour in national income.

The capital share is

$$\alpha = \frac{rK}{Y} = 0.1 \times 3 = 0.3.$$

Consequently, the labour share is 0.7

(b) Using standard growth accounting techniques, estimate the implied rate of growth in the effectiveness of labour in this economy.

The growth of output is given by

$$
\begin{aligned}
\frac{\Delta Y}{Y} &= \alpha\frac{\Delta K}{K} + (1-\alpha)\frac{\Delta A}{A} + (1-\alpha)\frac{\Delta L}{L} \\
\frac{\Delta A}{A} &= \frac{1}{1-\alpha}\frac{\Delta Y}{Y} - \frac{\alpha}{1-\alpha}\frac{\Delta K}{K} - \frac{\Delta L}{L}
\end{aligned}
$$

Since the capital output ratio is constant, the capital stock must be growing at the same rate as output. Assume the workforce is growing at the same rate as the population. Then

$$\frac{\Delta A}{A} = \frac{0.05}{0.7} - \frac{0.3 \times 0.03}{0.7} - 0.02 = 0.039.$$

Suppose that the effectiveness of labour is given by $A = Th$, where T is TFP and h is average human capital (H/L). The effect of an increase in schooling on an individual i's wage within the economy at a given level of technology is estimated to be given by

$$\Delta \ln w_i = 0.1\Delta s_i.$$

(c) Assuming that the economy is approximately competitive, how could this information be used to construct an index of aggregate human capital growth (see Hall and Jones, 1999)?

In a competitive equilibrium, the wage is given by the marginal product of a unit of human capital is:

$$v = (1-\alpha)K^{\alpha}T^{1-\alpha}(hL)^{-\alpha}$$

which depends on aggregate variables only. The wage of an individual with human capital h_i could then be expressed as $w_i = vh_i$. According to Mincerian wage regressions, the log of real wages is approximately linearly related to years of schooling, so we can think of a change in the log of human capital as being linearly related to a change in years of schooling: $\Delta \ln h_i = 0.1\Delta s_i$. It follows that a reasonable first approximation to an index of average human capital might be something like

$$h = e^{0.1E},$$

where E denotes the average years of schooling in the working population. Thus the growth rate of human capital is given by

$$\frac{\Delta h}{h} = 0.1\Delta E$$

(d) If average years of schooling increases by 0.1 years per year, decompose the growth in A into that component arising from TFP growth and that arising from human capital accumulation.

From the above, human capital grows at the rate $0.1 \times 0,1 = 0.01$, or 1% a year. It follows that must TFP grow at the rate

$$\frac{\Delta T}{T} = \frac{\Delta A}{A} - \frac{\Delta h}{h} = 0.039 - 0.01 = 0.029,$$

or 2% per year.