Department of Economics

Queen's University

Econ320: Macroeconomic Theory II

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Assignment 1

(Due: Wednesday, February 16th, 2005)

Section A: Read the following statements and indicate whether they are **True**, **False** or **Uncertain**. Briefly explain your answer. All questions have equal value. NO MARKS WILL BE GIVEN FOR UNSUPPORTED ANSWERS.

A1: The 1983 E*Economic report of the President* contained the following statement: "Devoting a larger share of national output to investment would help restore rapid productivity growth and rising living standards."

False - Suppose the economy begins with an initial steady-state capital stock below the golden rule level. The immediate effect of devoting a larger share of national output to investment is that the economy devotes a smaller share to consumption: that is, "living standards" as measured by consumption fall. The higher investment rate means that the capital stock increases more quickly, so the growth rate of output and output per worker rise. The productivity of workers is the average amount produced by each worker, that is, output per worker. So productivity growth rises. Hence, the immediate effect is that living standards fall but productivity growth rises.

In the new steady state, output grows at rate $n+g$, while output per worker grows at rate g. This means that in the steady state, productivity growth is independent of the rate of investment. Since we begin with an initial steady-state capital stock below the Golden Rule level, the higher investment rate means that the new steady state has a higher level of consumption. So, living standards are higher.

Thus, an increase in the investment rate increases the productivity growth rate in the short run but has no effect in the long run. Living standards, on the other hand, fall immediately and only rise over time. That is, the quotation emphasizes growth, but not the sacrifice to achieve it.

A2: In a pure Keynesian model (with fixed prices and interest rates), an increase in saving will reduce aggregate production.

True - The implication is known as the "paradox of thrift". It is a paradox in that in more classical models an increase in the supply of savings would lower the interest rate and induce greater investment thereby simulating aggregate output. This does not happen in the Keynesian model because output is demand-determined ultimately because of the assumed rigidity of prices and wages in the short run. In the pure Keynesain model, an exogenous (i.e. not driven by a change in income) increase in savings implies a decrease in consumption which reduces demand leading to a build up of inventories. In response firms reduce production until their planned level of investment equals the actual level again. Notice that during this process savings (and consumption fall even further in response to the decrease in Y. The adjustment is illustrated in Figure 1, which illustrates the impact of a shift up in the savings function when investment is sensitive to interest rates and income.

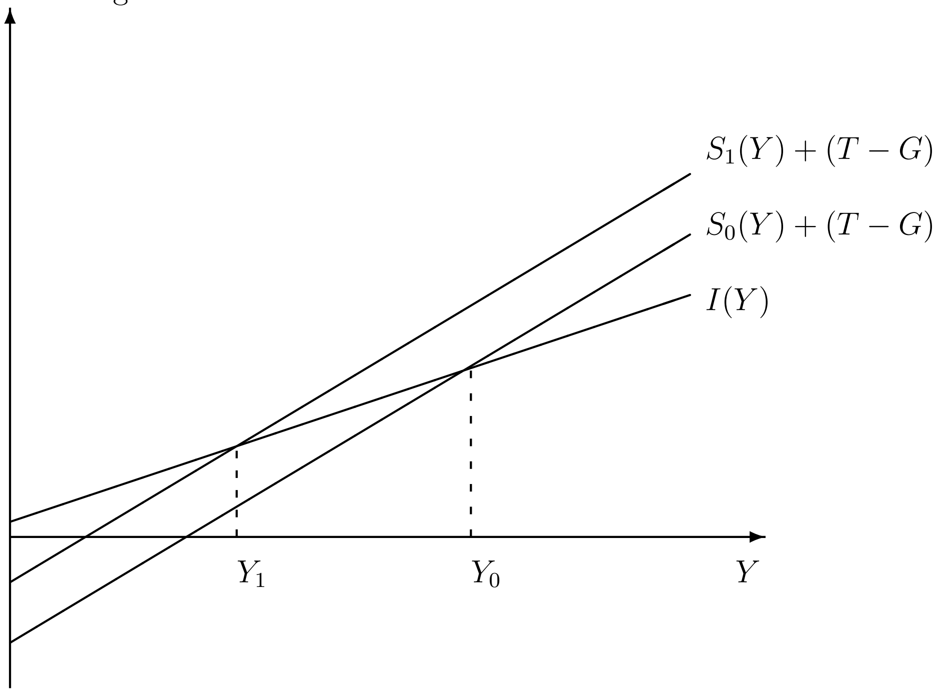


Figure 1: Investment-Savings Equilibrium in the Keynesian model

A3: In the augmented Solow model, with Cobb-Douglas technology:

$$Y_t = K_t^{\alpha} H_t^{\beta} (A_t L_t)^{1-\alpha-\beta}$$

and capital accumulation equations

$$\dot{K} = s_K Y_t \qquad and \qquad \dot{H} = s_H Y_t$$

the economy has a unique balanced growth path with positive growth.

True - let *k=K/AL, h=H/AL and y=Y/AL.* Then the intensive form production function is

$$y_t = k_t^{\alpha} h_t^{\beta}$$

The growth rate of physical capital is:

$$\dot{k}/k = \Delta k/k = \Delta K/K - \Delta A/A - \Delta L/L$$

So, the dynamics physical capital in intensive form is given by:

$$\dot{k}_t = \Delta k = \frac{\Delta K}{K} k - (n+g)k,$$

which implies,

$$\dot{k}_t = \frac{1}{AL} s_K Y_t - (n+g)k = s_K k_t^{\alpha} h_t^{\beta} - (n+g)k_t$$

Similarly, the dynamics human capital in intensive form is given by:

$$\dot{h}_t = s_H k_t^{\alpha} h_t^{\beta} - (n+g)h_t$$

Along a BGP $\dot{k}_t = \dot{h}_t = 0$, and so h and k must satisfy

$$s_K k_t^{\alpha} h_t^{\beta} = (n+g)k_t$$
$$s_H k_t^{\alpha} h_t^{\beta} = (n+g)h_t$$

We can express these two equations as

$$k_t = \left(\frac{s_K}{n+g}\right)^{\frac{1}{1-\alpha}} h_t^{\frac{\beta}{1-\alpha}}$$
$$k_t = \left(\frac{n+g}{s_H}\right)^{\frac{1}{\alpha}} h_t^{\frac{1-\beta}{\alpha}}$$

If we draw these two relationship in (k,h) space, then , since $1-\alpha-\beta<0$, the first expression is convex and the second is concave. Consequently, they have a unique intersection, given by

$$k=\left(\frac{s_K^{1-\beta}s_H^{\beta}}{n=g}\right)^{\frac{1}{1-\alpha-\beta}} \qquad and \qquad h=\left(\frac{s_K^{\alpha}s_H^{1-\alpha}}{n=g}\right)^{\frac{1}{1-\alpha-\beta}}$$

Section B: Answer any 2 of the following 3 Long Questions.

B1:Consider the basic Solow growth model with production technology

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$$

The savings rate is s, the rate of depreciation is δ, the population growth rate is n and the rate of technical change is g. Assume output and factor markets are competitive. Suppose the economy starts on a Balance Growth Path and then the savings rate increases permanently.

(a) What is the impact on the long-run values of the capital stock and output per unit of effective labor, and the long run level and growth rate of per capita output ?

The intensive form production function is

$$f(k_t) = k_t^{\alpha}$$

Figure 2 illustrate the impact of the increase in savings on the long run value of $k*$. As can be seen it increases. As a result, long-run output per unit of effective labour, $y*$, must also increase. Long run output per capita is given by

$$\frac{Y_t}{L_t} = A_t y*$$

Since the time path of A_t is unchanged, output per capita is higher at each point in time along the BGP than before. The long-run growth rate of output per capita is unchanged.

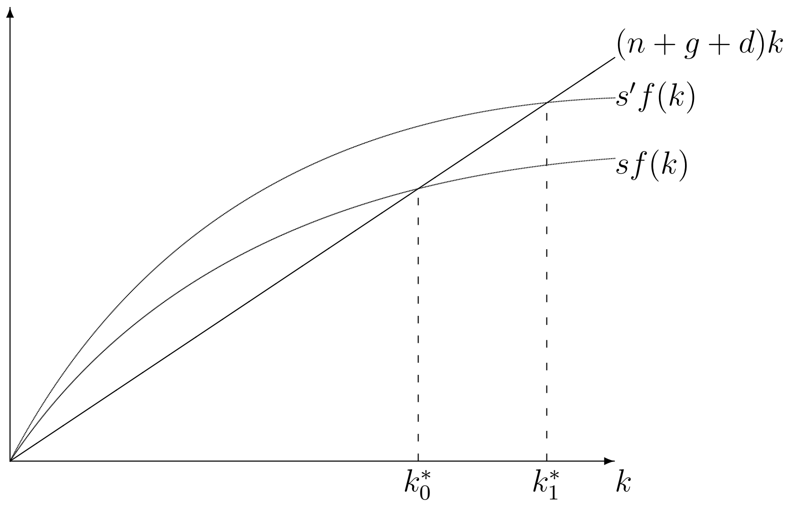


Figure 2

(b) Illustrate how these variable evolve towards their long-run values following the increase in *s*.

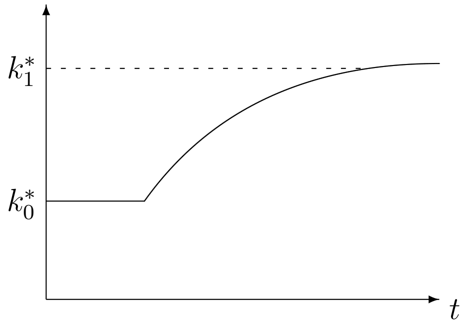
The capital stock per unit of effective labour accelerates initially and then converges towards its new long run level (see Figure 3). Output per unit of effective labour follows a similar path. Output per capita in logs is given by

$$ln\frac{Y_t}{L_t} = lnA_t + lny_t$$

Differentiating with respect to time yields the growth rate of per capita output

$$\frac{\Delta(Y_t/L_t)}{Y_t/L_t} = \frac{\Delta A_t}{A_t} + \frac{\Delta y_t}{y_t} = g + \frac{\Delta y_t}{y_t}$$

It follows that the time path for the growth rate and level of per capita income look like those given in Figure 3.



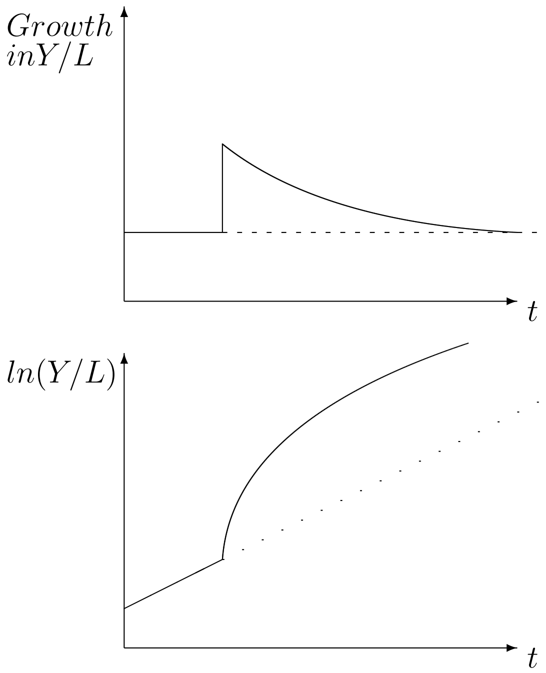


Figure 3

(c) How does consumption per unit of effective labour, c, evolve after the in crease in s. Does the long run value of c rise or fall relative to its initial value? Explain.

Consumption per unit of effective labour is given by

$$c = (1 - s)k^{\alpha}$$

Initially, k is fixed, so that c falls when s increases. Then c rises with k. To determine whether $c*$ is higher or lower in the long run we can write

$$c^* = (k^*)^{\alpha} - (n + g + \delta)k^*$$

Differentiating with respect to s yields

$$\frac{dc^*}{ds} = [\alpha(k^*)^{\alpha-1} - (n + g + \delta)]\frac{dk^*}{ds}$$

Thus, the impact is ambiguous depending on whether the economy is below or above the golden rule level of k.

(d) What savings rate would ensure that along the BGP the economy is dynamically efficient?

The golden rule value of k occurs where long-run c is maximized. That is where

$$f'(\hat{k}) = \alpha \hat{k}^{\alpha-1} = n + g + \delta$$
$$\hat{k} = \left(\frac{\alpha}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$$

The golden rule level will be achieved as the BGP if the savings rate satisfies

$$s\hat{k}^{\alpha} = (n+g+\delta)\hat{k}$$
$$s = (n+g+\delta)\hat{k}^{1-\alpha}$$

Substituting for k yields

$$s = \alpha$$

If follows that any savings rate less than α is dynamically efficient.

(e)The Solow model implicitly assumes that technical change in consumption and investment goods producing sectors is equal in the long run. What consequences would relaxing this assumption have for the aggregate resource constraint and the user-cost of capital?

The aggregate resource constraint would have to be expressed as

$$C_t + p_{It} I_t = Yt$$

where p_{It}denotes the relative price of investment to consumption goods. Consequently, if the savings rate were constraint, investment would be given by

$$I_t = K_{t+1} - (1-\delta)K_t = \frac{sY_t}{p_{It}}$$

The user-cost of capital would have to be expressed:

$$q_t = \left(r_t + \delta - \frac{p_{It} - p_{It-1}}{p_{It}}\right) p_{It}$$

B2: In the Cobb-Douglas version of the Solow model, suppose that output depends upon natural resources as well as capital and labor, so that:

$$Y_t = K_t^{\alpha} T_t^{\beta} (A_t L_t)^{1-\alpha-\beta}$$

where T_t represents the level of natural resources used in period *t*. Suppose that the "user cost" of a unit of natural resources is v_t, the user cost of capital is r_t and the wage per worker is w_t.

(a)Derive the intensive form production function. Assuming competitive factor markets, derive expressions for factor prices as a function of capital, k_t, and natural resources, τ_t, per effictive worker.

The intensive form of the production function is

$$y_t = k_t^{\alpha} \tau_t^{\beta}$$

and the factor pricing equations are

$$\begin{aligned} \alpha k_t^{\alpha-1} \tau_t^{\beta} &= r_t \\ \beta k_t^{\alpha} \tau_t^{\beta-1} &= v_t \\ (1-\alpha-\beta) k_t^{\alpha} \tau_t^{\beta} &= w_t \end{aligned}$$

(b)If Z_t represents the supply of resources, write down the equilibrium condition which determines v_t.

IN equilibrium, demand = supply. If $z_t = Z_t/A_t L_t$,then in equilibrium, $\tau_t = z_t$ and the steady state investment condition is:

$$\begin{aligned} s k_t^{\alpha} z_t^{\beta} &= (n+g+\delta) k_t \\ k(z_t) &= \left(\frac{s z_t^{\beta}}{n+g+\delta} \right)^{\frac{1}{1-\alpha}} \end{aligned}$$

Substituting into the first-order condition yields

$$\begin{aligned} v_t &= \beta \left(\frac{s z_t^{\beta}}{n+g+\delta} \right)^{\frac{\alpha}{1-\alpha}} z_t^{\beta-1} \\ v_t &= \beta \left(\frac{s}{n+g+\delta} \right)^{\frac{\alpha}{1-\alpha}} z_t^{-\left(\frac{1-\alpha-\beta}{1-\alpha}\right)} \end{aligned}$$

(c)Under what conditions will the intensive form production function have the same general form as that without natural resources (i.e. where output per effective worker can be expressed as a constant function of capital per effective worker).

The intensive form production function can be expressed as a constant function of capital work only if $z_t$is constant at some z, so that

$$y_t = k_t^\alpha z^\beta$$

This will only be true if Z_t grows at the same rate as$A_t L_t$, i.e. at the rate $n+g$.

(d)If the growth in natural resources is in fact given by $\dot{Z}_t/Z_t = -\eta < 0$, characterize the long-run growth path of the economy.

From part (b) we have

$$k_t = \left(\frac{s z_t^\beta}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$$

Taking logarithms and differentiating with respect to time we get

$$\frac{\dot{k}}{k} = \frac{\beta}{1-\alpha}\frac{\dot{z}}{z}$$

From the intensive form production function, growth in output per effective worker is

$$\frac{\dot{y}}{y} = \alpha\frac{\dot{k}}{k} + \beta\frac{\dot{z}}{z} = \left(\frac{\alpha}{1-\alpha}+1\right)\beta\frac{\dot{z}}{z}$$

So,

$$\frac{\dot{y}}{y} = \frac{\beta}{1-\alpha}\left(\frac{\dot{Z}}{Z} - n - g\right) = -\frac{\beta}{1-\alpha}(\eta+n+g)$$

It follows that productivity growth (or per capita growth) can be expressed as

$$\frac{\Delta(Y/L)}{Y/L} = g - \frac{\beta}{1-\alpha}(\eta+n+g) = \left(\frac{1-\alpha-\beta}{1-\alpha}\right)g - \frac{\beta}{1-\alpha}(\eta+n)$$

As long as β is not too large and the rate of technical change is large enough, per capita output growth may still be positive, but the decline in available resources represents a drag on growth.

B3: Consider a simple economy inhabited by N identical households. Each household has a one period planning horizon, is endowed with one unit of time which it can allocate between work at wage w_t or leisure, and is endowed at time t with a stock of assets k_t which earns a return q_t. Assume the household's preferences over consumption, c_t, savings, s_t, and fraction of hours worked, l_t, are given by:

$$u(c_t, s_t, h_t) = lnc_t + \beta lns_t + \gamma ln(1 - l_t).$$

The household faces a time t budget constraint given by:

$$c_t + s_t = (1 - \tau)t)w_t l_t + q_t k_t + T_t = y_t.$$

where y_t is total household income, $\tau$is a proportional wage tax and T_t is a lump sum transfer from the government. Assume that the government balances its budget each period, so that

$$\tau w_t l_t = T_t$$

but that the household takes the transfer T_t as given when making its decisions.

(a) Given the one-period planning horizon, show that consumption and savings are proportional to total income, y_t.

We can express the optimization problem in the form of a Lagrangian:

$$max \qquad L = lnc_t + \beta lns_t + \gamma ln(1 - l_t) + \lambda[(1 - \tau_t)w_t l_t + q_t k_t + T_t - c_t s_t]$$

The FOCs are

$$\frac{1}{c_t} = \lambda$$

$$\frac{\beta}{s_t} = \lambda$$

$$\frac{\gamma}{1 - l_t} = \lambda(1 - \tau_t)w_t$$

$$c_t + s_t = (1 - \tau_t)w_t l_t + q_t k_t + T_t = y_t$$

It follows from the first two conditions that

$$s_t = \beta c_t$$

Substituting into the budget constraint we get

$$c_t = \frac{y_t}{1+\beta}$$
$$s_t = \frac{\beta y_t}{1+\beta}$$

(b) Show that optimal household labor supply can be expressed as:

$$l_t = 1 - \frac{a y_t}{(1-\tau) w_t}$$

where a is constant term. What is the impact of an increase in τ on household labour supply?

Combining the first and third FOC we get

$$\frac{\gamma}{1-l_t} = \frac{1}{c_t}(1-\tau_t) w_t$$

Re-arranging yields

$$l_t = 1 - \frac{\gamma c_t}{(1-\tau) w_t} = 1 - \frac{(\frac{\gamma}{1+\beta}) y_t}{(1-\tau) w_t}$$

An increase in τ decreases household labour supply for a given wage and income.

Now suppose that we append this household model to a Solow growth model (with no technical change and no population growth). There are N identical households and the aggregate production function is given by:

$$Y_t = K_t^{\alpha} L_t^{1-\alpha},$$

where L_t denotes aggregate hours worked and the aggregate capital stock evolves according to

$$K_{t+1} = (1-\delta) K_t + I_t$$

where I_t denotes aggregate investment. Factor markets are competitive and clear every period.

(c) Derive expressions for the wage, w_t, and the user cost of capital, q_t, as functions of the capital stock per hour worked. Show that the economy's aggregate resource constraint (in per household terms) is equivalent to the household budget constraint.

Under competitive markets we know that factor prices equal their marginal products:

$$w_t = (1-\alpha)K_t^{\alpha}L_t^{-\alpha} = (1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha}$$

$$q_t = \alpha K_t^{\alpha-1}L_t^{1-\alpha} = \alpha\left(\frac{K_t}{L_t}\right)^{\alpha-1}$$

The aggregate resource constraint is

$$C_t + I_t = Y_t$$

The household budget constraint is

$$c_t + s_t = (1-\tau_t)w_t l_t + q_t k_t + T_t$$

Substituting in the government's budget constraint we get

$$c_t + s_t = w_t l_t + q_t k_t$$

Substituting in factor prices we get

$$c_t + s_t = (1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha} l_t + \alpha\left(\frac{K_t}{L_t}\right)^{\alpha-1} k_t$$

Multiplying by the number of households N and noting that in equilibrium $s_t N = I_t, L_t = N l_t, and K_t = N k_t$ yields

$$C_t + I_t = (1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha} L_t + \alpha\left(\frac{K_t}{L_t}\right)^{\alpha-1} K_t$$

$$and \qquad C_t + I_t = K_t^{\alpha}L_t^{1-\alpha} = Y_t$$

(d) Show that household labor supply is constant in general equilibrium. Why do you think the equilibrium labor supply is unrelated to the wage? (Hint: think about the income and substitution effects on labor supply for this example). In equilibrium, household labour supply is given by

$$l_t^* = 1 - \frac{\left(\frac{1}{1+\beta}\right) K_t^{\alpha}L_t^{1-\alpha}/N}{(1-\tau)(1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha}}$$

$$l_t^* = 1 - \frac{\left(\frac{1}{1+\beta}\right) l_t^*}{(1-\tau)(1-\alpha)}$$

Solving for l_t^* we get

$$\left(1 + \frac{1}{(1-\tau)(1-\alpha)(1+\beta)}\right) l_t^* = 1$$

$$l_t^* = l^*(\tau) = \frac{(1-\tau)(1-\alpha)(1+\beta)}{1 + (1-\tau)(1-\alpha)(1+\beta)}$$

The reason why labour supply is independent of wage (and hence the capital stock per worker) is because with logarithmic utility labour supply depends on ratio of income and substitution effects (y/w). Since in equilibrium, these variables move in proportion to one another ($w_t = (1-\alpha)y_t$)the income and substitution effects on labour supply exactly offset each other. Notice that l^* is a decreasing function of the tax rate.

(e) What is the capital stock per household and income per household along the balanced growth path? What is the impact of an increase in the tax rate τ on income per household along the balanced growth path? Using a diagram, characterize the transitional dynamics associated with this increase.

The aggregate labour force in equilibrium is given by $L = Nl^*(\tau)$. For a given tax rate, the steady state equation for the capital stock per worker solves

$$\frac{\beta}{1+\beta}\left(\frac{\hat{K}}{L}\right)^{\alpha} = \delta\left(\frac{\hat{K}}{L}\right)$$

$$\hat{K} = \left(\frac{\beta}{(1+\beta)\delta}\right)^{\frac{1}{1-\alpha}} L$$

Aggregate income is

$$\hat{Y} = \hat{K}^{\alpha} L^{1-\alpha} = \left(\frac{\beta}{(1+\beta)\delta}\right)^{\frac{\alpha}{1-\alpha}} L$$

Income per households is then

$$\hat{y} = \left(\frac{\beta}{(1+\beta)\delta}\right)^{\frac{\alpha}{1-\alpha}} l^*(\tau)$$

Since labour supply is decreasing in τ, so also is $\hat{y}$. For the diagram, see figure 4.

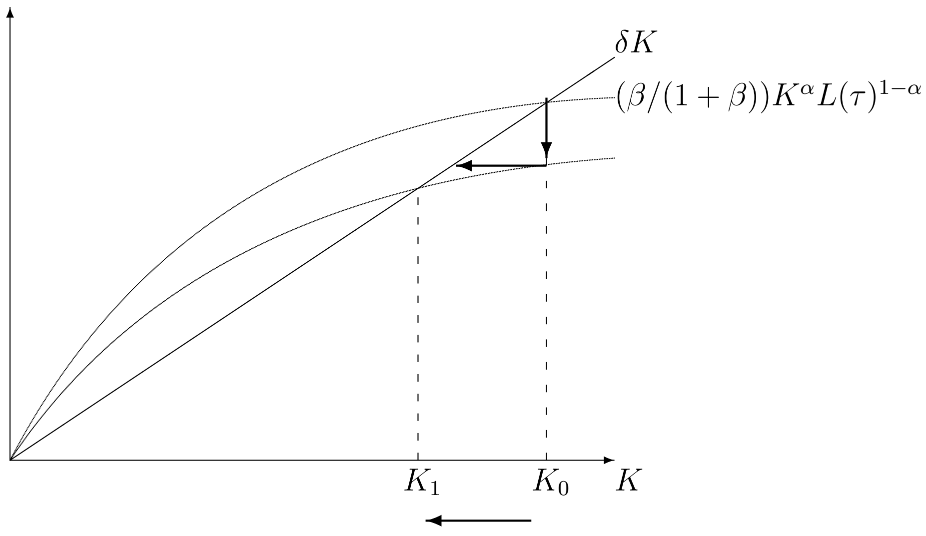


Figure 4