Economics 817 **Winter 2004**

Advanced Macroeconomic Theory II **Hiroyuki Kasahara**

Notes on Monotone Optimal Policies

1 Increasing Differences and Parametric Monotonicity

Consider a firm's profit maximization problem:

$$\max_k \pi(k, z) = zk^\alpha - rk,$$

where, $\alpha \in (0, 1)$, z is a technology level, k is capital stock. Here z is a parameter.

By solving this problem, we get:

$$k^*(z) = argmax_k \pi(k, z).$$

The issue of parametric monotonicity is if $k^*(z)$ is monotonic function of z or not.

The first order condition implies that

$$\pi_k(k^*(z), z) = 0.$$

In order to see the property of $k^*(z)$, we take a derivative of this first order condition with respect to z:

$$\begin{aligned} & \pi_{kz}(k^*(z), z) + \pi_{kk}(k^*(z), z)k^{*'}(z) = 0 \\ \Rightarrow \quad & k^{*'}(z) = -\pi_{kz}(k^*(z), z)/\pi_{kk}(k^*(z), z) > 0, \end{aligned}$$

if $\pi_{kz} > 0$ (strictly increasing differences) and $\pi_{kk} < 0$ (strict concavity).

Note that this is a general result. For any maximization problem with twice differentiable strictly concave objective functions, $\max_x h(x, \theta) : \mathbf{S} \times \Theta \to \mathbf{R}$, we can prove that the maximizer (which is unique by strict concavity) is monotonically increasing if $h_{xx} < 0$ and $h_{x\theta} > 0$. Here, we have $h_{xx} < 0$ in many economic applications since we often use strictly concave objective functions. Then, a key condition for parameteric monotonicity is $h_{x\theta} > 0$, which implies that x and θ are complementary to each other.

Sometimes, we do not have twice differentiability for objective functions. In particular, this is often the case in dynamic programming. How can we prove then the monotonicity of the maximizer? The key condition is "increasing differences" which is what we get if we generalize the condition $h_{x\theta} > 0$ in the case that objective functions are not differentiable.

A function $h : \mathbf{S} \times \Theta \to \mathbf{R}$ is said to satisfy *increasing differences* in (x, θ) if for all pairs (x, θ) in $\mathbf{S} \times \Theta$, it is the case that $x \geq x'$ and $\theta \geq \theta'$ implies:

$$h(x, \theta) - h(x', \theta) \geq h(x, \theta') - h(x', \theta').$$

Note that, if $h(x, \theta)$ is twice differentiable, then taking a limit $x' \to x$ and $\theta' \to \theta$ leads to $h_{x,\theta}(x, \theta) \geq 0$. If the inequality is strict whenever $x > x'$ and $\theta > \theta'$, the h is said to satisfy *strictly increasing differences* in (x, θ). With this definition, we get

Theorem 10.6 of Sundaram (1996)
Suppose that the optimization problem

$$\textit{Maximize } h(x, \theta) \textit{ subject to } x \in \mathbf{S}$$

has at least one solution for each $\theta \in \Theta$. Suppose also that h satisfies strictly increasing differences in (x, θ). Finally, suppose that $\mathbf{S} \subset \mathbf{R}$. Then optimal actions are nondecreasing in the parameter θ.

Proof Let $\theta_H > \theta_L$. Let $x_i \in argmax_x h(x, \theta_i)$ for $i = H, L$.

$$h(x_H, \theta_H) - h(x_L, \theta_H) \geq 0 \geq h(x_H, \theta_L) - h(x_L, \theta_L), \tag{1}$$

where the first inequality holds since x_H maximizes $h(x, \theta_H)$ and the second inequality holds since x_L maximizes $h(x, \theta_L)$.

It suffices to show that $x_H \geq x_L$. Suppose not, i.e., $x_H < x_L$. Then, since $h(x, \theta)$ satisfies strictly increasing differences in (x, θ),

$$h(x_L, \theta_H) - h(x_H, \theta_H) > h(x_L, \theta_L) - h(x_H, \theta_L).$$

This contradicts (1). Therefore, $x_H \geq x_L$. Q.E.D.

Remark: there are no concavity assumptions required.

2 Monotone Optimal Policy in Dynamic Programming under Cetainty

We may apply the above result for policy functions in dynamic programming problems. We consider the case of one state variable so that $\mathbf{X} = \mathbf{R}$.

Consider the Bellman equation:

$$V(x) = \max_{y \in \Gamma(x)} [F(x, y) + \beta V(y)] \tag{2}$$

Under Assumption 4.3-4.9 in Stokey and Lucas (1989), we can show that the fixed point of this functional equation is a function that is bounded, strictly concave, and continuously differentiable in the interior of support X. Let

$$W(x, y) \equiv F(x, y) + \beta V(y),$$

where $V(\cdot)$ is the fixed point of the Bellman equation (2). Suppose the following assumption holds.

Assumption SID (Strictly Increasing Differences) $F(x, y)$ *satisfies strictly increasing differences in* (x, y). *Under twice differentiability,* $\frac{\partial^2 F(x,y)}{\partial x \partial y} > 0$.

Then, it is easy to show that $W(x, y)$ satisfies strictly increasing differences under Assumption SID. Then, by Theorem 10.6 of Sundaram (1996), we know that the policy function $g(x)$, which is defined by

$$g(x) = argmax_x W(x, y), \tag{3}$$

is *nondecreasing* in x. Furthermore, we can prove that $g(x)$ is *strictly increasing* as the following Theorem states:

Theorem: Monotonicity of Policy Functions under Certainty

Suppose that Assumption 4.3-4.9 in Stokey and Lucas (1989) and Assumption SID hold. Suppose $X = \mathbf{R}$. *Then, the policy function defined by (3) is strictly increasing in* x.

Proof First, $W(x, y)$ satisfies strictly increasing differences [Prove yourself]. Then, by Theorem 10.6 of Sundaram (1996), $g(x)$ is nondecreasing in x.

It suffices to prove that, if $x^H > x^L$, then $g(x^H) \neq g(x^L)$. Suppose not, i.e., $g(x^H) = g(x^L) \equiv g^*$. Then, by differentiability of the value function (Theorem 4.11 of Stokey and Lucus), $g(x)$ satisfies the first order conditions:

$$\begin{aligned} W_2(x^H, g(x^H)) &= F_2(x^H, g^*) + \beta V'(g^*) = 0, \\ W_2(x^L, g(x^L)) &= F_2(x^L, g^*) + \beta V'(g^*) = 0. \end{aligned}$$

But since $F_2(x^H, g^*) > F_2(x^L, g^*)$ (Assumption SID), this is a contradiction. That is, if the first equation holds, then $0 = F(x^H, g^*) + \beta V(g^*) > F(x^L, g^*) + \beta V(g^*) = W(x^L, g(x^L))$ and hence the second equation does not hold. Therefore, $g(x^H) \neq g(x^L)$.

Thus, $g(x^H) > g(x^L)$ if $x^H > x^L$. Q.E.D.

General Reference for Monotone Optimal Policies: Serfozo, Richard F., "Monotone Optimal Policies for Markov Decision Processes," *Mathematical Programming Study* 6:202-215.