Economics 817 **Winter 2004**

Advanced Macroeconomic Theory II **Hiroyuki Kasahara**

Homework 3

(Due: February 9 by 5 pm)

Note: Study groups *discussing* the problems are strongly encouraged. But write your own answers.

1 Markov Chain

Let the state space be $S = \{s_1, s_2, s_3, s_4\}$. Let $\mathcal{S}$ be a family of all subsets of S. Denote a probability measure on $(S, \mathcal{S})$ by $p = (p_1, p_2, p_3, p_4)$. $\Delta^4 = \{p \in \mathbf{R}^4 : 0 \leq p_i \leq 1 \text{ for all i}, \sum_{i=1}^4 p_i = 1\}$ is the corresponding space of probability measures. Suppose that the transition function of S is characterized by the following Markov matrix:

$$\Pi = \begin{pmatrix} 1/3 & 0 & 0 & 2/3 \\ \alpha_1\gamma & 1-\gamma & \alpha_2\gamma & \alpha_3\gamma \\ 0 & 0 & 1 & 0 \\ 2/3 & 0 & 0 & 1/3 \end{pmatrix},$$

with $0 < \gamma, \alpha_1, \alpha_2, \alpha_3 < 1$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

1. How many ergodic sets are there? Identify all of them.
2. Is there a transient set? If so, identify the transient set?
3. Are there any ergodic sets that have cyclically moving subsets? If so, what are they?
4. What are the invariant distributions? Identify all of them.

2 Dynamic Programming under Uncertainty: Optimal Growth with Productivity Shocks

2.1 Theory

Consider the one-sector optimal growth problem with productivity shock z:

$$V(k,z) = \max_{0 \le k' \le f(k)} U[e^z f(k) - k'] + \beta \int V(k', z')\Phi(dz'), \tag{1}$$

where $\Phi(\cdot)$ is the distribution function of z; $\beta \in (0,1)$; $U : \mathbf{R}_+ \to \mathbf{R}$ is utility function that is continuous, strictly increasing, strictly concave, and twice differentiable; $f : \mathbf{R}_+ \to \mathbf{R}_+$ is a production function that is continuous, strictly increasing, strictly concave, and twice differentiable with $f(0) = 0$, $\lim_{k\to 0} f'(k) > 1$, and $\lim_{k\to\infty} f'(k) < 1$.

1. Prove that the fixed point V is strictly increasing in (z,k) and strictly concave in k.
2. Consider the policy correspondence $G(k,z) = \{k' \in [0, e^z f(k)] : V(k,z) = U[e^z f(k) - x] + \beta \int V(x,z')\Phi(dz')\}$. Prove that $G(z,k)$ is single-valued and strictly increasing in (z,k).

2.2 Numerical Dynamic Programming and Exercises on Markov Chain

It is very important to add your comments generously so that I can understand your codes. Unless it's really obvious, add comments on every smaller segments of code, or possibly every line, in this homework. Also, output of your program must be readable for me.

Choose:

$$U(c) = \ln c \tag{2}$$

$$f(k) = Ak^\alpha \tag{3}$$

so that we have a closed form solution to the policy function: $k_{t+1} = e^z \alpha\beta A k_t^\alpha$.

On the other hand, assume that z is distributed according to normal distribution with mean zero and variance σ^2.

Approximate the continuous state spaces of k and z by a finite set $K = \{k_1, ..., k_n\}$ and $Z = \{z_1, ..., z_m\}$. Given the finite set of Z, let q_j represents $Prob(z = z_j)$ $j = 1, 2, ..., m$. Then, the discretized Bellman equation is given by

$$V(i,j) = \max_{i'=1,2,\ldots,n} U[e^{z_j} f(k_i) - k_{i'}] + \beta \sum_{j'=1}^{m} q_{j'} V(i', j'), \tag{4}$$

where $V(i,j)$ represents the (i,j) element of a "n by m" matrix. Namely, the continuous function $V(k,z)$ that maps $[0,\bar{k}] \times \mathbf{R}$ into $\mathbf{R}$ is approximated by a function which maps $\{1,2,...,n\} \times \{1,2,...,m\}$ into $\mathbf{R}$. This approximated function space equipped with a metric $d_\infty(V,W) \equiv \max_{i,j} |V(i,j) - W(i,j)|$ is a complete metric space since it's just the nm-dimentional Euclidean space. The right hand side of (4) defines an operator that maps nm-dimentional Euclidean space into itself.[1]

We are going to solve this discretized Bellman equation using "value iteration" method using computers.

Set the parameter values to: $\beta = 0.96$, $\alpha = 0.25$. Also, $A = (\alpha\beta)^{-1}$.

Given the lognormal distribution of shocks, we discretize $\ln k$ instead of k. Our discretization for $\ln k$ will be $lnk_i = -5 + (i-1)ln\kappa$. We choose $\ln\kappa = 0.25$ with $n = 41$ so that $lnk_1 = -5$ and $lnk_n = 5$, or $k_1 = e^{-5}$ and $k_n = e^5$.

For z, which are normally distributed around zero with variance σ^2, we use a uniform grid consisting of equi-spaced points between -3σ and 3σ. By choosing $\Delta z = 6\sigma/(m-1)$, our discritization for z will be $z_j = -3\sigma + (j-1) * \Delta z$ for $j = 1,2,...,m$. We set $m = 41$. Accordingly, the normal distribution is approximated by multinomial distribution as follows:

$$q_j = \frac{\phi(z_j/\sigma)/\sigma}{\sum_{j'=1}^{m} \phi(z_{j'}/\sigma)/\sigma},$$

where ϕ is standard normal density function. The normalization insures that q_j is a well defined probability density. Let $\mathbf{q}$ be a m by 1 vector with the jth element equal to q_j. We set $\sigma = 0.5$.

1. Write a computer program that numerically solve the fixed point of the discretized Bellman equation (4) using "Value Function Iteration Algorithm":

   (a) Set the initial guess $V^0 = \mathbf{0}_{n,m}$, where $\mathbf{0}_{n,m}$ is a "n by m" matrix with each element is a scalar 0.[2]

   (b) After l value function iterations, we have a "n by m" matrix representing a value function, V^l. Given V^l, compute

$$V^{l+1}(i,j) = \max_{i'=1,2,\dots,n} U[e^{z_j} f(k_i) - k_{i'}] + \beta \sum_{j'=1}^{m} q_{j'} V^l(i',j')$$

   , for every $(i,j) \in \{1,2,...,n\} \times \{1,2,...,m\}$.

[1] Can you prove that the fixed point of this operator satisfies (i) strictly increasing and (ii) strictly concave? Can you prove the monotonicity of policy function?

[2] In Matlab, $v_0 = zeros(n,m)$ will create a "n by m" zero matrix.

One way to do this is to use "do-loop" twice. For $(i,j) = (1,1)$, compute $U[e^{z_1} f(k_1) - k_{i'}] + \beta \sum_{j'=1}^{m} q_{j'} V^l(i', j')$ for $i' = 1, 2, ..., n$ and store their values in a "n by 1" vector.[3] Then, find a max among those values. If the i^*th element is the max, then $V^{l+1}(1,1) = U[e^{z_1} f(k_1) - k_i^*] + \beta \sum_{j'=1}^{m} q_{j'} V^l(i^*, j')$. This is an "inner" loop. Repeat this for every pair of $(i,j) \in \{1,2,3,...,n\} \times \{1,2,3,...,m\}$ ("outer loop') to get $\{V^{l+1}(i,j)\}$ and store them in a "n by m" matrix; this is a function $V^{l+1} = T(V^l)$, where T is an operator defined by the right hand side of (4).

In practice, avoid loops as much as possible in Matlab, Gauss, and Ox since using them slows your program; instead, use vector or matrix operations. For this, it is convenient to "vectorize" the state space.[4] In this context, first create an "nm by n" utility matrix, say $\mathbf{U}$, where an $[(i + (j-1)m), i']$ element equal to $U[e^{z_j} f(k_i) - k_{i'}]$. The $(i + (j-1)m)$th row represents the current state (k_i, z_j) while the i'th column represents a choice of the next period's capital stock, $k_{i'}$. Second, compute an n by 1 vector $EV^l = V^l \mathbf{q}$, which represents the expected value function; the i'th element represents the expected value of the next period's value function when $k = k_{i'}$ is chosen. Third, given the utility matrix and the expected value function matrix, compute $W^{l+1} = \mathbf{U} + \beta[\mathbf{1}_{nm} \bigotimes EV^{l'}]$, where $\mathbf{1}_{nm}$ is nm by 1 vector with each element equal to 1 and $\bigotimes$ is the Kronecker product.[5] Fourth, for each *row*, take a max across columns to compute an "nm by 1" vector V^{l+1}. You can do this using only one command, max.[6] Finally, to get an n by m matrix, use a command $reshape(V, n, m)$.

(c) Repeat the recursive computation of Step (b) until $\max_{(i,j) \in \{1,2,...,n\} \times \{1,2...,m\}} |V^l(i,j) - V^{l+1}(i,j)| < \frac{\epsilon}{1-\beta}$, where we set $\epsilon = 0.001$. Let V^* be the computed fixed point.

(d) Compute a policy function which is an "nm by n" vector, P, of which $[(i + (j-1)m), i^*]$ element is 1 if $i^* = argmax_{i' \in \{1,2,...,n\}} U[e^{z_j} f(k_i) - k_{i'}] + \beta \sum_{j'} q_{j'} V^*(i', j')$, 0 otherwise.[7]

2. The policy function together with the distribution of z determines a transition function of the state space of (k_i, z_j). Specifically, an nm by nm matrix $\Pi \equiv \mathbf{q}' \bigotimes P$ defines a transition function.

[3]Some combination of (z, k, k') leads to "negative" consumptions. Since $U(c) = ln(c)$, you cannot evaluate the utility if c is negative. Note that strictly negative consumption is infeasible and also that a representative agent never chooses zero consumption since zero consumption leads her utility to minus infinity. One way to deal with this problem in numerical programming is to take $c = max(zf(k) - k', 0)$ so that you can evaluate $ln(c)$ for every choice of (z, k, k') and yet you never choose the tomorrow's capital stock k' that leads to negative or zero consumption.

[4]Ch. 14 of Ljungqvist and Sargent might be helpful.

[5]In Matlab, you can use the command $W = U + beta * kron(ones(n*m, 1), V_0)$.

[6]In Matlab, you can use the command $V_1 = \max(W, [\,], 2)$.

[7]Hint: in Matlab, use "max" command. Type "help max."

(a) Consider a vector p in the mn-dimensional unit simplex: $\Delta^{nm} = \{p \in \mathbf{R}^{nm} : p \geq 0, \sum_{i=1}^{nm} p_i = 1\}$, where p is 1 by nm vector. Let $||\cdot||_\Delta$ denote the norm on $\mathbf{R}^{nm}$ defined by $||p||_\Delta \equiv \sum_{i=1,2,\ldots,nm} |p_i|$. Argue that the metric space $(\mathbf{R}^{nm}, ||\cdot||_\Delta)$ is complete. [Use the fact that $\mathbf{R}$ is complete.]

(b) Show that the operator T_N^* defined by $T_N^* p = p\Pi^N$ maps Δ^{nm} into itself if you choose a large but finite N. Show that T_N^* is a contraction mapping with modulus $(1-\epsilon^N)$, where $\epsilon^N = \sum_j min_i \pi_{ij}^{(N)}$ and $\pi_{ij}^{(N)}$ is the (i,j) element of Π^N. [Hint. First, numerically check that $\epsilon^N > 0$ for a large but finite N. Then, use the argument found in Lemma 11.3.]

(c) Show that the state space of (k, z) has a unique ergodic set with no cyclically moving subsets. [Hint. Refer to Theorem 11.2 and Theorem 11.4.]

(d) Compute an invariant distribution of (k, z). [Hint. Compute Π^{kN} by choosing a large k so that $||\Pi^{kN} - \Pi^{(k+1)N}|| < 0.0001/(1 - \epsilon^N)$, where $||\cdot||$ is a norm you pick. Then, each row vector will be an invariant distribution of (k, z).]

3. To see if the value function is converged or not (or if your program is working or not), first compute the utility function under the policy function P, say $\mathbf{U}^P$, which is an "nm by 1" vector with the $(i + (j-1)m)$th element equal to $U[e^{z_j} f(k_i) - k_{i^*(i)}]$ where $k_{i^*(i)}$ is the optimal choice for the next capital stock given the current state is k_i. Next, compute $\mathbf{U}^P + \beta\Pi V^*$ and check whether it is equal to V^* or not. Explain briefly why $V^* = \mathbf{U}^P + \beta\Pi V^*$ must hold.

4. Suppose that this one-sector growth model is a good approximation of the reality. Furthermore, suppose that productivity shocks are independent across countries. Derive the model's prediction on the invariant distribution of the logarithm of capital stock per worker and the logarithm of output per worker across countries. Plot the steady state distribution on the graph using "bar" command in matlab.

5. Extra Question [Not Required but might be important for the Final Exam so you might want to try]: Suppose z follows a first order autoregressive process: $z_{t+1} = \rho z_t + \epsilon$ where $\rho = 0.9$ and ϵ normally distributed with $\sigma = 0.5$ as before. Can you do the similar programming exercise with this assumption? Hint: the transition function for z can be approximated by multinomial distribution as follows:

$$Prob(z_{t+1} = z_j | z_t = z_i) \equiv q_{ij} = \frac{\phi((z_j - \rho z_i)/\sigma)/\sigma}{\sum_{j'=1}^{m} \phi((z_{j'} - \rho z_i)/\sigma)/\sigma}.$$