Economics 817 **Winter 2004**

Advanced Macroeconomic Theory II **Hiroyuki Kasahara**

Homework 1

(Due: January 19 by 5 pm)

Note: Study groups *discussing* the problems are strongly encouraged. But write your own answers.

1 Metric Spaces and Normed Vector Spaces

1. Do Exercise 3.2(d) of Stokey and Lucas (1989) [Show that the properties a-h of vector spaces hold].
2. Do Exercise 3.3(c)-(d) of Stokey and Lucas (1989) [Show that the properties a-c of metric spaces hold].
3. Do Exercise 3.4(a)-(e) of Stokey and Lucas (1989) [Show that the properties a-c of normed vector spaces hold].
4. Do Exercise 3.5(a)-(c) of Stokey and Lucas (1989) [Hint: (a) Use a triangular inequality $\rho(x,y) \leq \rho(x,x_n)+\rho(x_n,y)$ together with the definition of $x_n \rightarrow x$ and $x_n \rightarrow y$ to show that $\rho(x,y) \leq \rho(x,x_n)+\rho(x_n,y) < \epsilon$. (b) By definition of $x_n \rightarrow x$, for any ϵ, there exists N_ϵ such that $\rho(x_m,x) < \epsilon/2$ and $\rho(x_n,x) < \epsilon/2$ for any $m,n \geq N_\epsilon$. Use a triangular in equality to show $\rho(x_m,x_n) < \epsilon/2+\epsilon/2 = \epsilon$. (c) $\rho(x_{N_\epsilon},x_0) < M$ for some $M < \infty$. Then, you can use a triangular inequality with the definition of a Cauchy sequence.]
5. Show that the metric spaces in Exercises 3.3(c) is not complete [Hint: Provide a counter-example]. Show that the metric spaces in Exercises 3.3(c) is complete if "strictly increasing" is replaced with "nondecreasing." [Hint: Mimic the proof of Theorem 3.1 of Stokey and Lucas or the proof of Theorem 12.9-10 of Sundaram line by line. You have to prove that the limit f is nondecreasing by arguing that f_n is nondecreasing and uniform convergence preserves the property of nondecreasing.]
6. Do Exercise 3.6(b). [Hint: You can use the fact that if E is a closed subset of a certain metric space, then it contains the limits of all convergent sequences in E.]

2 The Contraction Mapping Theorem

1. In Theorem 3.2 of Stokey and Lucas (1989), explain why the metric space (S, ρ) has to be complete. [Hint: Read the proof of Theorem 3.2. Note that a contraction mapping operator $T : S \to S$ only implies that $\{v_n\}$ is a Cauchy sequence. If (S, ρ) is not complete, $\lim_{n\to\infty} v_n$ may not be in....]

2. Consider a complete metric space $(\mathbf{R}^2, d_\infty)$, where $d_\infty(\mathbf{x}, \mathbf{y}) = \max_{i=1,2} |x_i - y_i|$ for $\mathbf{x} = (x_1, x_2)' \in \mathbf{R}^2$ and $\mathbf{y} = (y_1, y_2)' \in \mathbf{R}^2$. Consider an operator $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$T\mathbf{x} = \alpha + \beta\mathbf{P}\mathbf{x},$$

where $\alpha = (\alpha_1, \alpha_2)'$, $\beta \in (0, 1)$, and

$$\mathbf{P} = \begin{pmatrix} p_1 & 1 - p_1 \\ 1 - p_2 & p_2 \end{pmatrix},$$

where $p_1, p_2 \in [0, 1]$. We can interpret that this is a Bellman equation with two states. "Utilities" per period in state 1 and state 2 are α_1 and α_2, respectively. Transition across two states is governed by the transition matrix $\mathbf{P}$. β is a discount factor. Accordingly, we may think that $(\mathbf{R}^2, d_\infty)$ is the space of functions that have discrete support points $\{1, 2\}$ and a range $\mathbf{R}$, embedded with the sup norm $\rho(f, g) = \sup_{i=1,2} |f(i) - g(i)|$.

   (a) Prove that this operator has an unique fixed point by showing that it satisfies Blackwell's sufficient conditions and thus a contraction.

   (b) (Computer Programming)[1] Set $(\alpha_1, \alpha_2) = (2, 1)$, $\beta = 0.9$, and $(p_1, p_2) = (0.5, 0.8)$. Thus, the utility of state 1 is higher than that of state 2; and state 2 is more persistent than state 1. Set the initial value $\mathbf{x}_0 = (0, 0)$. Write a computer program that numerically solves the fixed point problem $T\mathbf{x} = \mathbf{x}$ by "successive approximation." That is, compute a sequence $\{\mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_n, \mathbf{x}_{n+1}, ...\}$ where, given the previous value $\mathbf{x}_n$, $\mathbf{x}_{n+1}$ is computed by:

$$\mathbf{x}_{n+1} = \alpha + \beta\mathbf{P}\mathbf{x}_n.$$

   Repeat this recursive computation until $\epsilon_n \equiv \max_{i=1,2} |x_{n,i} - x_{n-1,i}| < \epsilon$, where we set $\epsilon = 0.00001$.. Check whether your program works or not by directly computing the fixed point as: $\mathbf{x} = (I - \beta\mathbf{P})^{-1}\alpha$. Report (i) the first ten sequences $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_9, \mathbf{x}_{10}\}$ and (ii) the fixed point value of $\mathbf{x}$. Hand in your program, too.

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[1] It is very important to add your comments generously so that Zhen and I can understand your codes. Unless it's really obvious, add comments on every smaller segments of code, or possibly every line, in this homework.

3. Consider the complete metric space of the previous question. Now consider a different operator $T^* : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$\begin{aligned} T^*(x_1) &= \alpha_1 + \beta[p_1 x_1 + (1-p_1)x_2] \\ T^*(x_2) &= \max\{\alpha_1 - c + \beta x_1, \alpha_2 + \beta x_2\} \end{aligned}$$

where $p_1 \in [0,1]$ and $c \in \mathbf{R}_+$.

   (a) Prove that this operator has an unique fixed point by showing that it satisfies Blackwell's sufficient conditions and thus a contraction.

   (b) (Computer Programming) Set $(\alpha_1, \alpha_2) = (2, 1)$, $\beta = 0.9$, and $p_1 = 0.5$. Try different values of c between 0 and 5. Set the initial value $\mathbf{x}_0 = (0, 0)$. Write a computer program that numerically solves the fixed point problem $T^*\mathbf{x} = \mathbf{x}$ by "successive approximation." That is, compute a sequence $\{\mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_n, \mathbf{x}_{n+1}, ...\}$ where, given the previous value $\mathbf{x}_n$, $\mathbf{x}_{n+1}$ is computed by:

$$\mathbf{x}_{n+1} = T^*(\mathbf{x}_n)$$

   Repeat this recursive computation until $\epsilon_n \equiv \max_{i=1,2} |x_{n,i} - x_{n-1,i}| < \epsilon$, where we set $\epsilon = \frac{0.00001}{1-\beta}$. Report (i) the first ten sequences $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_9, \mathbf{x}_{10}\}$ and (ii) the fixed point value of $\mathbf{x}$ when $c = 2$. Hand in your program, too.