Answer Keys to HW1 (Econ 425)

1.(a)
$$\min_{K,L} \quad wL + rK$$
$$\text{s.t.} \quad \bar{X} = K^{\alpha} L^{1-\alpha}$$

$$\Rightarrow \quad \min_{k} \quad (w + rk) L$$
$$\text{s.t.} \quad \bar{X} = k^{\alpha} L$$
where $k = \frac{K}{L}$

$$\Rightarrow \quad \min_{k} \quad (w + rk) k^{-\alpha} \bar{X} \qquad (\text{use } L = \bar{X} k^{-\alpha})$$

$$\Rightarrow \quad \min_{k} \quad [w k^{-\alpha} + r k^{1-\alpha}] \bar{X}$$

The first-order condition:

$$k: \quad -\alpha w k^{-\alpha-1} + (1-\alpha) r k^{-\alpha} = 0$$

$$\Rightarrow \quad \boxed{k^* = \left(\frac{K^*}{L^*}\right) = \frac{\alpha}{1-\alpha}\left(\frac{w}{r}\right)}$$

Thus, the capita-labour ratio does not depend on the output level $\bar{X}$. This is because the production function is homogeneous. Homogenous production functions have the property that the slopes of isoquants are constant along a ray though the origin so the same captital-labor ratio is cost-minimizing for all levels of outputs.

(b). Using $\bar{X} = k^{\alpha} L$ and

$$k^* = \frac{\alpha}{1-\alpha}\left(\frac{w}{r}\right),$$

$$L^* = \bar{X} k^{-\alpha} = \bar{X} \left(\frac{\alpha}{1-\alpha}\right)^{-\alpha} \left(\frac{w}{r}\right)^{-\alpha}$$

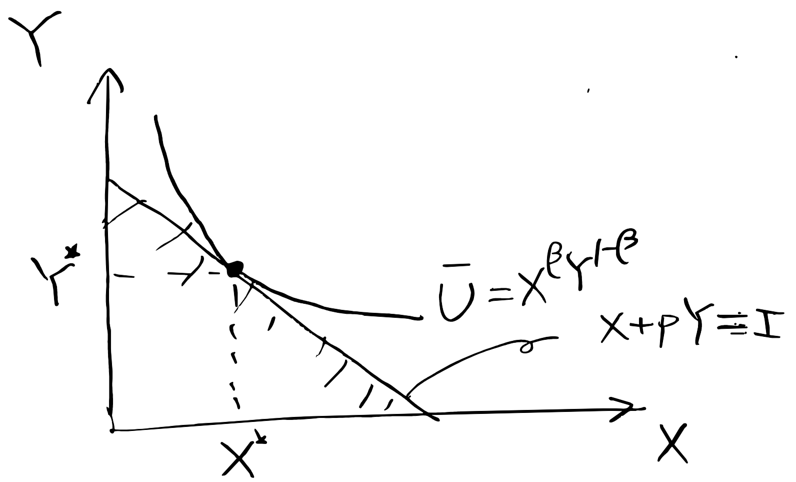
or $$\boxed{L^* = \left(\frac{\alpha}{1-\alpha}\frac{w}{r}\right)^{-\alpha} \bar{X}}$$

$$\boxed{K^* = k^* L^* = \left(\frac{\alpha}{1-\alpha}\frac{w}{r}\right)^{1-\alpha} \bar{X}}$$

2.(a)

$$\max_{X,Y} \quad X^{\beta} Y^{1-\beta}$$

$$\text{s.t.} \quad X + pY \le I$$



The utility maximization is characterized by

$$MRS = \frac{P_x}{P_y} = \frac{1}{p}$$

where MRS is determined by

$$MRS = \frac{\Delta U / \Delta X}{\Delta U / \Delta Y} = \frac{\beta X^{\beta-1} Y^{1-\beta}}{(1-\beta) X^{\beta} Y^{-\beta}} = \frac{\beta}{1-\beta}\left(\frac{Y}{X}\right)$$

Therefore,

$$\frac{\beta}{1-\beta}\left(\frac{Y}{X}\right) = \frac{1}{p} \quad - ⓐ$$

Also, from the budget constraint

$$X + pY = I \quad - ⓑ$$

As shown in the class, solving ⓐ & ⓑ for (X, Y), ④)

we get:

$$X = \beta I$$

$$Y = (1-\beta)\frac{I}{P}$$

The above analysis implies that each consumer's demands are given by:

$$X_1 = \beta I_1$$

$$X_2 = \beta I_2$$

$$Y_1 = (1-\beta) I_1/P$$

$$Y_2 = (1-\beta) I_2/P$$

Aggregate demands are

$$X = X_1 + X_2 = \beta (I_1 + I_2) = \beta I$$

$$Y = Y_1 + Y_2 = (1-\beta)\frac{(I_1+I_2)}{P} = (1-\beta)\frac{I}{P},$$

which does not depend on the distribution of income.

2 (b). Let $s = \frac{I_1}{I}$ $(1-s) = \frac{I_2}{I}$

the share of consumer 1's income in aggregate income.

Then,

$$X_1 = \beta I_1 = \beta s I$$

$$X_2 = \gamma (1-s) I$$

Aggregate Demand

$$X = X_1 + X_2 = \beta s I + \gamma(1-s) I$$

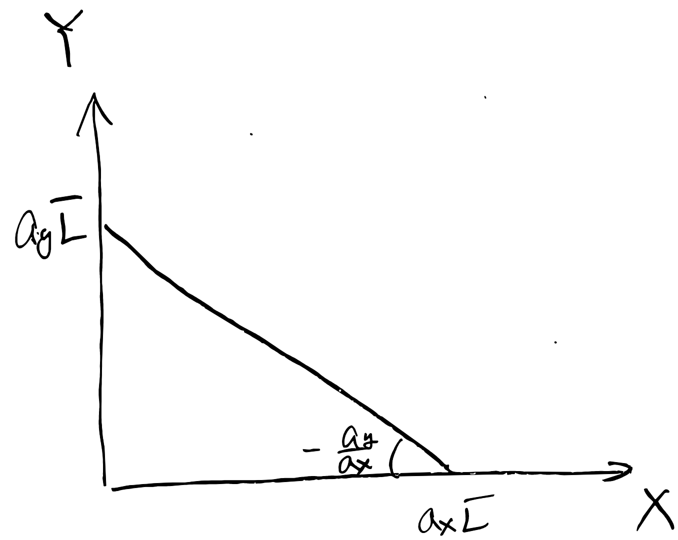
$$\Rightarrow \quad X = [\beta s + \gamma(1-s)] I$$

$$\Rightarrow \quad X = [\gamma + (\beta - \gamma) s] I,$$

Then, since $\gamma > \beta$, X is strictly decreasing in s.

Namely, a redistribution of income from consumer 1 to consumer 2 (i.e., $s \downarrow$) leads to an increase in X.

3(a).



3(b). – (d)

- Utility Maximization

$\Rightarrow$ Demand Functions :

$$X(p, w\bar{L}) = \beta w \bar{L}$$
$$Y(p, w\bar{L}) = (1-\beta)\frac{w\bar{L}}{p}.$$

$$\Rightarrow \quad \frac{X}{Y} = \frac{\beta}{1-\beta} p.$$

$$\Rightarrow \quad \boxed{\underbrace{\frac{1-\beta}{\beta}\frac{X}{Y}}_{MRS} = p} \quad -\text{ⓐ}$$

- Profit Maximization

$$\max_{L_x} \; a_x L_x - w L_x$$
$$\Rightarrow a_x = w$$
$$\max_{L_y} \; p a_y L_y - w L_y$$
$$\Rightarrow p a_y = w$$

$$\Rightarrow \quad \boxed{\underbrace{\frac{a_x}{a_y}}_{MRT} = p} \quad -\text{ⓑ}$$

- Labor Market

$$\bar{L} = L_x + L_y$$

$$\Rightarrow \boxed{\bar{L} = \frac{X}{a_x} + \frac{Y}{a_y}} \quad -\text{ⓒ}$$

Solving ⓐ – ⓒ (Three equations for three unknowns),

we get,

$$\boxed{P^* = \frac{a_x}{a_y}}$$

$$\text{ⓐ} \Rightarrow X = \frac{\beta}{1-\beta} P^* Y = \frac{\beta}{1-\beta} \frac{a_x}{a_y} Y$$

Plug this into ⓒ,

$$\bar{L} = \frac{\beta}{1-\beta} \frac{Y}{a_y} + \frac{Y}{a_y} = \frac{1}{1-\beta} \cdot \frac{Y}{a_y}$$

$$\Rightarrow \boxed{Y^* = (1-\beta)\, a_y \bar{L}}$$

Similary

$$\boxed{X^* = \beta\, a_x \bar{L}}$$

$$\Rightarrow \begin{cases} L_x^* = \beta \bar{L} \\ L_y^* = (1-\beta)\bar{L} \end{cases}$$

3.(e) Utility Maximization

$$\max_{X,Y} \quad X^{\beta} Y^{1-\beta}$$

$$\text{s.t.} \quad X + (1+t)pY \leq I$$

Solving this leads to

$$\boxed{\frac{X}{Y} = \frac{\gamma}{1-\gamma}(1+t)p} \quad - ⓐ'$$

Equilibrium is characterized by ⓐ', ⓑ, ⓒ.

$$\boxed{p^* = \frac{a_x}{a_y}}$$

$$ⓐ' \Rightarrow \quad X = \frac{\beta}{1-\beta}(1+t)p^* Y = \frac{\beta}{1-\beta}(1+t)\frac{a_x}{a_y} Y$$

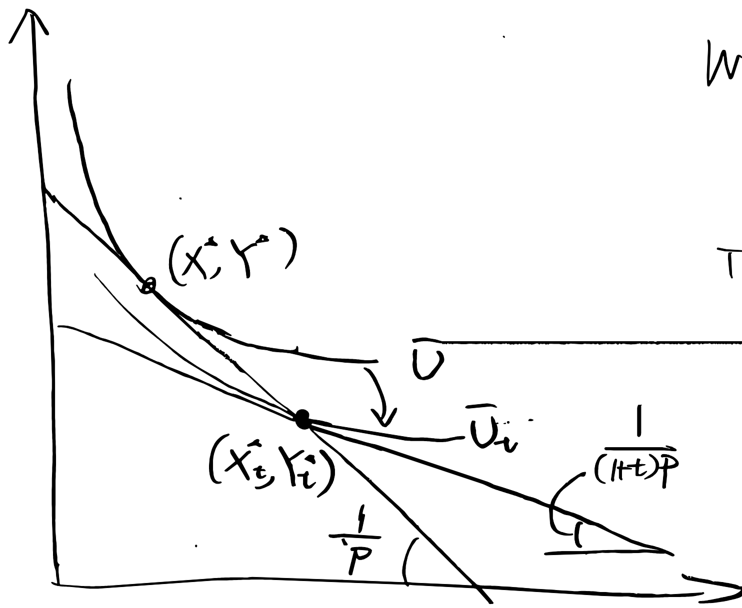
Plug this into ⓒ

$$\bar{L} = (1+t)\frac{\beta}{1-\beta}\frac{Y}{a_y} + \frac{Y}{a_y} = \frac{1+t\beta}{1-\beta}\frac{Y}{a_y}$$

$$\Rightarrow \quad \boxed{Y_t^* = \frac{1-\beta}{1+t\beta} a_y \bar{L}}$$

$$X^* = \frac{\beta}{\cancel{1-\beta}}(1+t)\frac{a_x}{\cancel{a_y}}\left(\frac{\cancel{1-\beta}}{1+t\beta}\right)\cancel{a_y}\bar{L}$$

$$\Rightarrow \boxed{X_t^* = \frac{1+t}{1+t\beta}\beta a_x \bar{L}}$$



With $t > 0$,

MRS $\neq$ MRT

The utility level goes down with the tax.

4. (a) – (d)

- Utility Maximization $\Rightarrow$ MRS $= P$.

i.e., $\boxed{P = \frac{1-\beta}{\beta} \frac{X}{Y}}$ — (1)

- Profit Max

$$\max_{K_x, L_x} \; K_x^{\alpha_x} L_x^{1-\alpha_x} - wL_x - rK_x.$$

F.O.C.:

$$\alpha_x K_x^{\alpha_x - 1} L_x^{1-\alpha_x} = r \quad \text{— ⓐ}$$

$$(1-\alpha_x) K_x^{\alpha_x} L_x^{-\alpha_x} = w. \quad \text{— ⓑ}$$

Similarly, for Industry Y.

$$\alpha_y K_y^{\alpha_y - 1} L_y^{1-\alpha_y} = r \quad \text{— ⓒ}$$

$$(1-\alpha_y) K_y^{\alpha_y} L_y^{-\alpha_y} = w. \quad \text{— ⓓ}$$

$$\frac{ⓐ}{ⓑ} \Rightarrow k_x = \frac{\alpha_x}{1-\alpha_x}\frac{w}{r}$$

$$\frac{ⓒ}{ⓓ} \Rightarrow k_y = \frac{\alpha_y}{1-\alpha_y}\frac{w}{r}$$

$$\Rightarrow \boxed{k_x = \frac{\alpha_x}{\alpha_y}\left(\frac{1-\alpha_y}{1-\alpha_x}\right)k_y} \quad -(2)$$

$\Downarrow$

If $\alpha_x > \alpha_y$, then $k_x < k_y$

$\downarrow$

Answer to 4(a).

$$\frac{ⓓ}{ⓑ} \Rightarrow \boxed{P = \frac{(1-\alpha_x)\, k_x^{\alpha_x}}{(1-\alpha_y)\, k_y^{\alpha_y}}} \quad -(3)$$

- Labor & Capital Market Equilibrium.

$$\bar{L} = L_x + L_y \Rightarrow 1 = \underbrace{\frac{L_x}{\bar{L}}}_{s} + \underbrace{\frac{L_y}{\bar{L}}}_{1-s}$$

$$\bar{K} = K_x + K_y$$

$$\Rightarrow \frac{\bar{K}}{\bar{L}} = \frac{L_x}{\bar{L}}\frac{K_x}{L_x} + \frac{L_y}{\bar{L}}\frac{K_y}{L_y}$$

$$\Rightarrow \boxed{\bar{k} = s\, k_x + (1-s)\, k_y} \quad -(4)$$

Note $\frac{X}{L_x} = k_x^{\alpha_x}$, $\frac{Y}{L_y} = k_y^{\alpha_y}$.

Then, (1) $\Rightarrow$

$$p = \frac{1-\beta}{\beta} \frac{X/L_x}{Y/L_y} \cdot \frac{L_x/L}{L_y/L}$$

$$\Rightarrow \boxed{p = \frac{1-\beta}{\beta} \frac{s}{1-s} \frac{k_x^{\alpha_x}}{k_y^{\alpha_y}}} \quad -(1)'$$

There are four equations [(1)', (2)–(4)]
for four unknowns (p, s, k_x, k_y).

Thus, we can solve these equations.

From (3) & (1)',

$$\frac{1-\alpha_x}{1-\alpha_y} \frac{\cancel{k_x^{\alpha_x}}}{\cancel{k_y^{\alpha_y}}} = \frac{1-\beta}{\beta} \frac{s}{1-s} \frac{\cancel{k_x^{\alpha_x}}}{\cancel{k_y^{\alpha_y}}}$$

$$\Rightarrow \left(\frac{1-\alpha_x}{1-\alpha_y} \frac{\beta}{1-\beta}\right)(1-s) = s \qquad \left(\frac{1-s}{(1-\alpha_y)(1-\beta)} = \frac{s}{(1-\alpha_x)\beta}\right)$$

$$\Rightarrow \left(\frac{1-\alpha_x}{1-\alpha_y} \frac{\beta}{1-\beta}\right) = \left(1 + \frac{1-\alpha_x}{1-\alpha_y} \frac{\beta}{1-\beta}\right) s$$

$$\Rightarrow \boxed{s^* = \frac{\frac{1-\alpha_x}{1-\alpha_y} \frac{\beta}{1-\beta}}{1 + \frac{1-\alpha_x}{1-\alpha_y} \frac{\beta}{1-\beta}}} \quad \times (1-\alpha_y)(1-\beta) \quad \Rightarrow \boxed{s^* = \frac{(1-\alpha_x)\beta}{(1-\alpha_y)(1-\beta) + (1-\alpha_x)\beta}}$$

From (2) & (4)

$$\bar{k} = s\,\frac{d_x}{d_y}\left(\frac{1-d_y}{1-d_x}\right)k_y + (1-s)\,k_y$$

$$\Rightarrow \bar{k} = \left[s\,\frac{d_x}{d_y}\left(\frac{1-d_y}{1-d_x}\right) + (1-s) \right] k_y$$

$$\boxed{\Rightarrow k_y^* = \frac{\bar{k}}{s^*\,\frac{d_x}{d_y}\left(\frac{1-d_y}{1-d_x}\right) + (1-s^*)}}$$

$$\boxed{k_x^* = \frac{\bar{k}}{(1-s^*)\,\frac{d_y}{d_x}\,\frac{1-d_x}{1-d_y} + s^*}}$$

Answer to (c)

$$L_x^* = s^*\,\bar{L}$$

$$L_y^* = (1-s^*)\,\bar{L}$$

$$K_x^* = k_x^* \cdot L_x^* = \frac{s^*\,\bar{K}}{(1-s^*)\,\frac{d_y}{d_x}\,\frac{1-d_x}{1-d_y} + s^*}$$

$$K_y^* = k_y^* \cdot L_y^* = \frac{(1-s^*)\,\bar{K}}{(1-s^*)\,\frac{d_y}{d_x}\,\frac{1-d_x}{1-d_y} + s^*}$$

(b)
$$\begin{cases} X^* = k_x^{*\alpha} L_x = \left(\dfrac{\bar{k}}{s^* \frac{\alpha_x}{\alpha_y} \left(\frac{1-\alpha_y}{1-\alpha_x}\right) + (1-s^*)} \right)^{\alpha} s^* \bar{L} \\ Y^* = \left(\dfrac{\bar{k}}{(1-s^*) \frac{\alpha_y}{\alpha_x} \frac{1-\alpha_x}{1-\alpha_y} + s^*} \right)^{\alpha} (1-s^*) \bar{L} \end{cases}$$

(d) $$p^* = \frac{1-\beta}{\beta} \frac{X^*}{Y^*}$$

$$\Rightarrow \boxed{p^* = \frac{1-\beta}{\beta} \frac{s^*}{1-s^*} \left(\frac{(1-s^*) \frac{\alpha_y}{\alpha_x} \frac{1-\alpha_x}{1-\alpha_y} + s^*}{s^* \frac{\alpha_x}{\alpha_y} \frac{1-\alpha_y}{1-\alpha_x} + 1-s^*} \right)^{\alpha}}$$

$$\left(\frac{w^*}{r^*}\right) = \frac{1-\alpha_x}{\alpha_x} k_x^*$$

$$\Rightarrow \boxed{\frac{w^*}{r^*} = \frac{\bar{k}}{(1-s^*) \frac{\alpha_y}{1-\alpha_y} + s^*}}$$

or

$$\left(= \frac{\bar{k}}{s^* \frac{\alpha_x}{1-\alpha_x} + 1-s^*} \right)$$

(e) Instead of (1)', we have

$$\boxed{(1+t)p = \frac{1-\beta}{\beta}\,\frac{s}{1-s}\,\frac{k_x^{d_x}}{k_y^{d_y}}} \quad \text{— (5)}$$

The rest is the same.

Solving (2)–(5), we get

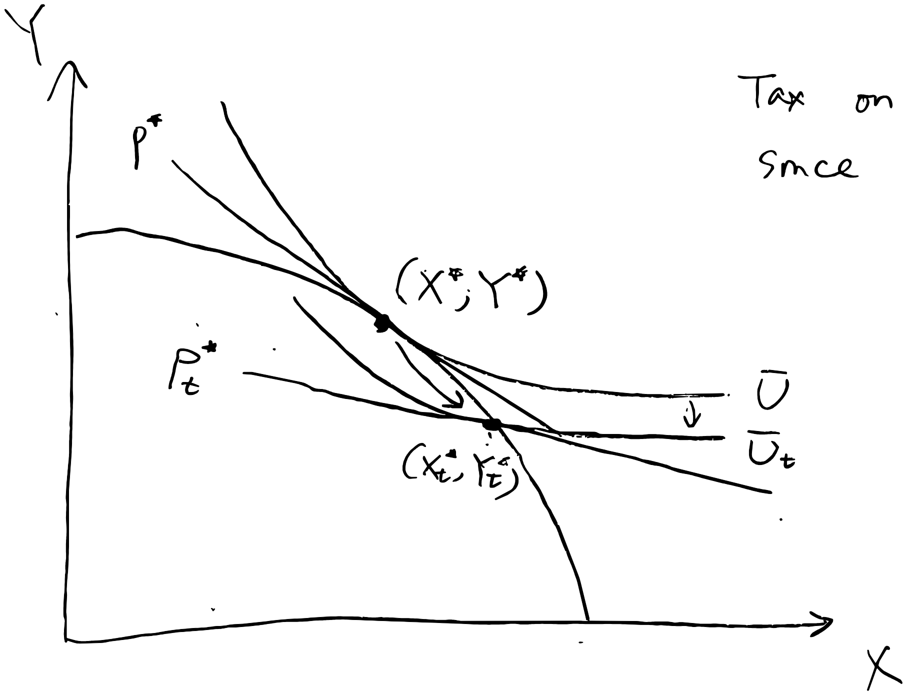
$$s_t^* = \frac{(1-d_x)\beta(1+t)}{(1-d_y)(1-\beta) + (1-d_x)\beta(1+t)}$$

(the higher the tax rate t, the higher the labor share of X)

$$p_t^* = \frac{1-\beta}{\beta(1+t)}\,\frac{s_t^*}{1-s_t^*}\left(\frac{(1-s_t^*)\frac{d_y}{d_x}\frac{1-d_x}{1-d_y} + s_t^*}{s_t^*\frac{d_x}{d_y}\frac{1-d_y}{1-d_x} + 1-s_t^*}\right)^{\alpha}$$

$$X_t^* = \left(\frac{\bar{k}}{s_t^*\frac{d_x}{d_y}\left(\frac{1-d_x}{1-d_y}\right) + (1-s_t^*)}\right)^{\alpha} s_t^*\,\bar{L}$$

$$Y_t^* = \left(\frac{\bar{k}}{(1-s_t^*)\frac{d_y}{d_x}\frac{1-d_x}{1-d_y} + s_t^*}\right)^{\alpha}(1-s_t^*)\,\bar{L}$$



Tax on Y Reduces the welfare since MRS ≠ MRT