Dynamic Discrete Choice Models

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Example: Fertility Decision

Per period utility

Having a child: $uX - C + \epsilon_1$
Not having a child:$uX + \epsilon_2$
X_t: Number of children: state vector
u: utility parameter of having a child
c: cost of having a child.

Next period state variable

Have a child: $X_{t+1} = X_t + 1$
Not have a child: $X_{t+1} = X_t$

Final period

Utility function:

$$V_T(X_T) = uX_T$$

Final period expected utility function:

$$E_{T-1}\left[V_T(X_T)\right] = uX_T$$

Calculate $V_T(X_T)$ for each $X_T = 0, ..., 20$ Calculate $E_{T-1}\left[V_T(X_T)\right]$ for each $X_T = 0, ..., 20$

Period $T-1$

- Value of having a child:

$$V_{CT-1}(X_{T-1}, \epsilon_{T-1}) = uX_{T-1} - c + \beta EV(X_T) + \epsilon_{1T-1}$$

where

$$X_T = X_{T-1} + 1$$

- value of not having a child:

$$V_{NT-1}(X_{T-1}, \epsilon_{T-1}) = uX_{T-1} + \beta EV(X_T) + \epsilon_{2T-1}$$

where

$$X_T = X_{T-1}$$

Then, the value function at period $T-1$ is

$$V_{T-1}(X_{T-1},\epsilon_{T-1}) = Max\left\{V_{CT-1}(X_{T-1},\epsilon_{T-1}), V_{NT-1}(X_{T-1},\epsilon_{T-1})\right\}$$

Derive the expected value function $EV_{T-2}(X_{T-1},\epsilon_{T-1})$
By simulation:

- Given the distribution of ϵ, draw ϵ^m_{T-1}.
- Calculate $V_{T-1}(X_{T-1},\epsilon^m_{T-1}), m = 1, ..., M$
- $\widehat{E}V_{T-1}(X_{T-1},\epsilon_{T-1}) \approx \frac{1}{M}\sum_{m=1}^{M} V_{T-1}(X_{T-1},\epsilon^m_{T-1})$
- Calculate the above expected value function for $X_{T-1} = 0, ..., 20$

Repeat the above algorithm from $T-1$ to 1.
After the dynamic Programming algorithm, we know

$$EV(X_T), EV(X_{T-1}, \epsilon_{T-1}), ..., EV(X_1, \epsilon_1)$$

. for all $X_t = 0, ..., 20$.

Simulated Likelihood Calculation

Simulate choice probability for an indidividual at age t who has X_t children, and decided to have one more child.

1. Draw $\epsilon_{1t}^m, \epsilon_{2t}^m$
2. Derive the value function.

$$V_{Ct}(X_t, \epsilon_t^m) = uX_t - c + \beta EV(X_{t+1}, \epsilon_{t+1}) + \epsilon_{1t}^m$$

where $X_{t+1} = X_t + 1$

$$V_{Nt}(X_t, \epsilon_t^m) = uX_t + \beta EV(X_{t+1}, \epsilon_{t+1}) + \epsilon_{2t}^m$$

where $X_{t+1} = X_t$

3. if $V_{Ct}(X_t, \epsilon_t^m) \geq V_{Nt}(X_t, \epsilon_t^m)$ then $I_C(X_t, \epsilon_t^m) = 1$, i.e. decides to have a child.
   Otherwise $I_C(X_t, \epsilon_t^m) = 0$, i.e. decides not to have a child.

Repeat 1-3 for $m = 1, ..., M$ and calculate the simulated choice probability.

$$\widehat{P}_{Ct}(X_t) = \frac{1}{M} \sum_{m=1}^{M} I_C(X_t, \epsilon_t^m)$$

If the person actually chose to have teh child, then
log likelihood increment: $ln\left[\widehat{P}_{Ct}(X_t^d)\right]$ If she did not, then
log likelihood increment: $ln\left[1 - \widehat{P}_{Ct}(X_t^d)\right]$

The Computational Burden

- Integration over the taste shock has to be carried out at the Dynamic Programming stage of the estimation, as well as the likelihood calculation.
- Integrations at the DP stage has to be carried out at every possible state space point.

Rust, Econometrica (1987)

Denote the deterministic values of each choice as follows

$$\overline{V}_{Ct}(X_t) = uX_t - c + \beta EV(X_{t+1}, \epsilon_{t+1})$$

$$\overline{V}_{Nt}(X_t) = uX_t + \beta EV(X_{t+1}, \epsilon_{t+1})$$

Then,

$$V_{Ct}(X_t, \epsilon_t) = \overline{V}_{Ct}(X_t) + \epsilon_{Ct}$$

$$V_{Nt}(X_t, \epsilon_t) = \overline{V}_{Nt}(X_t) + \epsilon_{Nt}$$

and

$$V(X_t, \epsilon_t) = Max\left\{V_{Ct}(X_t, \epsilon_t), V_{Nt}(X_t, \epsilon_t)\right\}$$

If $\epsilon_{Ct}, \epsilon_{Nt}$ are i.i.d. extreme value distributed, then the expected value function (Emax function) is

$$EV(X_t, \epsilon_t) = ln \left[exp(\overline{V}_{Ct}(X_t)) + exp(\overline{V}_{Nt}(X_t)) \right]$$

and the fertility choice probability can be expressed as a logit:

$$P_{Ct}(X_t) = \frac{exp(\overline{V}_{Ct}(X_t))}{exp(\overline{V}_{Ct}(X_t)) + exp(\overline{V}_{Nt}(X_t)}$$

Unobserved Heterogeneity

Heckman and Singer (1984)

- High fertility type: low c: c_H
- Low fertility type: high c: c_L
- Solve the DP problem for each fertility type.
- Get $EV_t(X_t, u, c_H)$, $EV_t(X_t, u, c_L)$

The likelihood increment for person i is:

$$
\begin{aligned}
&\pi l_{Hi} + (1-\pi) l_{Li} \\
= \; &\pi \prod_{t=t_1}^{T} \left\{ \left[\frac{exp(\overline{V}_{Ct}(X_{it}, u, C_H))}{exp(\overline{V}_{Ct}(X_{it}, u, C_H) + exp(\overline{V}_{Nt}(X_{it}, u, C_H)} \right]^{I_C} \right. \\
&\left. + \left[\frac{exp(\overline{V}_{Nt}(X_{it}, u, C_H))}{exp(\overline{V}_{Ct}(X_{it}, u, C_H) + exp(\overline{V}_{Nt}(X_{it}, u, C_H)} \right]^{1-I_C} \right\} \\
&+(1-\pi) \prod_{t=t_1}^{T} \left\{ \left[\frac{exp(\overline{V}_{Ct}(X_{it}, u, C_L))}{exp(\overline{V}_{Ct}(X_{it}, u, C_L) + exp(\overline{V}_{Nt}(X_{it}, u, C_L)} \right]^{I_C} \right. \\
&\left. + \left[\frac{exp(\overline{V}_{Nt}(X_{it}, u, C_L))}{exp(\overline{V}_{Ct}(X_{it}, u, C_L) + exp(\overline{V}_{Nt}(X_{it}, u, C_L)} \right]^{1-I_C} \right\}
\end{aligned}
$$

Parameter to be estimated: u, c_L, c_H, π.

- Suppose the data is divided into women who have many children and who do not.
- Then, for the first type, l_H is low and l_L is high. But the estimated weighted sum $\pi l_{Hi} + (1 - \pi) l_{Li}$ is higher than the estimated likelihood without heterogeneity l_i.
- For the second type, l_H is high and l_L is low. But the estimated weigthed sum is higher than the likelihood without heterogeneity. l_i
- Because if there is no heterogeneity, the model predicts that women have neigher many nor no children, which is not seen in the data.

Simulation

- Draw the unobserved type. Draw uniform distribution $\epsilon \sim U(0,1)$.
  $c = c_H$ if $\epsilon \leq \pi$. $c = c_L$ otherwise.
- Start with $X_{t_1} = 0$. Draw uniform distribution $\epsilon \sim U(0,1)$
- $X_{t_1+1} = X_{t_1} + 1$ if $\epsilon \leq P_{Ct}(X_t)$ $X_{t_1+1} = X_{t_1}$ if otherwise.
- Repeatedly simulate X_t for $t = t_1 + 1, ..., T$