Practice Question for the Final Exam, Answer Key

Question 1

Consider the model of bus engine replacement problem of Rust (1987).

1. Carefully write down the Bellman equation of the model, together with the transition probabilities of the state variables.

$$V(x_t) = Max\{-mc(x_t) + \beta \int_0^\infty V(y)p(dy|x_t) + \epsilon_m, -RC + \beta \int_0^\infty V(y)p(dy|0) + \epsilon_r\}$$

where $mc(x) = 0.01\theta_{11}x$ is the maintenance cost function and RC is the replacement cost. also $p(dy|x)$ is the transition density which is defined as follows.

$$\begin{aligned} p(y|x) &= \theta_2 exp\{\theta_2(y-x)\} \; if \; y \geq x \\ &= 0 \; otherwise \end{aligned}$$

Also,ϵ_m, ϵ_r are i.i.d. extremely value distributed.

2. Given you have estimated the parameters of the model, carefully explain how you would obtain the steady state replacement probability.

1. Given the replacement cost and other parameters that are estimated, solve for the Dynamic programming problem and derive the value function.

$$V(x, \epsilon, \theta) = V^{(n+1)}(x, \epsilon, \theta)$$

2. Derive the replacement probability.

$$\begin{aligned} & Pr(replace|x, \theta) \\ = \; & Pr\left(\epsilon(nr) - \epsilon(r) \leq u(x, r, \theta_1) - u(x, nr, \theta_1)\right. \\ & +\beta V(y, \epsilon')q(d\epsilon')p(dy \mid x, r, \theta_2) \\ & \left. -\; \beta V(y, \epsilon')q(d\epsilon')p(dy \mid x, nr, \theta_2)\right) \end{aligned}$$

3. Derive the mileage density.

$$p(x_{t+1} = x_j \mid x_t = x_i, i_t, \theta_2) =$$

$$\begin{cases} \theta_{j-i} & if \; i_t = 0 and \; x_{t+1} \geq x_t \\ \theta_j & if \; i_t = 1 and \; x_{t+1} \geq 0 \\ 0 & otherwise \end{cases}$$

They together are the transition probability of a Markov Chain from x_t to x_{t+1}.

4. Derive the stationary distribution of the Markov Chain of x:$\pi(x)$.

5. Stationary replacement probability:

$$\int Pr(replace \mid x, \theta)\pi(x)dx$$

Question 2

Suppose there are N participants of the sealed bid first price auction. Each bidder's private valuation v is independently and identically distributed with the distribution function being $F(v)$.

1. What is the probability that player i wins the auction when her bid is x_i? Express it in terms of $F(v)$. Write down the profit function of the bidder i.

$$F^{N-1}(b^{-1}(x_i))$$

where $b()$ is the bid function. That is, $x_i = b(v_i)$ where v_i is person i's valuation.

2. Derive the formula for player i's value maximizing bid given his private value being v_i as a function of $F(v)$.
The profit maximizing bid of individual i is

$$x_i = b(v_i) = Argmax_{x_i \in R^+}(v_i - x_i)F^{N-1}(b^{-1}(x_i))$$

The first order condition for profit maximization is:

$$-F^{N-1}(b^{-1}(x)) + (v - x)\frac{\partial F^{N-1}}{\partial v}(b^{-1}(x))/b'(v) = 0$$

muptiplying $b'(v)$ on both sides, and using $x = b(v)$, we get

$$b'(v)F^{N-1}(b^{-1}(x)) + b(v)\frac{\partial F^{N-1}}{\partial v} = v\frac{\partial F^{N-1}}{\partial v}$$

The above equation results in,

$$\frac{d}{dv}\left[b(v)F^{N-1}(v)\right] = v\frac{d}{dv}F^{N-1}(v)$$

By integrating both sides, we get

$$b(v)F^{N-1}(v) = \int_{b_0}^{v} sd(F^{N-1}(s))$$

Hence,

$$b(v) = \frac{\int_{b_0}^{v} sd[F^{N-1}(s)]}{F^{N-1}(v)}$$

Now, denote $G(s) = F^{N-1}(s)$. Then, by Integration by Parts,

$$\int_{b_0}^{v} s dG(s) = [sG(s)]_{b_0}^{v} - \int_{b_0}^{v} G(s)ds = vG(v) - \int_{b_0}^{v} G(s)ds$$

Hence,

$$b_i = v_i - \frac{1}{F(v_i)^{N-1}} \int_{b_0}^{v_i} [F(u)]^{N-1} du$$

3. Suppose $F(v)$ is uniformly distributed with support $[0,1]$. What is the optimal bid for player i if his valuation is v_i? Can you say something about the change in otimal bids and his profit when the number of players increase to infinity?

$$\begin{aligned} b_i &= v_i - \frac{1}{v_i^{N-1}} \int_0^{v_i} u^{N-1} du \\ &= v_i - \frac{1}{v_i^{N-1}} \frac{1}{N} v_i^N \\ &= \frac{N-1}{N} v_i \end{aligned}$$

which converges to v_i as N goes to infinity. That is, optimal bid increases with number of bidders and converges to the individual's true valuation. Hence, as the number of bidders increases, the profit of each bidder decreases to zero.

4. Given the bid of player i being b_i and the distribution of bids being $G(b)$, derive the private value of player i, v_i.
Use the formula in the lecture note to derive

$$v_i = b_i + \frac{G(b_i)}{(N-1)g(b_i)}$$

$g()$ can be obtained by taking a derivative of $G(b)$ with respect to b.