# Production Function Estimation

## Production function

$$Q_i = f(L_i, K_i, m_i)$$

$L$: labor input
$K$: capital input
$m$: other input

Example, Cobb-Douglas Production function

$$Q_i = AL_i^{\alpha} K_i^{\beta} exp(\epsilon_i)$$

$$ln(Q_i) = ln(A) + \alpha ln(L_i) + \beta ln(K_i) + \epsilon_i$$

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- Both $L_i$ and $K_i$ are endogenous, i.e. correlated with the error term $\epsilon_i$ because firms decide their inputs so that their profits are maximized, i.e. cost is minimized.
- Both labor and capital input will change depending on the productivity shock. If the productivity is high, firms will use less input given the same output. Firms perhaps will use more inputs and produce more output.

# Firm Cost Minimization Problem

Given output $Y$

$$MinC = wL + rK$$

$$s.t.\ Y = AL^{\alpha}K^{\beta}exp(\epsilon)$$

Lagrangian:

$$wL + rK + \lambda\left[Y - AL^{\alpha}K^{\beta}e^{\epsilon}\right]$$

F.O.C

$$\begin{aligned} wL &= \lambda\alpha Y \\ rK &= \lambda\beta Y \\ L &= \frac{\alpha r}{\beta w}K \end{aligned}$$

By substitution, and using $Y = AL^{\alpha}K^{\beta}e^{\epsilon}$ we get cost minimizing $K$, $L$ given $Y$

$$K = \left\{ \left[ A \left( \frac{\alpha r}{\beta w} \right)^{\alpha} e^{\epsilon} \right]^{-1} Y \right\}^{\frac{1}{\alpha+\beta}}$$

$$L = \left\{ \left[ A \left( \frac{\beta w}{\alpha r} \right)^{\beta} e^{\epsilon} \right]^{-1} Y \right\}^{\frac{1}{\alpha+\beta}}$$

Both $K$ and $L$ depend on the error term $\epsilon$. Hence, OLS of the production function estimation results in bias.

# Instrumental Variables Estimation

Input prices can be considered as instruments. That is,

$$ln(Q_i) = ln(A) + \alpha ln(L_i) + \beta ln(K_i) + \epsilon_i$$

with $w_i$, $r_i$ as instruments for $ln(L_i)$, $ln(K_i)$. One can assume that the market level input prices are not correlated with the productivity shock of the individual firm $\epsilon_i$. The two stage least squares would look like

$$ln(Q_i) = ln(A) + \alpha\left[\hat{\gamma}_{wL} w_i + \hat{\gamma}_{rL} r_i\right] + \beta\left[\hat{\gamma}_{wK} w_i + \hat{\gamma}_{rK} r_i\right] + \epsilon_i \quad (1)$$

where $\hat{\gamma}$'s are the first stage regression coefficients i.e.

$$\begin{aligned} L_i &= \gamma_{wL} w_i + \gamma_{rL} r_i + v_L \\ K_i &= \gamma_{wK} w_i + \gamma_{rK} r_i + v_K \end{aligned}$$

- However, equation 1 is not a relationship between the wages and rental rate of capital to output implied by the firm's cost minimization and profit maximization problem.
- Profit maximization does not say that if wage changes by $\Delta w$, firms should change output by

$$[\alpha\gamma_{wL} + \beta\gamma_{wK}]\,\Delta_w$$

- It is better to estimate the relationship between input prices and ouput derived directly from the cost minimization problem.

# Cost Function

$$
\begin{aligned}
C(w, r, p_E, Q) &= Min_{\{L,K\}} wL + rK + p_E E \\
s.t.\ Q &= f(L, K, E)
\end{aligned}
$$

- $E$: energy input.
- Notice that given output, the variables in the cost functions are all input prices, which are exogenous.
- From Shephard's Lemma

$$
\frac{\partial C}{\partial w} = L,\ \frac{\partial C}{\partial r} = K,\ \frac{\partial C}{\partial p_E} = E
$$

or

$$
\frac{\partial lnC}{\partial lnw} = \frac{wL}{C},\ \frac{\partial lnC}{\partial lnr} = \frac{rK}{C},\ \frac{\partial lnC}{\partial lnp_E} = \frac{p_E E}{C}
$$

# Functional Form

$$
\begin{aligned}
C(w,r,Q) = \; & exp\,[\beta_0 + \beta_w lnw + \beta_r lnr + \beta_E lnp_E \\
& + \beta_{w2}(lnw)^2 + \beta_{r2}(lnr)^2 + \beta_{E2}(lnp_E)^2 \\
& + \beta_{wr}(lnw)(lnr) + \beta_{wE}(lnw)(lnE) + \beta_{rE}(lnr)(lnp_E) \\
& + \beta_{wQ}(lnw)(lnQ) + \beta_{rQ}(lnr)(lnQ) + \beta_{EQ}(lnp_E)(lnQ) \\
& + \beta_Q lnQ + \beta_{Q2}(lnQ)^2 + \epsilon]
\end{aligned}
$$

or

$$
\begin{aligned}
lnC(w,r,Q) = \; & \beta_0 + \beta_w lnw + \beta_r lnr + \beta_E lnp_E \\
& + \beta_{w2}(lnw)^2 + \beta_{r2}(lnr)^2 + \beta_{E2}(lnp_E)^2 \\
& + \beta_{wr}(lnw)(lnr) + \beta_{wE}(lnw)(lnE) + \beta_{rE}(lnr)(lnp_E) \\
& + \beta_{wQ}(lnw)(lnQ) + \beta_{rQ}(lnr)(lnQ) + \beta_{EQ}(lnp_E)(lnQ) \\
& + \beta_{Q2}(lnQ)^2 + \epsilon
\end{aligned}
$$

Express all the input prices as $p_i, i = 1, 3$. Then,

$$
\begin{aligned}
lnC &= \beta_0 + \sum_i \beta_i lnp_i + \beta_Q lnQ \\
&+ \frac{1}{2} \sum_i \sum_j \gamma_{ij} lnp_i lnp_j + \sum_i \gamma_{iQ} lnp_i lnQ \\
&+ \frac{1}{2} \gamma_{QQ} (lnQ)^2
\end{aligned}
$$

# Restriction on the Cost Function Coefficients

An advantage of the cost function approach is that one can impose theoretical restrictions implied by the cost minimization on the cost function.

## Symmetry

$$\frac{\partial lnC}{\partial p_i \partial p_j} = \frac{\partial lnC}{\partial p_j \partial p_i}$$

That is,

$$\gamma_{ij} = \gamma_{ji}$$

## Homogeneity of degree one

If all the input prices increase by $s$, cost also increases by $s$ as well. That is,

$$
\begin{aligned}
& ln(sC) = lns + lnC \\
= \quad & \sum_i \beta_i (lns + lnp_i) + \frac{1}{2} \sum_i \sum_j \gamma_{ij} (lns + lnp_i)(lns + lnp_j) \\
& + \sum_i \gamma_{iQ} (lns + lnp_i) lnQ + terms\ without\ (lns + lnp_i)
\end{aligned}
$$

Hence, RHS equals $lns + lnC$ for any $p_i$, $Q$ if and only if

$$
\begin{aligned}
\sum_i \beta_i &= 1 \\
\sum_j \gamma_{ij} &= 0 \ \forall i \\
\sum_i \gamma_{iQ} &= 0
\end{aligned}
$$

# Estimation

## Cost function

One can estimate the cost equation by OLS, given one has data on cost, input price and output.

$$
\begin{aligned}
lnC &= \beta_0 + \sum_i \beta_i lnp_i + \beta_Q lnQ \\
&+ \frac{1}{2}\sum_i \sum_j \gamma_{ij} lnp_i lnp_j + \sum_i \gamma_{iQ} lnp_i lnQ \\
&+ \frac{1}{2}\gamma_{QQ}\left(lnQ\right)^2 + \epsilon_1 \qquad (1)
\end{aligned}
$$

But there are other equations that can be included in the estimation.

## Share Equations

3 Share functions: $i \in \{L, K, E\}$

$$\frac{p_i i}{C} = \beta_i + \sum_j \gamma_{ij} lnp_j + \gamma_{iQ} lnQ + \epsilon_{1+i} \tag{2}$$

Notice that since the cost share sum up to one, i.e.

$$\sum_i \frac{p_i i}{C} = 1$$

only 2 of the 3 cost share equations are independent. Therefore, we use the cost function and the 2 of the 3 cost share functions to estimate the parameters of the cost function, with the restrictions of the symmetry and homogeneity of degree 1.

# Maximum Likelihood Estimation

Assume that the error terms of the cost function and the share functions are distributed jointly normal. Then, the log likelihood increment of firm $k$ is

$$
\begin{aligned}
l_k &= -\frac{1}{2} ln(\pi) - \frac{1}{2} ln(|\Omega|) - \frac{1}{2} \widehat{\epsilon_k} \Omega^{-1} \widehat{\epsilon_k}' \\
\widehat{\epsilon_{k1}} &= lnC_k - \left[ \beta_0 + \sum_i \beta_i lnp_{ik} + \beta_Q lnQ_k \ldots \right] \\
\widehat{\epsilon_{k,i+1}} &= \frac{p_i i}{C} - \left[ \beta_i + \sum_j \gamma_{ij} lnp_j + \gamma_{iQ} lnQ \right]
\end{aligned}
$$

where $\Omega$ is the variance covariance matrix. log likelihood is

$$l = \sum_k l_k$$

Choose parameters of the model subject to symmetry and homogeneity of degree 1 restrictions to maximize the likelihood.

# GMM Estimation

GMM objective function is

$$F = \left[\sum_k \widehat{\epsilon_k}\right] W \left[\sum_k \widehat{\epsilon_k}\right]$$

- Choose parameters to minimize the objective function, subject to the restriction of symmetry and homogeneity of degree 1.
- The weighting function is set to be the estimate of the variance covariance matrix of $\sum_k \widehat{\epsilon_k}$.