Structural Estimation of Sealed Bid First Price Auction

The Model of Sealed Bit First Price Auction

- There are N participants.
- Each bidder has valuation of the auctioned object v_i, $i = 1, ..., N$ which are independently and identically distributed with distribution function $F(v)$.
- Individual i knows her own valuation v_i but not those of the others v_j, $j \neq i$.
- The bid, x of the individual i is a function of her valuation v_i.

Profit Function of the Bidder

Symmetric equilibrium: Suppose everybody's bidding function is $b(v)$.

$$\pi_i = (v_i - x_i) Pr(win \mid bid\ x, opponents\ bid\ b(v_j), j \neq i)$$

Then, given bidder i's valuation being v_i, the profit maximizing strategy is

$$x_i = b(v_i) = Argmax_{x_i \in R^+} (v_i - x_i) Pr(win \mid bid\ x_i, opponents\ bid\ b(v_j), j \neq$$

i winning the bid means that everybody elses' bids were lower than x_i, i.e.

$$x_i = b(v_i) = Argmax_{x_i \in R^+} (v_i - x_i) Pr(b(v_j) < x_i, \forall j \neq i)$$

Given the bid function being nondecreasing in valuation, $b(v_j) < x$ if and only if $v_j < b^{-1}(x)$

$$= Argmax_{x_i \in R^+} (v_i - x_i) Pr(v_j < b^{-1}(x_i), \forall j \neq i)$$

Now, because for bidder j,

$$Pr(v_j < b^{-1}(x_i)) = F(b^{-1}(x_i))$$

That is, in a symmetric equilibrium, the bidder i wins the bid when everybody elses' valuations are lower than his valuation which makes x_i his optimal bid.

$$x_i = b(v_i) = Argmax_{x_i \in R^+}(v_i - x_i)F^{N-1}(b^{-1}(x_i))$$

F.O.C.

$$0 = -F^{N-1}(b^{-1}(x)) + (v - x)\frac{\partial F^{N-1}}{\partial v}(b^{-1}(x))/b'(v)$$

$$b'(v)F^{N-1}(b^{-1}(x)) + b(v)\frac{\partial F^{N-1}}{\partial v} = v\frac{\partial F^{N-1}}{\partial v}$$

The above equation results in,

$$\frac{d}{dv}\left[b(v)F^{N-1}(v)\right] = v\frac{d}{dv}F^{N-1}(v)$$

By integrating both sides, we get

$$b(v)F^{N-1}(v) = \int_{b_0}^{v} sd(F^{N-1}(s))$$

Hence,

$$b(v) = \frac{\int_{b_0}^{v} sd[F^{N-1}(s)]}{F^{N-1}(v)}$$

Now, denote $G(s) = F^{N-1}(s)$. Then, by Integration by Parts,

$$\int_{b_0}^{v} sdG(s) = [sG(s)]_{b_0}^{v} - \int_{b_0}^{v} G(s)ds = vG(v) - \int_{b_0}^{v} G(s)ds$$

Optimal Bid Function

Hence,

$$b_i = v_i - \frac{1}{F(v_i)^{N-1}} \int_{b_0}^{v_i} [F(u)]^{N-1} \, du$$

But this is not very useful for estimation of the valuation because there are so many unknowns, valuation v_i, its distribution $F()$ and its density f. We need to eliminate as many unknowns as possible. Now, define $h(b)$ as the density of the bid b. By change of variable, we can relate the density of private value v and the that of the bid $h(b)$ as follows.

$$h(b)db = f(v) \left| \frac{\partial v}{\partial b} \right| db$$

Also, take the total derivative of the bid equation.

$$db_i = dv_i + \left[\frac{N-1}{F(v_i)^N} \int_{b_0}^{v_i} G(u)du \right] f(v_i)dv_i - \frac{G(v_i)}{G(v_i)} dv_i$$

substituting the bid equation, we get

$$db_i = \frac{N-1}{F(v_i)}\left[v_i - b_i\right] f(v_i) dv_i$$

$$v_i - b_i = \frac{F(v_i)}{(N-1)f(v_i)} \frac{db_i}{dv_i}$$

Now, because the bid is an increasing function of the valuation, number of people who bid less than b_i equals the number of people whose valuation is less than v_i,

$$F(v_i) = H(b_i)$$

Furthermore, because $\frac{db_i}{dv_i} > 0$ we know from change of variables,

$$f(v_i)\frac{dv_i}{db_i} = f(v_i)\left|\frac{dv_i}{db_i}\right| = h(b_i)$$

Together, we obtain

$$v_i - b_i = \frac{H(b_i)}{(N-1)h(b_i)}$$

or

$$v_i = b_i + \frac{H(b_i)}{(N-1)h(b_i)}$$

- Notice that all the terms of the RHS can be derived from the data. b_i is the bid of individual i. N is the number of bidders, and $H()$ is the distribution of bids submitted by the bidders, and $h()$ is the density of bids submitted by the bidders.
- The data we need to estimate H and h is the data on repeated auctions with exactly N players. We need to assume that in all those auctions, the distribution of the valuation of bidders are the same.