Neighborhoods, House Prices and Homeownership*

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Abstract

A model of a city is developed that features heterogeneous neighborhoods with differing levels of amenities and a population of households differing in income. Households make location choices and sort between renting and owning. Houses are constructed by a competitive development industry and either rented or sold to households through a process of competitive search. Along a balanced growth path, both the composition of the city and the rate of homeownership depend on the distributions of income, neighborhood amenities and construction costs. Homeownership is determined by the demand and supply sides of the market sorting optimally between competitive rental markets and frictional owner-occupied markets. Even in the absence of down-payment constraints, the model generates interesting patterns of homeownership: higher income households live in better neighborhoods and are more likely on average (but not strictly so) to be homeowners than lower income ones.

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1 Introduction

In this paper, we construct a model of a city comprised of heterogeneous long-lived households in which the rate of homeownership is endogenous and the value of housing assets determines the distribution of wealth. All households require housing, and each may either rent or own a house. Houses are of a finite number of different types, and all are built by a construction industry comprised of a large number of firms with free entry. Vacant houses may be rented competitively or sold through a process of competitive search. Using this environment, we consider relationships among income, city composition, homeownership, house prices, and "time-on-the-market" for houses of different types. We emphasize that ownership patterns are driven not by binding down-payment constraints, but rather by the optimal decisions of households faced with a choice between competitive rental markets and frictional owner-occupied markets. Higher income households live in better neighborhoods and are more likely on average (but not strictly so) to be homeowners than lower income ones.

Understanding the relationship between the characteristics and values of houses within and across cities is a long-standing issue in urban and real estate economics. Moreover, as houses account for a very large share of wealth for most households, their value and saleability are important for macroeconomic purposes. Recently, it has been documented that within cities, houses of different characteristics (or in different *market segments*) exhibit different house price movements (Landvoigt, Piazzesi, and Schneider, 2012) and sell at different rates (Piazzesi, Schneider, and Stroebel, 2013). Similarly, it has been observed that house prices have behaved very differently across cities over time (Gyourko, Mayer, and Sinai, 2006; Van Nieuwerburgh and Weill, 2010; Head, Lloyd-Ellis, and Sun, 2012). In contrast, we consider the extent to which the distribution of income may account for both the rate of homeownership and the speed with which houses in different locations sell (*i.e.*, their *liquidity*) in a setting where both are determined endogenously as results of buyers', renters' and sellers' decisions to enter particular segments of the housing market.

A number of other studies emphasize the role of search and matching frictions in housing markets (Wheaton, 1990; Krainer, 2001; Albrecht et al., 2007; Díaz and Jerez, 2013; Head, Lloyd-Ellis, and Sun, 2012). With only a few exceptions, past studies have ignored issues related to homeownership by omitting the rent-versus-own decision on the demand side and rent-versus-sell decision on the supply side of the market. Perhaps more importantly, it is often the case that search models of housing markets assume that all buyers are identical and/or that houses are homogeneous. These setups are therefore appropriate for studying only individual market segments. Incorporating heterogeneity in terms of buyers' permanent incomes, house characteristics, and neighborhood qualities is essential for extending the theory to study interactions between market segments at the city-level.

An important part of the proposed analysis is the decision of households regarding whether to rent or buy. For the most part, the existing literature posits buyers' willingness to buy either by assumption (Ortalo-Magné and Rady, 2006; Ríos-Rull and Sánchez-Marcos, 2008) or by embedding it in preferences (Iacoviello and Pavan, 2013; Kiyotaki, Michaelides, and Nikolov, 2010). All households want to own, and rent only because they have to; either they have no opportunity to buy (say, due to time-consuming matching between buyers and sellers) or they cannot afford to (say, due to a credit constraint). In our framework, we show that some households will choose to rent permanently despite wanting (to an extent) to own and facing no credit constraints *per se*. Moreover, these households may not be only those with the lowest income. In cases in which the lowest income households do choose to rent permanently, they will do so because it maximizes utility, rather than by assumption or because they are forced to by binding constraints.

In our model, households are differentiated permanently by income. Similarly, housing units come in different types, each associated with a different level of amenities, which we loosely interpret as reflecting location or "neighborhood" quality. Construction costs are higher for higher quality houses/neighborhoods, a feature which we interpret as them requiring more or better land. Households of all income levels enter the city exogenously, and choose first a neighborhood in which to rent. While renting, they may also choose to search for a house to buy in the same neighborhood in which they are renting, in another neighborhood, or not at all.

If a searching household (*i.e.*, a potential buyer) finds a match and buys the house, beginning the next period they stop renting, move into their house and receive each period an ownership premium.[1] A high income household ends up with a low marginal utility from non-housing consumption, has a high willingness to spend resources on housing, and therefore chooses to locate in a better (and more expensive) neighborhood. High income households do not, however, necessarily all choose to search and become homeowners. Those that do search, match successfully, and buy a

[1] As in Kiyotaki, Michaelides, and Nikolov (2010), the utility premium from owning relative to renting may arise because customizing a home is entirely within the owner's discretion, whereas landlords limit tenants' freedom to modify a rental unit for fear that their alterations will adversely affect it's market value or saleability. As such, the occupier of a house derives additional housing services when owning.

house, randomly receive shocks which render them unhappy with their current house (as in Wheaton, 1990). In this case they sell the house, return to renting, and again decide whether to search for a new suitable house to buy. In this way, households of each particular type (income level) cycle between renting and owning throughout their infinite lives.

The economy has a stationary balanced growth path in which all active neighborhoods (that is, all types of houses which developers actually choose to build and either rent or sell) grow at the rate of city population growth. This equilibrium is characterized by distributions of households across neighborhoods, ownership status and housing wealth. Houses in different neighborhoods take different lengths of time to sell owing to differences in the relative measures of buyers and sellers. Thus, houses of different types differ in their liquidity, and their prices reflect neighborhood-specific "liquidity discounts": the difference between the price at which a house is actually sold and that at which it would trade if there existed competitive markets in which households could simply buy houses without having to go through the time-consuming search process (Piazzesi, Schneider, and Stroebel, 2013). The existence of the liquidity discount depends crucially on search and matching frictions. Households searching for a home to buy take into account that if they find a house they like and buy, eventually they will no longer want it. The price at which they buy therefore reflects the time it will take at that point for the house to be matched with a buyer who likes it. To our knowledge, Head, Lloyd-Ellis, and Sun (2012) and Halket and Pignatti (2013) are the only others to consider this important distinction between buying a home and renting.

This version of the paper is preliminary and incomplete. Computed examples with a small number of house types indicate that the income distribution has a significant effect on composition of the city with regard to the relative sizes of neighborhoods. While higher-income households will typically choose to live in better neighborhoods, it is worthwhile to show that some high-income households may, in some circumstances, choose to remain renters. Moreover, as a consequence of the search frictions in the market for owner-occupied housing, renters who are searching to buy in lower quality locations can be found in all but the lowest quality neighborhood. The relationship between house quality and time-on-the market across neighborhoods will depend on parameters and reflect endogenous search decisions. As the distribution of income changes, the nature of the market equilibrium will adjust in response.

The remainder of the paper is organized as follows. Section 2 describes the environment and the competitive search process. Section 3 defines a stationary balanced growth path. Section 4 analyzes a series of examples in order to illustrate the relationships among income, wealth, the distributions of households across neighborhoods and between renting and owning, and the differences between liquidity discounts across neighborhoods. Section 5 concludes briefly and outlines future work.

2 The Economy

Consider an economy characterized by a single *city* and the *rest of the world* in discrete time. We assume that markets are complete and that the world interest rate is constant at net rate r.

2.1 The environment

The economy is populated by a growing number of infinitely-lived households. The aggregate (*i.e.*, world-wide) population is given by Q_t and grows at rate $\nu \geq 0$:

$$Q_t = (1 + \nu)Q_{t-1}. \tag{1}$$

Each period, a fraction ρ of the new households in the economy migrate to the city, keeping it constant in size relative to the rest of the world, with population ρQ_t. Households are of a large number of types, differing *ex ante* only with regard to their per period income, y. Household income is distributed on interval $[\underline{y}, \overline{y}]$ with cumulative distribution function F. The income distribution, F, is assumed to be continuous and have no mass points.

Households maximize their expected utility over their infinite lifetime,

$$U = \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + m(h_t)\right], \tag{2}$$

where $u(\cdot)$ and $m(\cdot)$ are both increasing and strictly concave, and $\beta = 1/(1+r)$. Here c_t is household consumption of a single non-storable good and $h_t \geq 0$ is household consumption of *housing services*.

Households in the city require housing and at each date must live in a single *house*. Houses are differentiated by location, with each being situated in a particular *neighborhood* indexed by $i = 1, \ldots, n$. A house located in neighborhood i yields a_i units of housing services per period to the household that lives in it. Here we have in mind that houses within a neighborhood share a certain number of characteristics we will refer to as amenities.

Households may either rent or own their homes. A household renting in neighborhood i in period t receives the amenity benefit, a_i, and pays rent, κ_{it}, to the owner, which may be either another household, or a *development firm* (described below). A household may also purchase the house in which it lives. In this case it pays no rent and each period realizes an *ownership premium*, z_t. The ownership premium is not specific to the location of the house. At any point in time, each individual household *likes* a small number of houses. If the household owns and lives in one of these, its ownership premium is positive; that is, $z_t = z > 0$. If the household owns a house that it does not like, then its ownership premium equals zero. In each period, the total housing services enjoyed by a household may then be represented

$$h_t = a_t + z_t, \tag{3}$$

where $a_t \in \{a_1, \ldots, a_n\}$ and $z_t \in \{0, z\}$. Without loss of generality, we assume that neighborhoods are ordered such that $a_i > a_{i-1}$ for $i = 2, \ldots, n$. Each period, with probability π_i, a well-matched homeowner (that is, a household which owns the house in which it lives and derives ownership premium $z > 0$) in neighborhood i receives an idiosyncratic shock which causes them to no longer like their home. In this case, the ownership premium they derive from that particular house falls permanently to zero. The household can, however, obtain an ownership premium in the future by finding, buying, and occupying a different suitable house. Mobility risk may be specified to differ across neighborhoods. For example, a case in which $\pi_i < \pi_{i-1}$ for $i = 2, \ldots, n$ is consistent with Piazzesi, Schneider, and Stroebel's (2013) finding cheaper market segments tend to be less "stable" (*i.e.*, moving shocks occur more frequently).

At each point in time, the total stock of housing in the city is given by H_t. Each of these houses may be owned by its occupant, rented to an occupant, or held vacant for sale. Let N_t denote the measure of owner-occupied houses (or, equivalently, the measure of homeowners), H_t^R denote the measure of houses for rent (or of renting households), and S_t the measure of houses vacant for sale. Among the renting households, we distinguish between those which are searching for a house to buy (measure B_t) and those who have chosen not to search and remain renters (measure R_t). We then have

$$H_t = N_t + S_t + H_t^R \tag{4}$$

$$\rho Q_t = N_t + B_t + R_t. \tag{5}$$

Thus, in each period, the measure of houses in the city exceeds that of resident households by vacancies, S_t.

The construction of a new house located in neighborhood i takes one period, and comes at marginal cost T_{it} which we interpret as the cost of neighborhood i land:

$$T_{it} = \tau_i \mathcal{T}\left(\frac{H_{it+1} - H_{it}}{H_{it}}\right). \tag{6}$$

Here we assume that $\tau_i > \tau_{i-1}$ for $i = 2, \ldots, n$, that $\mathcal{T}' > 0$, $\mathcal{T}'' > 0$, and that $\mathcal{T}(\nu) = 1$. The interpretation of the τ_i's differing across locations is that the quantity of land required for a house in each neighborhood is directly related to the level of neighborhood amenities. We also assume that the stock of available land for construction in each neighborhood grows at the same rate as the city's population. The per unit cost of land when development takes place at population growth is normalized to one.

The construction technology (6) is operated by a large number of development firms which produce houses, rent or sell them to households in need of residences, and buy existing houses from households who no longer like them. Firms are owned by households and remit their profits (if any) lump-sum. There is free entry into construction, so firms will construct homes until the discounted future value of a vacant house equals its current marginal cost of production:

$$T_{it} = \beta \tilde{V}_{it+1}. \tag{7}$$

In each neighborhood, firms behave competitively in the rental market, and may also sell vacant houses through a process of competitive search akin to that studied by Moen (1997). Within each neighborhood, firms offer houses for sale at posted prices, thereby creating *sub-markets* distinguished by price. Households observe all prices and direct their search to a particular sub-market (in a particular neighborhood), taking as given the entry of sellers and other competing buyers. When posting prices, firms take into account that households' decisions regarding in which sub-market to search are influenced by both the posted price and their beliefs regarding the number of houses offered for sale relative to the number of prospective buyers in each sub-market.

Within a sub-market, the number of successful matches is given by the matching function

$$M = \mathcal{M}(B, S), \tag{8}$$

where B and S indicate the measures of searching buyers and sellers present in the sub-market. Here $\mathcal{M}$ exhibits constant returns to scale and is increasing and strictly concave in both its arguments. We refer to $\theta \equiv B/S$ as the *tightness* of the particular sub-market. Given the properties of $\mathcal{M}$, the probabilities with which individual searchers and vacant houses, respectively, are successfully matched may be written as functions of θ:

$$\lambda(\theta) = \frac{\mathcal{M}(B,S)}{B} \qquad \text{and} \qquad \gamma(\theta) = \frac{\mathcal{M}(B,S)}{S} = \theta\lambda(\theta). \tag{9}$$

Households direct their search to a particular sub-market knowing both the price they will pay in a successful match and the probability of attaining such a match, given the matching function and their beliefs regarding the searching behavior of other prospective buyers.

Assumption 1. *The matching probabilities satisfy the following properties:*

i. $\lambda(\theta) \in [0,1]$ *and* $\gamma(\theta) \in [0,1]$ *for all* $\theta \in [0,\infty]$*;*

ii. $\lim_{\theta\to\infty} \lambda(\theta) = \lim_{\theta\to 0} \gamma(\theta) = 0$*;*

iii. $\lim_{\theta\to 0} \lambda(\theta) = \lim_{\theta\to\infty} \gamma(\theta) = 1$*; and*

iv. $\lambda'(\theta) < 0$ *and* $\gamma'(\theta) > 0$*.*

An example we will make use of is the so-called telephone line matching function:

$$\mathcal{M}(B,S) = \left(\frac{\alpha BS}{\alpha B + S}\right), \tag{10}$$

where $\alpha \in (0,1]$ is a matching efficiency parameter to reflect the fact that, as noted above, a household can only be *well-matched* with (and receive an ownership premium from buying) only certain houses at each point in time. A successful match therefore depends not only on the competition between buyers and sellers for matches, but also on the level of z within a particular match. The likelihood of suitability within each match may in principle differ across neighborhoods. Within a neighborhood, however, it is assumed to be constant. For this reason, the matching function may be specified to differ across neighborhoods. For example, consider a case in which $\alpha_i < \alpha_{i-1}$, $i = 2, \ldots, n$. Fewer successful matches (for a given tightness) in a sub-market within a high quality neighborhood could reflect the houses being more diverse and thus specifically appealing to a smaller fraction of buyers.

Finally, there exist competitive markets in a complete set of state-contingent claims that pay off in units of the non-storable consumption good. These enable households to insure fully their matching risk in the housing market (associated with λ) and the risk of losing their ownership premium (associated with π). Households face no financial constraint on purchasing a house beyond that implied by their budgets.

2.2 Competitive search

Let $\tilde{V}_{it}$ denote the value of an unoccupied neighborhood i house at the beginning of period t. Such a house may be owned by a development firm or a household and may be either rented or held vacant for sale in the current period. As such, its value is given by

$$\tilde{V}_{it} = \max\Big\{ \underbrace{\kappa_{it} + \beta\tilde{V}_{it+1}}_{\text{rent}}, \underbrace{V_{it}}_{\text{sell}} \Big\}. \tag{11}$$

Neighborhood i houses held vacant-for-sale at posted price P_{it} represent a *sub-market*. More specifically, each sub-market is characterized by a neighborhood, i, a common price, P_{it}, and a ratio of searching buyers to sellers, θ_{it}, which effectively responds to the price. Each period, a vacant-for-sale house either sells or it does not:

$$V_{it} = \max_{(P_{it},\theta_{it})} \left\{ \gamma(\theta_{it})P_{it} + (1-\gamma(\theta_{it}))\,\beta\tilde{V}_{it+1} \right\}. \tag{12}$$

Since all houses and sellers within a neighborhood are identical, sellers must be indifferent across active sub-markets (those with entry by a positive measure of sellers). This gives rise to an equilibrium relationship between price and tightness across active sub-markets within a neighborhood:

$$\gamma(\theta_{it}) = \frac{V_{it} - \beta\tilde{V}_{it+1}}{P_{it} - \beta\tilde{V}_{it+1}}, \qquad i = 1,\ldots,n. \tag{13}$$

Given (13), we can identify uniquely sub-markets within a neighborhood using only the posted price, P, and write tightness for any active sub-market in neighborhood i as $\theta_{it}(P)$. Moreover, as sellers may freely decide whether to rent or hold a house vacant-for-sale, in all active sub-markets we have $\tilde{V}_{it} = V_{it}$ and

$$\kappa_{it} = V_{it} - \beta V_{it+1}, \qquad i = 1,\ldots,n. \tag{14}$$

2.3 The household decision problem

Given complete markets, households can carry out financial market transactions to smooth consumption. In particular, at the beginning of the period, a household searching for a home to buy in sub-market P in neighborhood j can purchase $s^B_{jt}(P)$ units of insurance that each pay one unit of the consumption good in period t contingent on buying a house, and $s^R_{jt+1}(P)$ units of a security that each pay one unit of

$t+1$ consumption contingent on still renting at time $t+1$ and zero otherwise. For a homeowner at time t, let s^S_{jt+1} denote claims to one unit of $t+1$ consumption contingent on becoming mismatched with their current home. Finally, there is a riskless asset; s^F_{t+1} units of a discount bond pay s^F_{t+1} units of the consumption good at time $t+1$. The prices in period t for these securities are denoted $\{q^B_{jt}(P), q^R_{jt}(P), q^S_{jt}, q^F_t\}$.

At the beginning of each period, a household is either a renter or a homeowner. Renters, depending on their income and wealth, choose the following: (i) a neighborhood in which to live while renting; (ii) whether or not to search for a house to buy; and, if searching, (iii) a particular sub-market (of a particular neighborhood) within which to search. To simplify the notation, the decision not to search for a house to purchase will be represented by the choice of searching in neighborhood 0 in sub-market $P=0$. Accordingly, $\theta_{0t}(0)=\infty$ and $\lambda(\theta_{0t}(0))=0$. A renter with income y and wealth s_t therefore has value:

$$V^R_t(y,s_t) = \max_{\substack{i\in\{1,\dots,n\},\\ j\in\{0,\dots,n\},P,\\ c_t,s^B_t,s^R_{t+1},s^F_{t+1}}} \Bigg\{ m(a_i) + \lambda(\theta_{jt}(P))\Big(u(c_t + s^B_{jt}(P) - P) + \beta V^{N_j}_{t+1}(y, s^F_{t+1})\Big) \\ + \Big[1-\lambda(\theta_{jt}(P))\Big]\Big(u(c_t) + \beta V^R_{t+1}(y, s^F_{t+1} + s^R_{t+1})\Big)\Bigg\} \tag{15}$$

subject to

$$c_t + q^B_{jt}(P)s^B_{jt}(P) + q^R_{jt}(P)s^R_{jt+1}(P) + q^F_t s^F_{t+1} + \kappa_{it} = y + s_t,$$

where j indexes the neighborhood and P the particular sub-market in which the buyer would search, and $V^{N_j}_{t+1}(y, s_{t+1})$ is the value of being a neighborhood j homeowner with income y and wealth s_{t+1} from the beginning of the next period. In particular,

$$V^{N_j}_t(y,s_t) = \max_{c_t,s^S_{t+1},s^F_{t+1}} \Bigg\{ u(c_t) + m(a_j + z) + \beta\Bigg(V^{N_j}_{t+1}(y, s^F_{t+1}) \\ + \pi_j\Big[V^R_{t+1}(y, s^F_{t+1} + s^S_{jt+1} + V_{jt+1}) - V^{N_j}_{t+1}(y, s^F_{t+1})\Big]\Bigg)\Bigg\} \tag{16}$$

subject to

$$c_t + q^S_{jt}s^S_{jt+1} + q^F_t s^F_{t+1} = y + s_t.$$

3 A stationary equilibrium

We focus on a stationary equilibrium in which the relative sizes of neighborhoods within the city remain constant, with all growing at the rate of population growth, ν. Households of a given type make the same search decisions in every period and the distributions of sub-markets within and across neighborhoods remain constant.

3.1 Construction

In a stationary equilibrium, as the stock of houses of a given type grows at the population growth rate, ν, (7) becomes

$$\tau_i = \beta V_i, \qquad i = 1, \ldots, n. \tag{17}$$

In this case, the cost of renting (from equation (14)) is

$$\kappa_i = (1-\beta)V_i = \frac{(1-\beta)\tau_i}{\beta} = r\tau_i, \qquad i = 1, \ldots, n. \tag{18}$$

3.2 Price posting

The relationship between price and tightness, (13), may be written

$$\gamma(\theta_i) = \frac{(1-\beta)V_i}{P - \beta V_i}, \qquad i = 1, \ldots, n. \tag{19}$$

Using (19) we will express tightness within any active sub-market as a function of the posted price, $\theta_i(P)$.

3.3 Consumption

Appendix A contains the solution to the household's portfolio allocation problem and the derivation of the optimal consumption sequence. With complete markets, households achieve perfect consumption smoothing. For a household that, when renting in neighborhood i, searches for a home to buy in sub-market P of neighborhood j, (constant) per period consumption is

$$c(y,i,j,P) = y - \frac{\big[1-\beta(1-\pi_j)\big]\big(\kappa_i + \lambda(\theta_j(P))P\big) - \beta^2\pi_j\lambda(\theta_j(P))V_j}{1-\beta\big[1-\pi_j-\lambda(\theta_j(P))\big]}, \tag{20}$$

which is the highest attainable constant consumption sequence satisfying the present value budget constraint given the cost of insuring against all expenditures and proceeds from housing-related transactions.

3.4 Within neighborhood search

With regard to the search decision, each household searching *within neighborhood j while renting in neighborhood i* chooses their preferred sub-market. The optimal choice of sub-market satisfies

$$\begin{aligned} P = \beta V_j + \frac{\beta\big[1-\eta\big(\theta_j(P)\big)\big]}{1-\beta\big[1-\pi_j-\eta\big(\theta_j(P)\big)\lambda\big(\theta_j(P)\big)\big]}\Bigg\{&\kappa_i - (1-\beta)\,V_j \\ &+ \frac{m(a_j+z)-m(a_i)}{u'\big(c(y)\big)}\Bigg\}, \end{aligned} \tag{21}$$

where $\eta(\theta) = \gamma'(\theta)\theta/\gamma(\theta)$. This equation is derived in Appendix B from the optimal within-neighborhood search decisions of buyers and price posting strategies and sellers. Define $P(y,i,j)$ as the solution to (21) if $j>0$, and $P(y,i,j)=0$ otherwise.

3.5 Location choice

In light of (20) and (21),[2] the stationary value function for a household that is currently renting in neighborhood i and searching in neighborhood j simplifies to

$$\begin{aligned} V^R(y) = \max_{\substack{i\in\{1,\dots,n\},\\ j\in\{0,\dots,n\}}} \Bigg\{&u\big(c(y,i,j)\big) + m(a_i) \\ &+ \beta\bigg(V^R(y) + \lambda(\theta_j(P))\left[V^{N_j}(y) - V^R(y)\right]\bigg)\Bigg\}, \end{aligned} \tag{22}$$

[2]By combining $P(y,i,j)$ from (21) with $c(y,i,j,P)$ from (20), we can write $c(y,i,j) = c\big(y,i,j,P(y,i,j)\big)$.

where $P = P(y, i, j)$, $\theta_j(P)$ satisfies (19), and

$$V^{N_j}(y) = u\big(c(y, i, j)\big) + m(a_j + z) + \beta\Big(V^{N_j}(y) + \pi_j\Big[V^R(y) - V^{N_j}(y)\Big]\Big). \tag{23}$$

Combining (22) and (23) yields the value of moving into a suitable home (*i.e.*, the buyer's surplus from a match):

$$V^{N_j}(y) - V^R(y) = \frac{m(a_j + z) - m(a_i)}{1 - \beta(1 - \pi_j - \lambda(\theta_j(P))}. \tag{24}$$

The value of renting for a household with income y is thus

$$\begin{aligned} V^R(y) = \frac{1}{1-\beta} \max_{\substack{i \in \{1,\ldots,n\},\\ j \in \{0,\ldots,n\}}} \Bigg\{ & u\big(c(y, i, j)\big) + m(a_i) \\ & + \beta\lambda(\theta_j(P)) \frac{m(a_j + z) - m(a_i)}{1 - \beta(1 - \pi_j - \lambda(\theta_j(P))} \Bigg\}. \end{aligned} \tag{25}$$

Define the function $i(y)$ as indicating each household's preferred location to live in *as a renter*. Next, define $j(y)$ as the preferred neighborhood in which to search for a home *to buy*. For any household which chooses not to search (because the surplus from buying a home is negative in all neighborhoods), $j(y) = 0$. Conditional on searching, $j(y)$ identifies the neighborhood(s) within which the household would be willing to search, and $P(y)$ identifies the preferred sub-market(s) according to (21) with $i = i(y)$ and $j \in j(y)$. We then have the following:

Proposition 1. *Given that the income distribution function $F(\cdot)$ has no mass points, no positive measure of households is indifferent between searching in two or more active sub-markets.*

Given Proposition 1, we can write the searching household's optimal decisions as $i(y)$, $j(y)$ and $P(y)$; their optimal consumption level as $c(y) = c(y, i(y), j(y))$; and the resulting market tightness as a function of the income of households searching there, $\theta(y) = \theta_{j(y)}(P(y))$.

The decision rule $i(y)$ indicating where to *rent* can be represented by the following functional equation:

$$i(y) = \operatorname*{argmax}_{i'} \left\{ \frac{u\Big(c(y) - (1-\beta)\left(\kappa_{i'} - \kappa_{i(y)}\right)\Big)}{1-\beta} + m(a_{i'}) \right\}. \tag{26}$$

According to (26), $i(y)$ is the neighborhood for which it is not optimal to deviate even temporarily to any other neighborhood, paying for the difference in rental expense with a perpetuity. Except indirectly via its impact on consumption, a household's choice in this regard has no direct bearing on their decision where or even if they search for a house to *buy*.

We may then define the function $j(y)$ indicating where to *buy* as

$$j(y) = \begin{cases} 0 & \text{if } \max_{j'} \Psi_{j'}(y) < 0 \\ \underset{j'}{\operatorname{argmax}}\, \Psi_{j'}(y) & \text{otherwise} \end{cases} \tag{27}$$

where, letting $P_j(y) = P(y, i(y), j)$,

$$\begin{aligned} \Psi_{j'}(y) = & \frac{u\big(c(y) - (1-\beta)\left[\lambda(\theta_{j'}(P_{j'}(y)))P_{j'}(y) - \lambda(\theta(y)P(y)\right]\big)}{1-\beta} \\ & + \beta\lambda(\theta_{j'}(P_{j'}(y)))\left(\frac{m(a_{j'}+z) - m(a_{i(y)})}{1-\beta(1-\pi_{j'})} - \beta\lambda(\theta(y))\left[V^N(y) - V^R(y)\right]\right). \end{aligned} \tag{28}$$

The problem in (27) is the choice of where to search in the current period, conditional on searching optimally thereafter. The difference in the cost of insuring against the uncertainty about finding a suitable house in the owner-occupied market in the current period is again financed with a perpetuity.

3.6 Homeownership and income

We now establish that the returns to neighborhood amenities and homeownership are increasing in income under the restrictions that $\alpha_i = \alpha$ and $\pi_i = \pi$ for $i = 1, \ldots, n$.

Proposition 2. *Given income levels,* $y, y' \in [\underline{y}, \overline{y}]$ *with* $y' > y$,

i. $i(y') \geq i(y)$.

ii. if $j(y) > 0$ *and* $j(y') > 0$*, then* $j(y') \geq j(y)$

Part *i.* of Proposition 2 states that if a given neighborhood is the preferred rental location of a household with income y, then a household with income $y' > y$, when renting, will reside in a neighborhood with at least as high a level of amenities. Part *ii.* states a similar result for two different *searching* households: If a household with income y searches in a given neighborhood, then a higher income household will search in a neighborhood with at least as high an amenity level, *if it searches at all.*

It is also convenient to assume the following regarding the relative importance of neighborhood amenities and the homeownership premium:

Assumption 2. *$a_i - a_{i-1} > z$ for $i = 2, \ldots, n$.*

Under this condition, we have that a household searching to buy in a given neighborhood will rent in a neighborhood with at least as high an amenity level:

Proposition 3. *Under Assumption 2, given income level $y \in [\underline{y}, \overline{y}]$, $i(y) \geq j(y)$.*

Propositions 2 and 3 establish that households sort between neighborhoods according to income, with respect to both renting and searching/buying decisions. If there are non-searching renters in the lowest amenity neighborhood, they comprise the lowest income households, although in equilibrium there might not actually be any such households (which we refer to as *permanent renters*). Moreover, Proposition 1 does not rule out the possibility that a household with income y searches in neighborhood i, but a household with income $y' > y$ rents permanently in neighborhood $j > i$. Accordingly, we show below that examples can be constructed with such features.

3.7 Balanced growth path

The stocks of permanent renters, searchers, and homeowners by income level evolve according to:

$$R_{t+1}(y) = \begin{cases} R_t(y) + \rho\nu Q_t f(y) & \text{if } j(y) = 0 \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

$$B_{t+1}(y) = \begin{cases} 0 & \text{if } j(y) = 0 \\ \left[1 - \lambda(\theta(y))\right] B_t(y) + \rho\nu Q_t f(y) + \pi(y) N_t(y) & \text{otherwise} \end{cases} \tag{30}$$

$$N_{t+1}(y) = \begin{cases} 0 & \text{if } j(y) = 0 \\ (1 - \pi(y)) N_t(y) + \lambda(\theta(y)) B_t(y) & \text{otherwise} \end{cases} \tag{31}$$

where $\pi(y) = \pi_{j(y)}$ and $f(\cdot)$ is the density of $F(\cdot)$. Dividing all quantities by Q_t and using lower case letters to represent the normalized values, the relative quantities along the balanced growth path can be expressed as

$$r(y) = \begin{cases} \rho f(y) & \text{if } j(y) = 0 \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

$$b(y) = \begin{cases} 0 & \text{if } j(y) = 0 \\ \frac{\rho[\nu+\pi(y)]}{\nu+\pi(y)+\lambda(\theta(y))} f(y) & \text{otherwise} \end{cases} \tag{33}$$

$$n(y) = \begin{cases} 0 & \text{if } j(y) = 0 \\ \frac{\rho\lambda(\theta(y))}{\nu+\pi(y)+\lambda(\theta(y))} f(y) & \text{otherwise} \end{cases} \tag{34}$$

The (normalized) stocks of rental housing, vacant houses sale, and total housing in neighborhood k, denoted $h_k^R = H_{kt}^R/Q_t$, $s_k = S_{kt}/Q_t$ and $h_k = H_{kt}/Q_t$, are

$$h_k^R = \int_{i(y)=k} b(y) + r(y)dy \tag{35}$$

$$s_k = \int_{j(y)=k} b(y)/\theta(y)dy \tag{36}$$

$$h_k = \int_{j(y)=k} n(y) + b(y)/\theta(y)dy + \int_{i(y)=k} b(y) + r(y)dy \tag{37}$$

We may then define an *Equilibrium Balanced Growth Path*:

Definition 1. *An Equilibrium Balanced Growth Path is time invariant*

i. *household values and decision rules conditional on income, $y \in [\underline{y}, \overline{y}]$: $V^R(y)$, $V^N(y)$, $i(y)$, $j(y)$, $P(y)$ and $c(y)$;*

ii. *house values, V_i for $i = 1, \ldots, n$;*

iii. *neighborhood rental rates, κ_i for $i = 1, \ldots, n$;*

iv. *a tightness function $\theta(y)$;*

v. *shares of households renting, buying, and owning by income: $r(y)$, $b(y)$, and $n(y)$ for $y \in [\underline{y}, \overline{y}]$:*

and

vi. *shares of rental, vacant and total housing across neighborhoods: h_k^R, s_k and h_k for $k = 1, \ldots, n$;*

such that

1. *the values $V^R(y)$ and $V^N(y)$ satisfy the Bellman equations,* (15) *and* (16) *with the associated policies $i(y)$, $j(y)$, $P(y)$ and $c(y)$ satisfying* (26), (27), (21) *and* (20)*;*
2. *free entry in the construction of new housing results in $\tau_i = \beta V_i$ for $i = 1, \ldots, n$;*
3. *owner's of vacant houses choose optimally between renting them and holding them vacant for sale: $\kappa_i = (1-\beta)V_i = r\tau_i$ for $i = 1, \ldots, n$;*
4. *the tightness function reflects optimal search and price posting strategies: $\theta(y) = \theta_{j(y)}(P(y))$ satisfies* (19)*;*
5. *the measures of households, $r(y)$, $b(y)$ and $n(y)$, satisfy* (32), (33) *and* (34)*;*

and

6. *housing stocks grow at the rate of population growth, ν, by neighborhood, and h_k^R, s_k and h_k satisfy* (35), (36) *and* (37) *for $k = 1, \ldots, n$.*

At this stage we conjecture that there exists a unique stationary balanced growth path satisfying this definition. At this point, we do not attempt to establish this formally and move on, in the next section, to consider some examples (which demonstrate existence, but not uniqueness). The purpose of these examples is to investigate the basic mechanisms at work in the economy.

3.8 Liquidity discounts

The liquidity discount reflects the difference between the price at which houses are actually sold and the value of the house if there were no frictions in the housing market. In the absence of search frictions and separation shocks, the value of a house to a household with income y is equal to the amount that a household would be willing to pay to immediately switch from being a permanent renter to a perpetual homeowner in neighborhood $j(y)$. This amount, denoted $\hat{V}^1(y)$, satisfies the indifference condition

$$u\big(y - \kappa_{j(y)}\big) + m(a_{j(y)}) = u\big(y - (1-\beta)\hat{V}^1(y)\big) + m(a_{j(y)} + z). \tag{38}$$

The liquidity discount is the difference between this imputed value and the actual transaction price, normalized by dividing by the imputed value so that the liquidity discount is between 0 and 1. Since both the imputed value and the transaction price household-specific, so is the liquidity discount:

$$l^1(y) = \frac{\hat{V}^1(y) - P(y)}{\hat{V}^1(y)}.$$

Note, however, that the household with income y may not choose to rent and buy in the same neighborhood; that is, it could be that $i(y) \neq j(y)$. Renting permanently in neighborhood $j(y)$ may not be the most appropriate benchmark for computing the value of owning in neighborhood $j(y)$. Another measure is the amount that a household would be willing to pay to immediately switch from their current situation of renting in $i(y)$ while searching to buy in neighborhood $j(y)$ to owning permanently in neighborhood $j(y)$. This amount, denoted $\hat{V}^2(y)$ satisfies the following indifference condition:

$$V^R(y) = \frac{u\big(y - (1-\beta)\hat{V}^2(y)\big) + m(a_{j(y)} + z)}{1-\beta}. \tag{39}$$

The liquidity discount according to this measure of the imputed value of a house to a household with income y is

$$l^2(y) = \frac{\hat{V}^2 - P(y)}{\hat{V}^2(y)} = 1 - \frac{(1-\beta)P(y)}{y - u^{-1}\big((1-\beta)V^R(y) - m(a_{j(y)} + z)\big)}.$$

The average liquidity discount in each neighborhood is computed by weighting the household-specific liquidity discount by the measure of transactions at each income level:

$$l_j^1 = \int_{j(y)=j} l^1(y)\mathcal{M}\big(b(y), s(y)\big)dy \quad \text{and} \quad l_j^2 = \int_{j(y)=j} l^2(y)\mathcal{M}\big(b(y), s(y)\big)dy. \tag{40}$$

4 Numerical Examples

In this section we compute some numerical examples to illustrate the basic workings of the model. Here we work with economies in which there are three levels of house/neighborhood quality. This is done both for simplicity and because in many cases it is sufficient to illustrated the workings of the model. It is not particularly difficult to compute equilibria in which cities are comprised of a larger number of neighborhoods. This will be the approach we take when we take up quantitative analysis later.

In all of the examples presented here, we use the matching function specified in (10) and assume that the inherent matching frictions and mobility risk embedded in parameters α and π are the same across neighborhoods: $\alpha_i = \alpha = 1$ and $\pi_i = \pi$, for $i = 1, \ldots, n$. We keep the population growth rate set to $\nu = 0.00125$ throughout. The fraction of new households entering the city, ρ, is set to one. In all cases, we also set $\beta = .99$ so that the interest rate is 1.01%, functional forms $u(c) = \log(c)$ and $m(h) = h$, and the utility premium from homeownership, $z = .001$. For the income distribution, we choose income cut-offs between quintiles, $\{Q(0), Q(0.2), Q(0.4), Q(0.6), Q(0.8), Q(1)\}$, and impose uniform distribution within each quintile. Our examples will be distinguished by these income distribution parameters, the vector of amenity values, $\{a_1, a_2, a_3\}$, and the vector of construction cost parameters, $\{\tau_1, \tau_2, \tau_3\}$.

In a stationary equilibrium, in order for households to live in any neighborhood but that yielding the highest possible level of amenities, it must be the case that construction costs are sufficiently high relative to the income of at least some households. Similarly, construction costs must be sufficiently low relative to income for people to afford to buy or rent in high quality neighborhoods. In all our examples, we choose the τ_i's such that this is true, and the a_i's and z to maintain Assumption 2.

4.1 Low and high(er) income permanent renters

In this example, we consider households' decisions to become home-owners. Table 1 contains the parameters of the economy. Income here is distributed uniformly on the inverval $[\underline{y}, \overline{y}] = [1, 5]$.

Table 1: Example 4.1: Parameter Values

description	parameters	values
income distn.	$\{\underline{y}, \overline{y}\}$	$\{1.0, 5.0\}$
	$\{Q(0.2), Q(0.4), Q(0.6), Q(0.8)\}$	$\{1.8, 2.6, 3.4, 4.2\}$
amenity levels	$\{a_1, a_2, a_3\}$	$\{0.0100, 0.0217, 0.0364\}$
construction costs	$\{\tau_1, \tau_2, \tau_3\}$	$\{4.6349, 6.4175, 12.8351\}$

As above, we refer to households which rent and choose not to search in equilibrium as *permanent renters*. If housing costs are sufficiently high, then the poorest households will reside in the worst and least expensive neighborhood (neighborhood 1). Moving up the income distribution, the assignment rules $i(y)$ and $j(y)$ describe the composition of the city by indicating, respectively, in which neighborhoods households live as renters and in which they choose to live as owners in the event that they search and successfully match.

Given their monotonicity properties (established in Proposition 2) and the fact that there are a finite number of neighborhoods these functions can be represented by a series of income cut-offs. In this example, the equilibrium assignment of renters across neighborhoods is given by:

$$i(y) = \begin{cases} 1 & \text{if } y \in [1.000, 1.595) \\ 2 & \text{if } y \in [1.595, 4.474) \\ 3 & \text{if } y \in [4.474, 5.000]. \end{cases} \tag{41}$$

With regard to search, we have

$$j(y) = \begin{cases} 0 & \text{if } y \in [1.000, 1.300) \cup [1.636, 1.800) \\ 1 & \text{if } y \in [1.300, 1.636) \\ 2 & \text{if } y \in [1.800, 4.637) \\ 3 & \text{if } y \in [4.637, 5.000]. \end{cases} \tag{42}$$

Considering (41) and (42), it is clear that the poorest households not only live in the lowest quality neighborhood, but also do so as permanent renters. Low-income households rent permanently because renting is cheaper than buying, owing to fact that house sellers must be compensated for the opportunity cost of leaving their house vacant and for sale until matched with a buyer – a delay of at least one period. Households with income $y \in [1.300, 1.636)$ search for houses to buy in neighborhood 1, but of these, those with income of $y = 1.595$ or higher do so while renting in neighborhood 2 (and receiving amenities of a_2 rather than a_1).

In this example, amenities in neighborhood 2, a_2, are sufficiently appealing relative to the amenities and ownership premium in neighborhood 1, $z + a_1$, that higher income households (specifically those with incomes $y \in [1.636, 1.800)$) stop searching in neighborhood 1 and rent *permanently* in neighborhood 2. Thus, this example illustrates that it is possible in a stationary equilibrium for some households to choose not to search even though other, lower income, households do. Further up the income distribution are households whose optimal strategy is to search in neighborhood 2, and finally in neighborhood 3.

Table 2 contains several statistics regarding housing in this example. Neighborhood 1 is dominated by renters, with almost 50% of them choosing not to search.

Neighborhoods 2 and 3, in contrast are populated mainly by home-owners and those households that are renting while searching. All households renting in neighborhood 3 are searching, although some are searching in neighborhood 2. The overall home-ownership rate is 70.00%, a number not too far from that observed in the U.S. in recent years, and 11.59% of households never search for a house to buy.

Table 2: Neighborhood Statistics in Example 4.1

neighborhood	1	2	3	aggregate
share of city popn.	0.1545	0.7492	0.0963	1.0000
perm. rentership rate	0.4856	0.0546	0.0000	0.1159
rentership rate	0.6791	0.2275	0.2558	0.3000
homeownership rate	0.3209	0.7725	0.7442	0.7000
vacancy rate	0.0059	0.0152	0.0141	0.0137
avg. selling prob.	0.9644	0.8953	0.9320	0.9035
avg. price	4.6834	6.4899	12.9742	7.0261
avg. price-rent ratio	25.0092	25.0292	25.0182	25.0262
liquidity discount 1	0.0294	0.0463	0.0342	0.0439
liquidity discount 2	0.0276	0.0312	0.0292	0.0308

Figure 1 contains house prices and matching probabilities by income for this example. The average sale price is very closely tied to the construction cost, which is not surprising given that there is free entry into construction. Of course, the average price *must* exceed the construction cost because it takes at least one period to sell a vacant house. House prices also vary within a neighborhood because, conditional on searching, households with different incomes enter different sub-markets. Recall that sub-markets, indexed by either price or market tightness through (19), are also associated with the particular income level of the buyers who visit them.

Given (19), within each neighborhood a higher price is associated with a higher matching rate for buyers, and a lower matching rate (or longer *time-on-the-market* for sellers). The relationship between income and price in the sub-market targeted is not necessarily monotonic, in spite of the fact that $v(\cdot)$ is increasing. This can be seen in Figure 1 as the matching rate for sellers typically falls and then rises with income in each neighborhood. Note also that there are no active sub-markets associated with income levels $y \in [1.000, 1.300) \cup [1.637, 1.800)$, as these households choose to rent permanently.

In Figure 1, price variation within neighborhoods is effectively invisible, simply

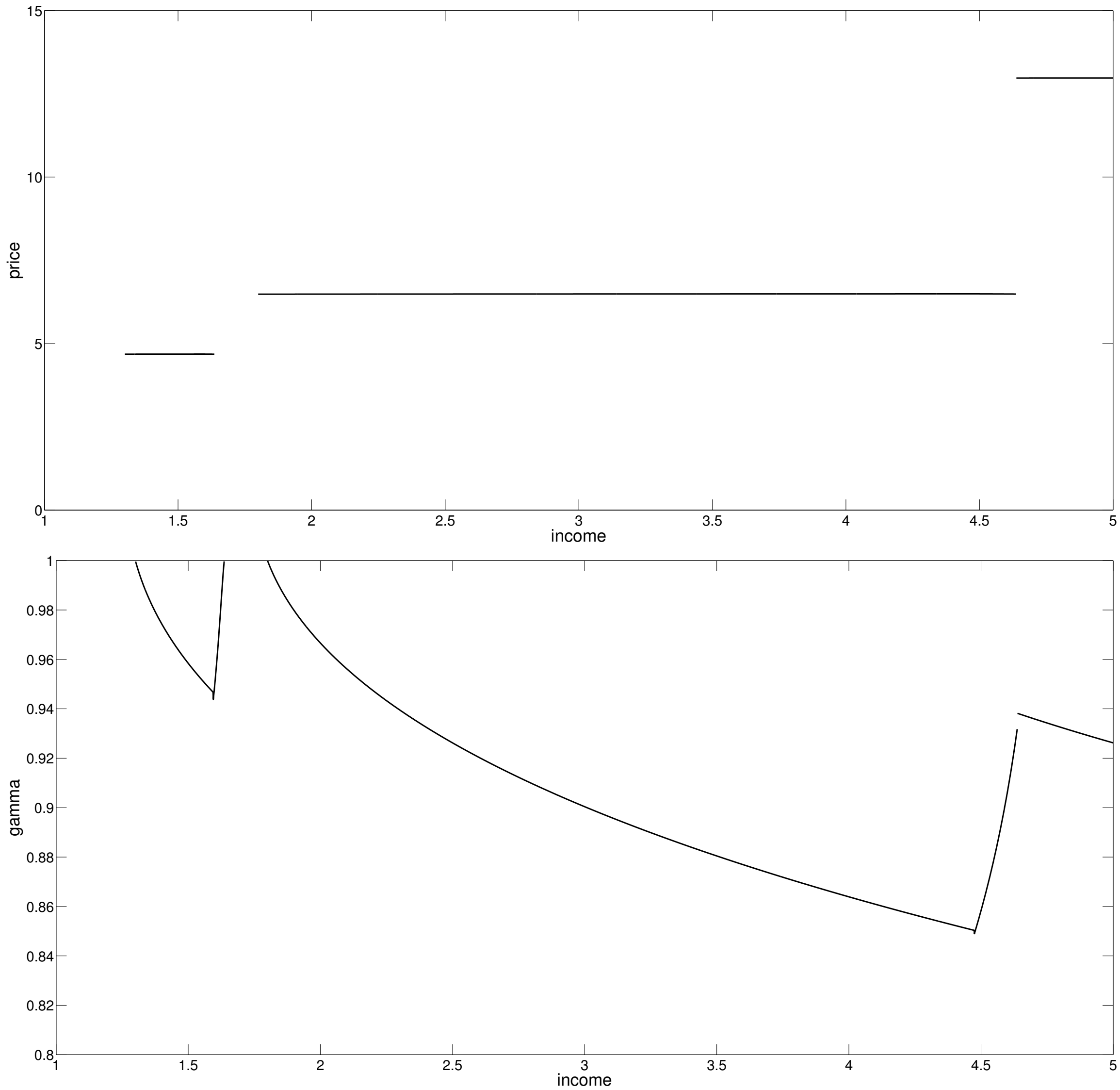

Figure 1: Prices and Matching Probabilities in Example 4.1

because it is very small relative to that across neighborhoods. Figure 2 illustrates price variation by income within neighborhood 2. Here, the lowest income households who search (those with income $y = 1.800$) are indifferent between owning and renting. As such, they cannot be induced to pay more than the value of a vacant house, V_2. Sellers, of course, are willing to offer a house at such a price only if they expect to sell it with probability one. This is only true as tightness goes to infinity; thus, sellers effectively match with certainty, while buyers never match.

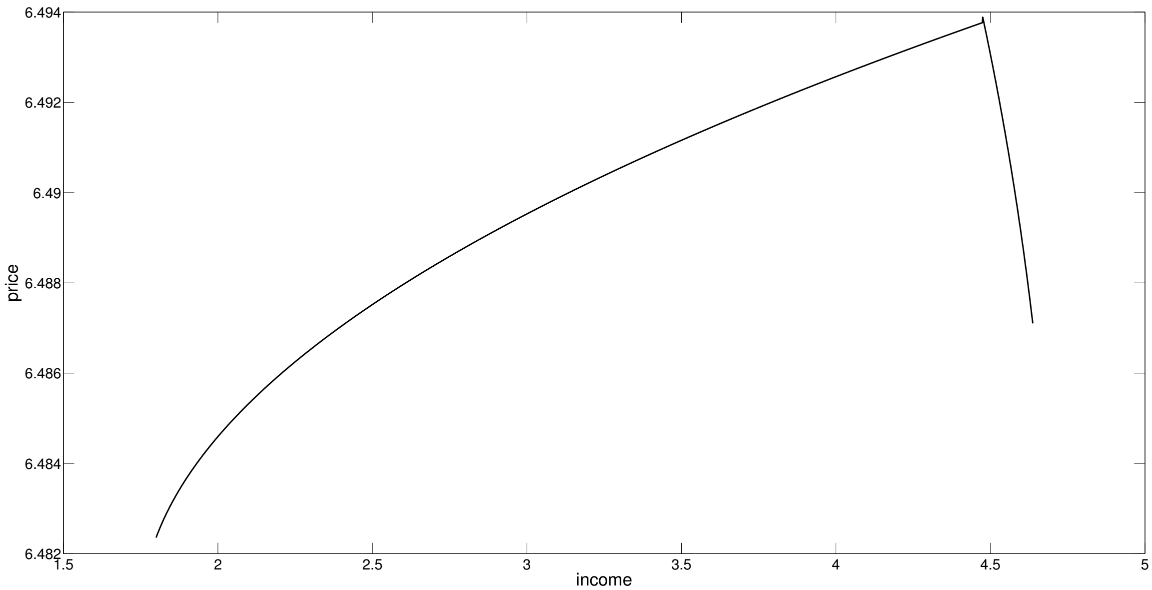

Figure 2: Neighborhood 2 House Prices in Example 4.1.

As income rises, buyers strictly prefer owning, and so they are willing to pay a higher price in exchange for finding a match more quickly. At an income level of $y = 4.474$, households choose to live in neighborhood 3 while renting, while continuing to search for a house to buy in neighborhood 2. For these households, matching with a seller and becoming a home-owner leads to a *reduction* in amenities, with housing services dropping from a_3 to $a_2 + z$ as per Assumption 2. Because households with higher income value housing services more relative to non-housing consumption, for these households the utility gain from ownership *in neighborhood 2* drops as income rises. As such they target sub-markets with lower prices and higher matching rates for sellers. This continues until $y = 4.638$, at which point the households prefer to search in neighborhood 3. There are no permanent renters in neighborhood 3. Thus, no household living there is indifferent between renting and owning, and as such all are willing to pay a price above V_3 in order to have a strictly positive probability of finding a match in the housing market. Thus, the lowest price sub-market in neighborhood 3 does not have a matching rate for sellers equal to one.

There is no clear theoretical link between the average selling price and the average probability of selling across neighborhoods. The former, of course, is closely related to the construction cost in each neighborhood. The latter is associated with the *liquidity* *discount*, which is determined endogenously given the search decisions of households.

The next example demonstrates that the relationship between the liquidity discount and neighborhood quality (and the associated average prices) depends on parameters.

4.2 A monotonic relationship between price and liquidity

In their detailed analysis of search behaviour and housing market activity in the San Francisco Bay area, Piazzesi, Schneider, and Stroebel (2013) find a decreasing relationship between house prices and liquidity discounts across housing segments. While such a relationship does not arise in the previous example, it is not inconsistent with our basic theory.

In our second example we consider a case in which the liquidity discount declines monotonically with neighborhood quality, and thus with the average neighborhood house price. This arises in equilibrium owing to the sorting of buyers and sellers across neighborhoods and does not require exogenously imposed differences in the efficiency of matching across neighborhoods (*i.e.*, heterogeneity in the α's). Table 3 contains the economy parameters for this example:

Table 3: Example 4.2: Parameter Values

description	parameters	values
income distn.	$\{\underline{y}, \overline{y}\}$	$\{1.0, 5.0\}$
	$\{Q(0.2), Q(0.4), Q(0.6), Q(0.8)\}$	$\{1.8, 2.6, 3.4, 4.2\}$
amenity levels	$\{a_1, a_2, a_3\}$	$\{0.0100, 0.0223, 0.0315\}$
construction costs	$\{\tau_1, \tau_2, \tau_3\}$	$\{4.9914, 8.5567, 12.1220\}$

In the stationary equilibrium of this example, the assignments of renters and

Table 4: Neighborhood Statistics in Example 4.2

neighborhood	1	2	3	aggregate
share of city popn.	0.5187	0.2547	0.2266	1.0000
perm. rentership rate	0.1928	0.0000	0.0000	0.1000
rentership rate	0.3532	0.1753	0.2327	0.2806
homeownership rate	0.6468	0.8247	0.7673	0.7194
vacancy rate	0.0126	0.0158	0.0145	0.0139
avg. selling prob.	0.9087	0.9179	0.9319	0.9169
avg. price	5.0469	8.6509	12.2534	7.8407
avg. price-rent ratio	25.0251	25.0223	25.0183	25.0216
liquidity discount 1	0.0417	0.0383	0.0343	0.0389
liquidity discount 2	0.0305	0.0299	0.0292	0.0300

searchers across neighborhoods are given by

$$i(y) = \begin{cases} 1 & \text{if } y \in [1.000, 2.986) \\ 2 & \text{if } y \in [2.986, 3.988) \\ 3 & \text{if } y \in [3.988, 5.000]. \end{cases} \tag{43}$$

$$j(y) = \begin{cases} 0 & \text{if } y \in [1.000, 1.400) \\ 1 & \text{if } y \in [1.400, 3.091) \\ 2 & \text{if } y \in [3.091, 4.118) \\ 3 & \text{if } y \in [4.118, 5.000] \end{cases} \tag{44}$$

Table 4 contains housing statistics by neighborhood for this example and Figure 3 depicts prices and matching probabilities by neighborhood. In this case, the worst neighborhood houses more than half of the city's population. More importantly, it has a much higher rate of home-ownership than in the previous example. Buyers in this neighborhood with (relatively) high income search in sub-markets with low buyer-seller ratios, bringing down the average selling probability. Houses in better neighborhoods sell at higher prices and, depending on parameter values, with higher probability. The result is a negative relationship between the liquidity discount and either neighborhood quality or the neighborhood average house price. It is also the case that permanent renters live only in neighborhood 1. The overall home-ownership rate is, however, not dramatically higher here than in Example 4.1.

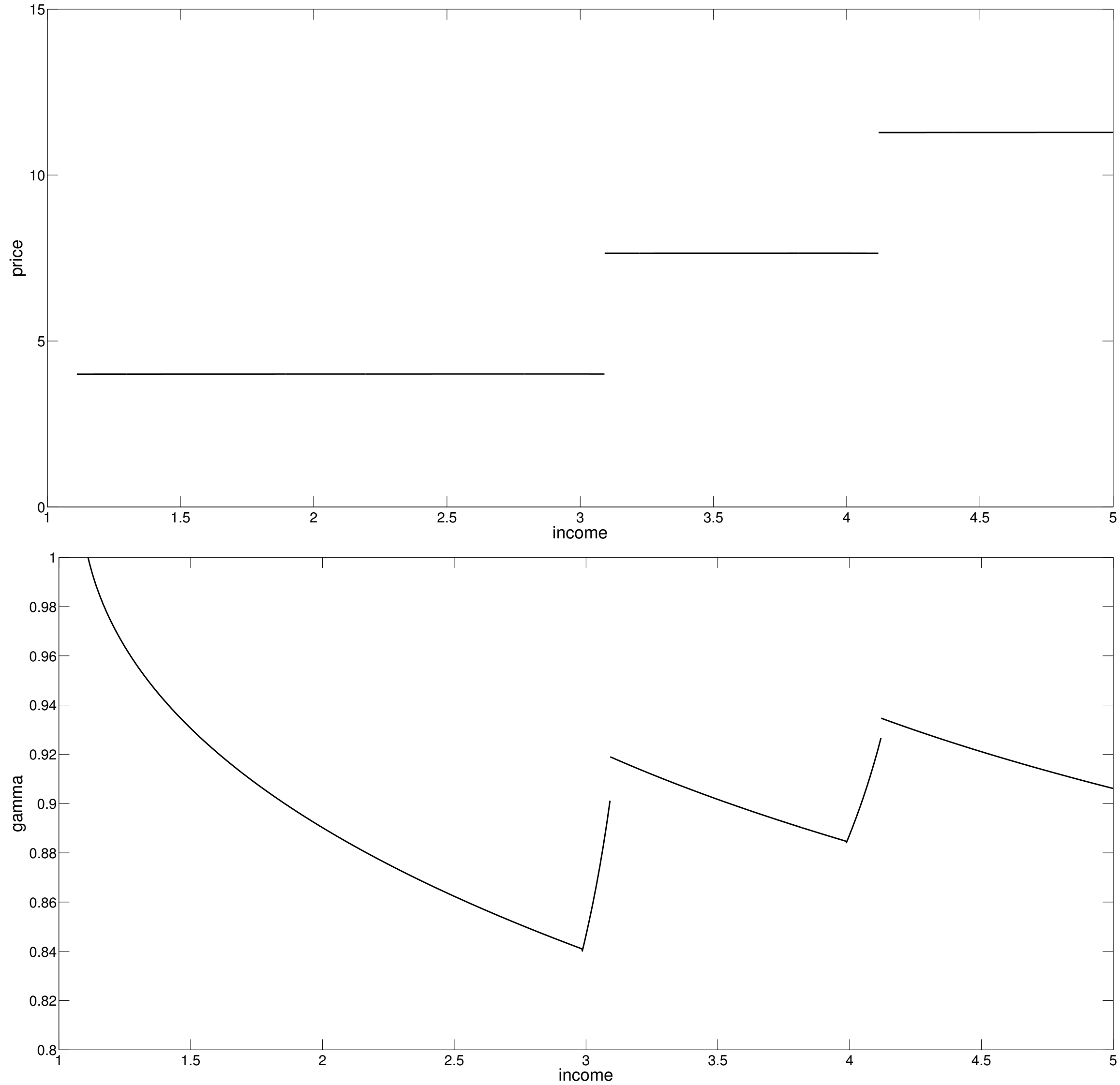

Figure 3: Prices and Matching Probabilities in Example 4.2.

4.3 An Alternative Income Distribution

We now consider the effect of changing the income distribution in the previous example to one which is first-order stochastically dominated by the original income distribution. All that we change are the income thresholds separating each quintile of the distribution (see Table 5). Figure 4 compares the income distributions in

Examples 4.2 and 4.3.

Table 5: Example 4.3: Parameter Values

description	parameters	values
income distn.	$\{\underline{y}, \overline{y}\}$	$\{1.0, 5.0\}$
	$\{Q(0.2), Q(0.4), Q(0.6), Q(0.8)\}$	$\{1.5, 2.2, 3.0, 3.9\}$
amenity levels	$\{a_1, a_2, a_3\}$	$\{0.0100, 0.0223, 0.0315\}$
construction costs	$\{\tau_1, \tau_2, \tau_3\}$	$\{4.9914, 8.5567, 12.1220\}$

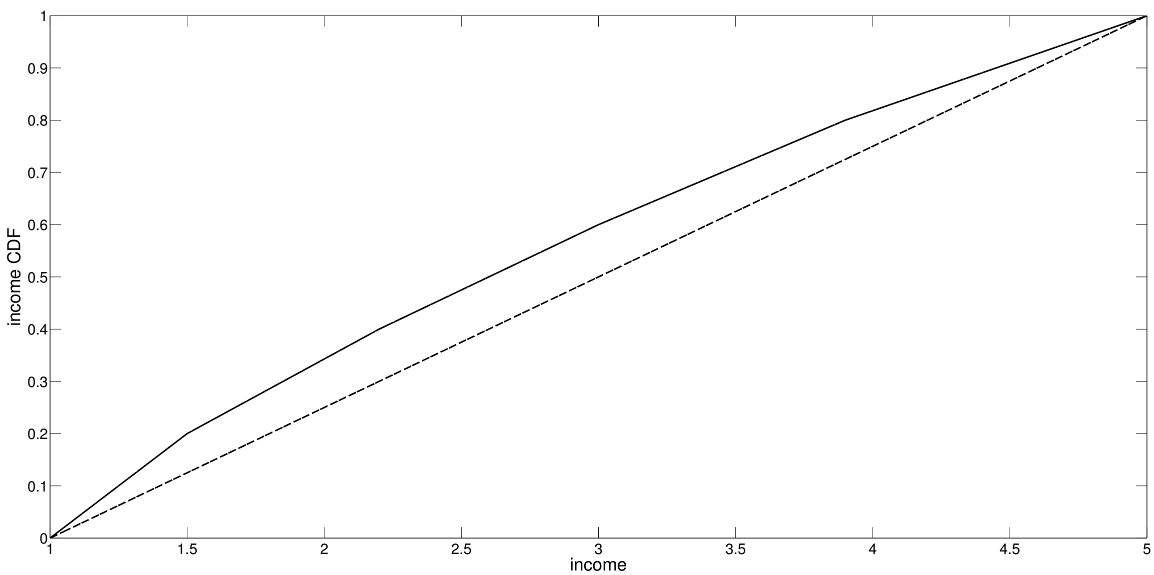

Figure 4: Income Distributions in Examples 4.2 (dashed line) and 4.3 (solid line).

The endogenous income cut-offs between neighborhoods and optimal decision rules for households do not change relative to Example 4.2. This is because prices and rents are ultimately determined by construction costs so that these income cut-offs are functions of construction costs and neighborhood amenities. Neighborhood statistics, in contrast, are affected because the relative sizes of neighborhoods (and the sub-markets within them) change according to the income distribution. Table 6 reports the summary statistics by neighborhood. Notice that prices, liquidity discounts, and selling probabilities are relatively similar to the numbers in Example 4.2. The distribution of households across neighborhoods as well as the rates of renting and

Table 6: Neighborhood Statistics in Example 4.3

neighborhood	1	2	3	aggregate
share of city popn.	0.6166	0.2186	0.1648	1.0000
perm. rentership rate	0.2595	0.0000	0.0000	0.1600
rentership rate	0.4186	0.1793	0.2327	0.3356
homeownership rate	0.5814	0.8207	0.7673	0.6644
vacancy rate	0.0113	0.0158	0.0145	0.0128
avg. selling prob.	0.9112	0.9182	0.9319	0.9170
avg. price	5.0467	8.6508	12.2534	7.3917
avg. price-rent ratio	25.0244	25.0223	25.0183	25.0218
liquidity discount 1	0.0410	0.0381	0.0343	0.0389
liquidity discount 2	0.0303	0.0299	0.0292	0.0300

home-ownership do change substantially under the new income distribution. There are relatively more low-income households, and hence more permanent renters and a lower homeownership rate.

4.4 An example with ten neighborhoods

The balanced growth path is not hard to compute with many neighborhoods. With ten neighborhoods, for example, there are many possible patterns of homeownership and neighborhood composition, depending on parameter values. For the parameter values in Table 7, there are permanent renters only in the high amenity, expensive neighborhoods (see Table 8 and Figure 5).

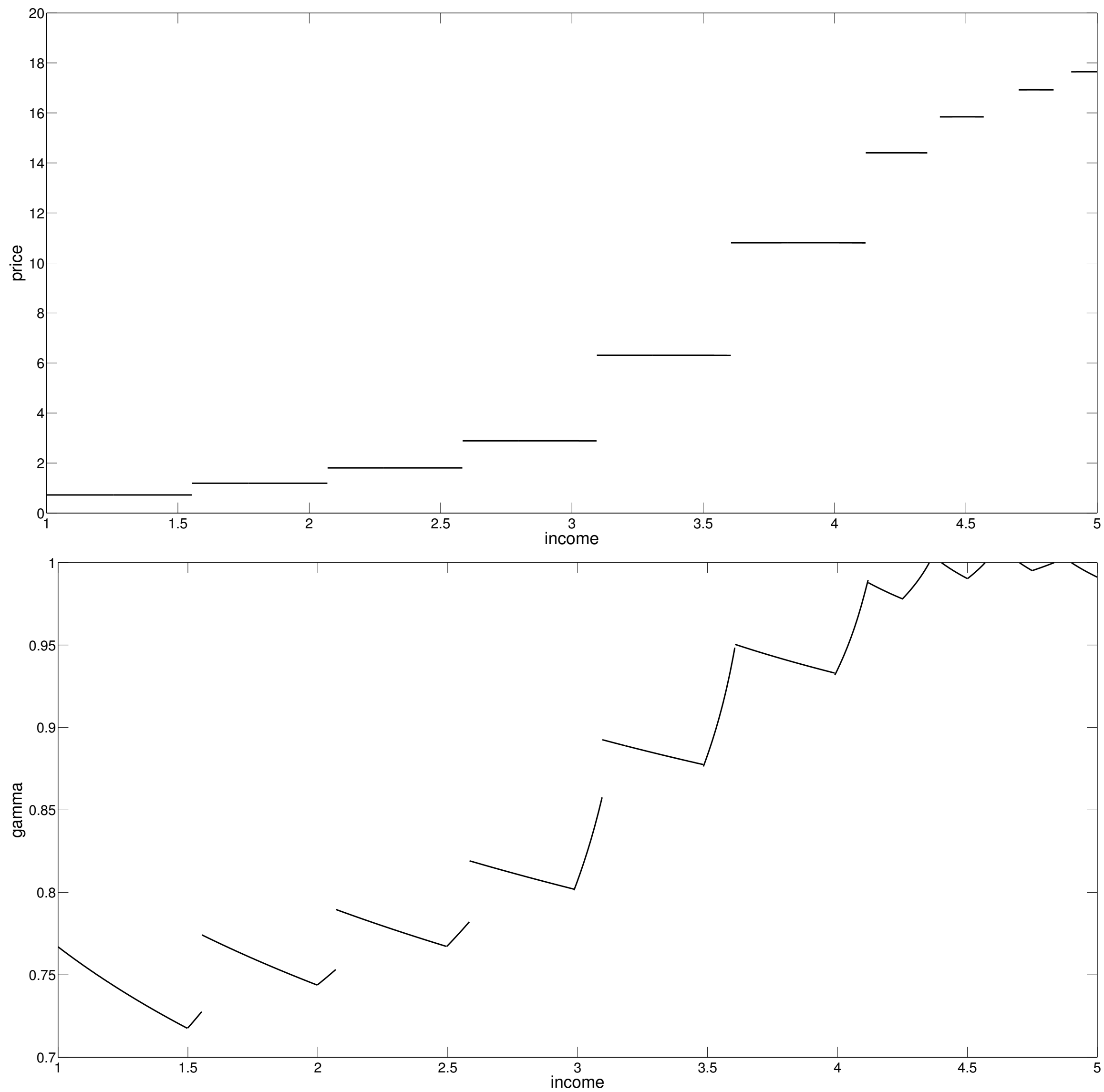

Figure 5: Prices and Matching Probabilities in Example 4.2.

Table 7: Example 4.4: Parameter Values

description	parameters	values
income distn.	$\{Q(0), Q(0.2), Q(0.4), Q(0.6), Q(0.8), Q(1)\}$	$\{1.0, 1.8, 2.6, 3.4, 4.2, 5.0\}$
amenity levels	$\{a_1, \ldots, a_5\}$	$\{0.0100, 0.0131, 0.0162, 0.0206, 0.0322\}$
	$\{a_6, \ldots, a_{10}\}$	$\{0.0454, 0.0506, 0.0582, 0.0607, 0.0622\}$
construction costs	$\{\tau_1, \ldots, \tau_5\}$	$\{0.7131, 1.1765, 1.7826, 2.8522, 6.2393\}$
	$\{\tau_6, \ldots, \tau_{10}\}$	$\{10.6959, 14.2612, 15.6873, 16.7569, 17.4700\}$

Table 8: Neighborhood Statistics in Example 4.4

neighborhood	1	2	3	4	5	6	7	8	9	10	aggregate
share of city popn.	0.1378	0.1272	0.1295	0.1266	0.1250	0.1225	0.0530	0.0562	0.0632	0.0590	1.0000
perm. rentership rate	0.0000	0.0000	0.0000	0.1000	0.000	0.000	0.0000	0.2223	0.5542	0.2965	0.0650
rentership rate	0.0582	0.0669	0.0711	0.0813	0.1259	0.2241	0.5081	0.8419	0.9367	0.9173	0.2668
homeownership rate	0.9418	0.9331	0.9289	0.9187	0.8741	0.7759	0.4919	0.1581	0.0633	0.0827	0.7332
vacancy rate	0.0223	0.0216	0.0210	0.0198	0.0173	0.0145	0.0089	0.0028	0.0011	0.0015	0.0158
avg. selling prob.	0.7385	0.7569	0.7775	0.8138	0.8897	0.9440	0.9837	0.9935	0.9967	0.9941	0.8165
avg. price	0.7228	1.1922	1.8058	2.8876	6.3101	10.810	14.408	15.847	16.927	17.648	4.3305
avg. price-rent ratio	25.089	25.080	25.072	25.057	25.031	25.015	25.004	25.002	25.001	25.002	25.029
liquidity discount 1	0.1467	0.1286	0.1110	0.0868	0.0484	0.0330	0.0274	0.0265	0.0264	0.0264	0.0919
liquidity discount 2	0.0434	0.0416	0.0397	0.0366	0.0315	0.0286	0.0268	0.0264	0.0262	0.0263	0.0367

5 Conclusions and Further Work

To this point it has been shown that the environment is capable of generating endogenous relationships among the distributions of income, housing wealth, and time-on-the-market, or the liquidity discount. While house prices in the long-run are determined by construction costs, the distribution of households across neighborhoods and the liquidity of housing are driven by the differing search behavior of households with different income. The theory is, in principle, consistent both with a non-strictly monotonic relationship between income and home-ownership and a decreasing relationship between the liquidity discount and average house values across neighborhoods or market segments.

It is our intention to use the model for quantitative experiments. Data on income, time-on-the-market, mobility, and construction costs at the city level can in principle be used to calibrate the model, in particular with regard to the nature of search frictions. Calibrated versions of the model will be used to make comparisons across cities. Ultimately, the goal is to do experiments considering changes in the level of income and the rate of population growth at the city level on the distribution of households across neighborhoods and ownership status and the distribution of house prices over time. The distribution of housing wealth is not only determined by the distribution of house prices, but also the endogenously determined distribution of homeownership. The purpose of this is to consider the distributional effects of city-level house price movements such as those studied by Head, Lloyd-Ellis, and Sun (2012), Díaz and Jerez (2013), and others.

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A The consumption-saving decision

The Euler equation associated with the problem of insuring against the cost of a house purchase is

$$\begin{aligned}\big(1-q_j^B(P)\big)\lambda(\theta_j(P))u'\big(c^R(y,s)+s_j^B(P)-P\big)\\ =q_j^B(P)\left[1-\lambda(\theta_j(P))\right]u'\big(c^R(y,s)\big).\end{aligned} \tag{A.1}$$

The absence of arbitrage implies that the equilibrium price adjusts until $q_j^B(P) = \lambda(\theta_j(P))$. By the concavity of $u(\cdot)$, the Euler equation (A.1) then implies that households fully insure, $s_j^B(P) = P$.

The remaining Euler equations associated with the solution to the portfolio allocation problems of renters and homeowners are

$$\begin{aligned} q_j^R(P) &= \beta\left[1-\lambda(\theta_j(P))\right]\frac{u'\left(c^R(y,s^F+s_j^R)\right)}{u'\big(c^R(y,s)\big)} \\ q^F &= \beta\frac{\lambda(\theta_j(P))u'\big(c^N(y,s^F)\big)+\left[1-\lambda(\theta_j(P))\right]u'\big(c^R(y,s^F+s_j^R\big)}{u'\big(c^R(y,s)\big)} \\ q^F &= \beta\frac{\pi_j u'\big(c^R(y,s^F+s_j^S+V_j)\big)+(1-\pi_j)u'\big(c^N(y,s^F)\big)}{u'\big(c^N(y,s)\big)} \\ q_j^S &= \beta\pi_j\frac{u'\big(c^R(y,s^F+s_j^S+V_j)\big)}{u'\big(c^N(y,s)\big)} \end{aligned}$$

In equilibrium, the absence of arbitrage requires that $q_j^R(P) = \beta[1-\lambda(\theta_j(P))]$, $q_j^S = \beta\pi_j$ and $q^F = \beta = 1/(1+r)$. By the concavity of $u(\cdot)$, the Euler equations then imply that the optimal investment plan yields a constant consumption stream.

To achieve constant consumption, c, the financial asset positions, $\{s^F, s^S, s^R\}$, must satisfy the following budget constraints:

$$\begin{aligned} y+(1-\beta)s^F+s^S+V-\beta(1-\lambda(\theta))s^R-\lambda(\theta)s^B-\kappa=c \\ y+(1-\beta)s^F+[1-\beta(1-\lambda(\theta))]s^R-\lambda(\theta)s^B-\kappa=c \\ y+(1-\beta)s^F-\beta\pi s^S=c \end{aligned}$$

where we have simplified the notation by removing the i and j subscripts and writing, for example, θ instead of $\theta_j(P)$. The first two budget constraints imply $s^R = s^S + V$. The first and last budget constraints then yield

$$s^R = \frac{\kappa+\lambda(\theta)P+\beta\pi V}{1-\beta(1-\lambda(\theta)-\pi)}. \tag{A.2}$$

We can now determine s^F from any of the budget constraints given c, $s^B = P$, $s^S = s^R - V$, and the expression for s^R in (A.2). To determine c, we first compute the present discounted values of net housing-related expenditures when currently owning and renting:

$$\begin{aligned} X^N &= \beta\left[(1-\pi)X^N + \pi(X^R - V)\right] \\ X^R &= \kappa + \lambda\left[P + \beta X^N\right] + \beta(1-\lambda)X^R. \end{aligned}$$

Solving for X^R yields

$$X^R = \frac{\left[1-\beta(1-\pi)\right](\kappa+\lambda P) - \beta^2\pi\lambda V}{(1-\beta)\left[1-\beta(1-\pi-\lambda)\right]}.$$

For a household that, when renting in neighborhood i, searches for a home to buy in sub-market P of neighborhood j, the (constant) per period consumption level for this household must satisfy the present value budget constraint, $(y-c)/(1-\beta) = X^R$, or

$$c(y,i,j,P) = y - \frac{\left[1-\beta(1-\pi_j)\right]\left(\kappa_i + \lambda(\theta_j(P))P\right) - \beta^2\pi_j\lambda(\theta_j(P))V_j}{1-\beta\left[1-\pi_j-\lambda(\theta_j(P))\right]}. \quad \text{(A.3)}$$

B The search problem

The first order condition for the household's within-neighborhood search problem is

$$\lambda(\theta_j(P))u'\big(c(y)\big) = \lambda'(\theta_j(P))\theta_j'(P)\left\{\beta\left[V^{N_j}(y) - V^R(y)\right] - u'\big(c(y)\big)\left[\beta s_j^R(P) - s_j^B(P)\right]\right\}.$$

The derivative $\theta_j'(P)$ is obtained by differentiating (19):

$$\theta_j'(P) = -\frac{\gamma(\theta_j(P))}{\gamma'(\theta_j(P))}\left(\frac{1}{P-\beta V_j}\right).$$

Combining these yields

$$\frac{\gamma'(\theta_j(P))}{\gamma(\theta_j(P))}\left[P-\beta V_j\right] = -\frac{\lambda'(\theta_j(P))}{\lambda(\theta_j(P))}\left\{\frac{\beta\left[V^{N_j}(y) - V^R(y)\right]}{u'\big(c(y)\big)} + \beta s_j^R(P) - s_j^B(P)\right\}.$$

Using the relationship $\gamma(\theta) = \theta\lambda(\theta)$ and the definition of $\eta(\theta)$, this becomes

$$\eta\big(\theta_j(P)\big)\left[P-\beta V_j\right] = \left[1-\eta\big(\theta_j(P)\big)\right]\left\{\frac{\beta\left[V^{N_j}(y) - V^R(y)\right]}{u'\big(c(y)\big)} + \beta s_j^R(P) - s_j^B(P)\right\}.$$

Finally, equation (21) is obtained by substituting $s_j^B(P) = P$, the expression for $s_j^R(P)$ from (A.2), and the buyer's surplus from a match in a stationary equilibrium, which is

$$V^{N_j}(y) - V^R(y) = \frac{m(a_j + z) - m(a_i)}{1 - \beta(1 - \pi_j - \lambda(\theta_j(P))}.$$