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# Equilibrium Wage Dispersion, Firm Size, and Growth

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This paper analyzes a model of equilibrium wage dynamics and wage dispersion across firms. It considers a labor market where firms set wages and workers use on-the-job search to look for better paid work. It analyzes a perfect equilibrium where each firm can change its wage paid at any time, and workers use optimal quit strategies. Firms trade off higher wages against a lower quit rate, and large firms (those with more employees) always pay higher wages than small firms. Non-steady-state dispersed price equilibria are also analyzed, which describe how wages vary as each firm and the industry as a whole grow over time. *Journal of Economic Literature* Classification Numbers: D43, J41. 

*Key Words:* on-the-job search, wage dispersion, perfect equilibrium

## INTRODUCTION

Labor markets are characterized by a surprising amount of wage dispersion and quit turnover. For example, Topel and Ward (1992) find that during the first 10 years in the labor market, a typical worker holds seven jobs, that the wage gains at job changes account for at least a third of early career wage growth, and that the wage is the key determinant of job changing decisions among young workers. The presence of search frictions can easily explain such quit turnover—an unemployed worker might accept relatively poorly paid work in the short-run, with the intention of continuing search on-the-job for something more rewarding. But the puzzle is why do some firms offer higher wages than others? Burdett and Mortensen (1998) describe a market with search frictions where optimal wage setting behavior by firms and on-the-job search by employees gener-

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ates wage dispersion as an equilibrium outcome. Individual firms trade off lower wages against higher quit rates, and steady state finds that large firms with many employees pay high wages (and so have a low quit rate), while small firms with few employees pay low wages (and have a high quit rate).

The Burdett and Mortensen (1998) framework (henceforth B & M) has become particularly influential in the labor market literature as it provides a structural model of both wage dispersion and quit turnover (see van den Berg, 1999, for a recent survey) and is broadly consistent with stylized facts (see Pencavel, 1970; Krueger and Summers, 1987, 1988; Dickens and Katz, 1987; Brown and Medoff, 1989). However, this paper explores a theoretical flaw in their argument. For simplicity, they assume firms can never change their posted wage. Although this assumption might seem reasonable in a steady-state environment, it is not reasonable outside of steady state where one should expect firms will change wages over time. Unfortunately this wage restriction plays a central role in their theory. If any firm were allowed to change wage and deviate from the B & M steady-state equilibrium, it would choose to do so by cutting its wage to the workers' reservation wage $R$. This suggests their explanation of wage dispersion may not be robust to allowing full wage flexibility.

This paper extends their framework to a dynamic equilibrium where firms can change the wage paid at any time, which also allows discussion of non-steady-state equilibria. Unfortunately this much complicates their problem as the optimal quit decision of any worker depends on expected future wages at their current employer, and those expectations must now be consistent with their employer's (dynamic) wage setting strategy. Tractability requires restricting attention to Markov strategies. In particular, at the start of each period, each firm announces an updated wage $w'$ conditional on its previous period wage $w$ and current employment level $n$. Assuming that a worker observes both $w'$ and $n$ (given contact), the worker uses this information to predict expected future wages. The worker's optimal quit strategy then compares expected future wage streams, given current offers $(w', n)$ at any contacted firms.

Although a firm does not observe the distribution of wage offers by other firms, it has beliefs on that wage distribution (as do workers). Given those beliefs and the (stationary) worker quit strategies, each firm uses an optimal wage strategy to maximize expected discounted profit. Of course equilibrium requires all agents are rational—that the firm and worker beliefs on the distribution of wages are consistent with the collective wage strategies of all firms in the market. Identifying such a market equilibrium is clearly complex. The central aim is to extend the B & M results (where possible) to this case of no future wage precommitment. The constructed equilibrium relies heavily on their insights.

One problem is immediately apparent. Suppose a B & M type equilibrium exists, where a firm with $n$ employees is supposed to announce a wage $w = w^*(n) > 0$. Suppose this firm deviates by setting a lower wage $w' < w^*(n)$. Search frictions imply that the only agents who observe this wage deviation are this firm's employees and any other workers who contact this firm through search. The issue is what wage do these workers anticipate in the future? It turns out that it cannot be an equilibrium that the firm returns to announcing $w = w^*(n)$. If it were, this wage deviation would not change expected future wages and the workers' optimal turnover decisions do not change (by much). The firm's optimal wage would then be $w = 0$; this low wage has (almost) no effect on employment as workers (foolishly) expect future wage $w^*(n)$. Supporting a B & M type equilibrium requires that if the firm cuts its wage below the equilibrium wage $w^*$, its employees expect lower wages in the future. Of course such beliefs must be rational; it has to be optimal for the firm to announce a lower wage, given deviation $w < w^*(n)$.

But there is a second problem. As workers use stationary (Markov) strategies, the firm has the option of returning to the equilibrium path by announcing $w = w^*(n)$. Equilibrium requires that this is also a profit maximizing strategy. A B & M type equilibrium therefore requires that for all (equilibrium values of) $n$, there exist (at least) two optimal pricing strategies. The firm must be indifferent between announcing the equilibrium wage $w^*(n)$ and some other low wage $w_0$. If the firm deviates and cuts wage below the equilibrium wage, its employees expect wage $w_0$ in the future. The obvious candidate for this wage floor $w_0$ is the worker's reservation wage $R$.

Hence unlike the B & M equilibrium requirement that all firms make equal profit, here the requirement is that each firm is indifferent between announcing its equilibrium wage $w^*(n)$ or announcing the worker reservation wage $R$. It is shown that this implies large firms (those with more employees) make greater profit than small firms. Equilibrium then has a simple dynamic structure. Should a firm cut its wage below $w^*(n)$ (to extract more search rents), its employees expect low wages in the future—they expect wage $w = R$. These lower wage expectations increase the quit rate of employees to alternative (better paid) employment. As the increased quit rate lowers employment, this turnover response reduces expected future profits. This loss then offsets the gain through paying lower wages.

By embedding this wage setting structure into the original B & M search framework, this paper shows that a B & M type equilibrium exists, even with no precommitment on future wages. Like B & M, it is shown that markets are characterized by equilibrium wage dispersion, where large firms set high wages and have low quit rates. However unlike B & M, this structure also implies large firms make greater profit.

At first sight, this latter result suggests an efficiency wage outcome—by paying higher wages and reducing employee quit rates a firm can increase profit. But if this were an efficiency wage result, one would predict that all firms should pay high wages. Characterizing the non-steady-state equilibrium dynamics clarifies the issue. In the non-steady-state equilibrium (where all firms are initially equally sized), there is wage dispersion where low wage firms at first make greater (flow) profit than high wage firms. But by gradually attracting more employees, these latter firms ultimately grow to become large firms which then make greater profit (as the market converges to the B & M type steady state).[2] All wage and employment paths make the same initial expected discounted profit, but the path of profits varies across high and low wage firms. Fundamentally speaking, paying a high wage today is an investment into a larger (better?) workforce tomorrow. The result that large firms make greater profit in a steady state is a reflection of prior investments they have made in the search market.

## 1. THE FRAMEWORK

This paper considers a discrete time, infinite horizon economy with period interval $\Delta > 0$, but focuses on the limiting equilibrium as $\Delta \to 0$. There is a continuum of firms with mass normalized to unity, and a continuum of workers with mass $P > 0$. All firms are infinitely lived but workers are not. A worker's stay in the labor market is described by a Poisson process with parameter $\delta > 0$. If a worker leaves the labor market, a new (unemployed) worker immediately enters.

All workers can be in one of two states, employed or unemployed. If $U_t$ denotes the measure of unemployed workers at time $t$, then $N_t = P - U_t$ are employed. Unemployment exists because there are matching frictions in the labor market. In particular, the probability that an unemployed worker contacts a firm in a given period is $\lambda_0 \Delta$, where $\lambda_0 > 0$. There is also on-the-job search, where the probability that an employed worker contacts another firm (in addition to the current employer) is $\lambda_1 \Delta$, where $\lambda_1 > 0$. Given the wage offer of the firm contacted, the worker can either accept employment at that firm or reject it. If the worker chooses to reject the job offer, there is no recall.

[2] In a consumer version of this model with repeat purchasers, this framework predicts that low price stores will have more customers (greater sales turnover) and outside of steady state, accumulate customers and hence grow more quickly. See Evans (1987a, b) and Hall (1987) for evidence on firm pricing and growth.

The critical difference to B & M is that each firm can change its wage at any stage.

Consider a firm which at the start of period $[t, t + \Delta)$, has $n_{t-\Delta}$ employees carried over from the previous period. The firm first announces a wage $w_t$ which is then fixed for this period. Each employee then generates beliefs on expected future wage levels at this firm. By restricting attention to Markov strategies, $(w_t, n_{t-\Delta})$ will be a sufficient statistic describing how wages evolve over time at this firm. Assuming each employee observes $(w_t, n_{t-\Delta})$, those workers then compute the expected value to staying at this firm. Depending on each worker's value of quitting (which depends on any outside offer $(w_t', n_{t-\Delta}')$ from some other firm), each employee then decides whether to stay or quit. It is assumed that the worker can quit freely (i.e., does not pay compensation to his current employer) but if the worker quits, there is no recall. Workers who contact this firm also observe $(w_t, n_{t-\Delta})$ and compute the expected value of accepting employment at this firm. Depending on the value of their outside option, they also decide whether or not to accept employment there (with no recall should they decline the offer). Given these turnover decisions, net firm turnover at time $t$ is denoted by $dn_t$ and so $n_t = n_{t-\Delta} + dn_t$. The firm makes profit $n_t[\pi - w_t]\Delta$ over this period, where $\pi > 0$. The process is then repeated in the next period given $n_t$ employees.

Given the turnover strategies of workers, each firm chooses a wage setting strategy to maximize expected discounted profits. Given those wage setting strategies, all workers choose turnover strategies to maximize their expected discounted utility, where the flow value of being unemployed is $b\Delta$, while the flow value of being employed at wage $w$ is $w\Delta$. There are no search costs. Assume that an unemployed worker accepts a job if indifferent to doing so, while a worker who is indifferent to switching firms stays at his current employer. All agents have the same discount rate $r > 0$.

The aim is to construct a perfect equilibrium to this market game. Clearly the game is complicated as the worker turnover strategies depend on the wage setting behavior of all of the firms, so that each firm's optimal wage policy depends on the wage policies of the other firms. Furthermore, computing the expected value of staying at a firm is complicated if future wages are expected to change in arbitrary ways. To simplify matters, this first section will only consider *market equilibria* where the (Markov) strategies form a perfect equilibrium and

(i) all firms and all workers use the same strategies (equilibria are symmetric), and

(ii) the market is in a "steady state,"

where steady state requires that each firm's equilibrium pricing strategy (along the equilibrium path) implies $dn_t = 0$ so that its number of employees does not change over time, and so its optimal wage offer does not change over time. Notice this also implies a constant unemployment level $U_t = U$.

Of course, even though firms use the same strategies, firms with different employment levels may announce different wages. In such market equilibria (assuming they exist) the market distribution of wages, denoted $F(w)$ with support $[\underline{w}, \overline{w}]$, is constant over time, as is the distribution of firm employment levels denoted $G(n)$.

We construct a market equilibrium in three steps. The first two steps ignore the issue of perfectness and instead construct a candidate search equilibrium using insights derived from standard search theory. Assuming firms cannot change price, B & M have shown that in a search equilibrium, unemployed workers will accept a job offer if and only if the wage offer exceeds some reservation wage $R$, where $R = R(F)$ depends on the distribution of market wages. Furthermore, given the reservation wage $R$, the equilibrium wage posting strategies of firms imply a market distribution $F = \hat{F}(w|R)$. A candidate search equilibrium requires finding $R$ which solves the fixed point condition $R = R(\hat{F}(\cdot|R))$. This fixed point problem is considered in Steps 1 and 2 below, imposing the restriction that firms are not allowed to change wage. Using this candidate search equilibrium as the equilibrium path, Step 3 identifies a market equilibrium by describing strategies off the equilibrium path which ensure that no firm wishes to change price.

### 1.1. *Step* 1: *A Candidate Steady State* (*Given* $R$)

In this step and in Step 2 below, assume *firms precommit to a fixed wage for the entire future*. In that case, optimal job search behavior by workers implies:

**(A1)** Each unemployed worker uses a reservation wage $R$, accepting a job offer if and only if $w$, the wage offered, satisfies $w \geq R$.

**(A2)** The optimal quit strategy of an employed worker with current wage $w \geq R$ is to quit if and only if contact is made with a firm offering a strictly higher wage.

Conditional on $R$, this section characterizes wage posting behavior assuming (A1) and (A2) describe worker turnover. Step 3 below relaxes the assumption that firms cannot change price, but ensures that firms do not choose to change price so that (A1) and (A2) will continue to describe optimal worker behavior. These (stationary) worker turnover rules imply that turnover at a given firm is of the form $dn_t = dn(n, w)$

We denote a candidate steady state as $\{\hat{w}(n, R),\ \hat{n}(w, R),\ \hat{G}(n, R),\ \hat{F}(w, R)\}$, where $\hat{w}(n, R)$ denotes the fixed wage set by a firm with $n$

employees when $R$ is the reservation wage of worker, $\hat{n}(w, R)$ is the number of employees at a firm offering wage $w$, $\hat{G}(n, R)$ denotes the steady-state distribution of firm size, and $\hat{F}(w, R)$ denotes the market distribution of wages. As $R$ is fixed throughout this step, we temporarily subsume reference to $R$ in these candidate functions. Let $S$ denote the set of firm employment sizes where if $n \in S$, a firm with $n$ employees exists in the candidate steady state, while such a firm does not exist if $n \notin S$. Further let $[\underline{n}, \bar{n}]$ denote the support of $S$.

Assuming $R < \pi$ and turnover rules (A1), (A2), a candidate steady state is required to satisfy two market properties.[3]

(M1) a firm with $n$ employees is indifferent to posting "$w = \hat{w}(n)$ forever" or posting "$w = R$ forever" for all $n \in S$, and

(M2) $dn(n, \hat{w}(n)) = 0$ for all $n \in S$.

Although at this stage no firm can change wage, (M1) requires that each firm would be indifferent to switching to "$w = R$ forever" if it were allowed to do so. As explained in the introduction, this property will be critical for dynamic consistency. B & M did not assume (M1). Their framework implied all firms would make the same profit. This will not occur here—Lemma 1 below implies that larger firms make greater profit. Obviously property (M2) is necessary for a steady state.

The following claim provides some basic structure for the candidate steady state.

*Claim* 1. Given $R < \pi$, then in any candidate steady state:

(i) $\hat{w}(n) \geq R$ for all $n \in S$, i.e., $\underline{w} \geq R$, and

(ii) steady-state unemployment $U = \delta P/(\delta + \lambda_0)$ and $\alpha = \lambda_0 \delta P/(\delta + \lambda_0)$ is the rate at which each firm is contacted by unemployed workers.

*Proof.* (i) Proof by contradiction. Suppose $\hat{w}(n) < R$ for some $n \in S$. As such a wage attracts no workers, (M2) implies $n = 0 \in S$. But such firms make zero profits and are strictly better off attracting workers with a wage $w = R$, which contradicts (M1).

(ii) By (i), all firms offer $w \geq R$ and so each unemployed worker accepts a job at the first firm they contact. Hence steady-state unemployment must satisfy $\lambda_0 U = \delta[P - U]$, and so $U = \delta P/(\delta + \lambda_0)$. With random search, each firm is contacted by unemployed workers at rate $\lambda_0 U$. This completes the proof of the claim.

[3] The other cases are trivial. If $R > \pi$, all firms prefer to offer $w \leq \pi < R$ and so no workers are ever hired. If $R = \pi$, those firms with a positive number of employees pay $w = R = \pi$.

Given this result, we first focus on (M1). Let $V(n)$ denote the value of being a firm with $n$ employees in the candidate steady state. (M2) implies

$$V(n) = n[\pi - \hat{w}(n)]/r. \tag{1}$$

Given the same value of $n$, suppose instead the firm posts "$w = R$ forever." In that case, unemployed workers will continue to accept work at this firm, but all employees who receive an outside offer $w > R$ will quit. Let $m = \hat{F}(R)$, which by Claim 1 is the mass of firms announcing wage $w = R$; clearly $m = 0$ if no such mass point exists. Given $(n, R)$, the steady-state turnover behavior of workers implies

$$dn(n, R) = [\alpha - \delta n - \lambda_1(1 - m)n]\Delta + 0(\Delta^2) \tag{2}$$

where $\alpha$ is the rate at which unemployed workers contact this firm (and accept the job offer), $\delta$ is the rate at which each employee leaves the labor market, and $\lambda_1(1 - m)$ is the rate at which each employee finds and then quits to a better paid job. If $\hat{V}(n)$ denotes the value of this alternative pricing strategy, then standard dynamic programming arguments imply

$$[1 + r\Delta]\hat{V}(n) = n[\pi - R]\Delta + \hat{V}(n + dn(n, R)) \tag{3}$$

As (M1) requires $V(n) = \hat{V}(n)$ for all $n \in S$, we now construct $\hat{w}(n)$

LEMMA 1. *In the limit as $\Delta \to 0$, a candidate steady state satisfies $(M1)$ if and only if*

$$\hat{w}(n) = R + \frac{[\lambda_1(1 - m) + \delta][\pi - R]}{r + \lambda_1(1 - m) + \delta}\left[\frac{n - n_0}{n}\right] \tag{4}$$

*for all $n \in S$, where*

$$V(n) = \frac{\pi - R}{r + \lambda_1(1 - m) + \delta}\left[n + \frac{\alpha}{r}\right] \tag{5}$$

*and $n_0 = \alpha/(\lambda_1(1 - m) + \delta)$.*

*Proof.* The proof is in Appendix A.

Notice that like B & M, the wage policy $\hat{w}(n)$ is a strictly increasing function of $n$—larger firms pay higher wages. However, unlike B & M (in the case where firms are equally productive) (5) shows that larger firms also make greater profit.

Now consider (M2). Suppose a firm announces wage $w \geq R$, and has current employment level $n$. The worker turnover rules imply:

$$dn(n, w) = \left[\alpha - \delta n - \lambda_1 n\left[1 - \hat{F}(w)\right] + \lambda_1 \int_{w' \in [\underline{w}, w)} \hat{n}(w') d\hat{F}(w')\right]\Delta + 0(\Delta^2) \tag{6}$$

where the $\alpha - \delta_n$ term is obvious, $\lambda_1 n[1 - \hat{F}(w)]$ is the rate at which employees quit for strictly better paid jobs, while the integral term is the rate at which the firm attracts workers from strictly lower paid jobs. Of course for this wage, and given $\hat{F}$, steady-state employment $\hat{n}(w)$ is defined by $dn(\hat{n}, w) = 0$. In the limit as $\Delta \to 0$, (6) implies

$$\alpha - \delta\hat{n}(w) - \lambda_1 \hat{n}(w)\left[1 - \hat{F}(w)\right] + \lambda_1 \int_{w' \in [\underline{w}, w)} \hat{n}(w') d\hat{F}(w') = 0 \tag{7}$$

Lemma 2 gives the (unique) solution to this integral equation for $\hat{n}$ (given $\hat{F}$).

LEMMA 2. *Given $\hat{F}, \hat{n}$ is uniquely defined by* (7) *and has closed form solution*[4]

$$\hat{n}(w) = \frac{\alpha(\delta + \lambda_1)}{\left[\delta + \lambda_1\left(1 - \hat{F}(w)\right)\right]\left[\delta + \lambda_1\left(1 - \hat{F}(w^-)\right)\right]} \tag{8}$$

Lemma 2 implies that the steady-state number of workers employed at a firm offering wage $w$ depends on its position in the market distribution of wages, and on whether there is a mass point of firms offering the same wage. This is intuitively obvious as the firm attracts workers from all firms offering strictly lower wages, and loses employees to all firms offering strictly higher wages. Although the intuition is clear, solving (7) for this expression is not straightforward. A formal proof is not provided as the appropriate details are given in B & M.

A candidate steady state requires solving the following fixed point problem. Fix $S$. (M1) requires that firms set wages $w = \hat{w}(n)$ for all $n \in S$, where $\hat{w}$ is given by (4). Given an announced wage $w$, (M2) implies a steady-state employment level $n = \hat{n}(w)$ given by (8). Hence (M1) and

[4]As $\hat{F}$ may contain mass points in its support, $\hat{F}(w^-)$ is defined as $\lim_{\varepsilon > 0,\, \varepsilon \to 0} \hat{F}(w - \varepsilon)$. If $F$ has a mass point $m(w) > 0$ at $w$, then $\hat{F}(w) = \hat{F}(w^-) + m(w)$. If there is no mass point, then $\hat{F}(w^-) = \hat{F}(w)$ and $\hat{F}$ is continuous at $w$.

(M2) require finding an $\hat{F}$ so that $\hat{n}(\hat{w}(n)) = n$ for all $n \in S$. Suppose such an $\hat{F}$ exists. As $w = \hat{w}(n)$ is a strictly increasing function of $n$, then $\hat{F}$ implies $\hat{G}(n) = \hat{F}(\hat{w}(n))$. Of course this implied distribution $\hat{G}$ must then imply the original set $S$. For now assume such a candidate steady state exists.

### 1.2. *Step* 2: *A Candidate Search Equilibrium*

Assuming workers have a reservation wage $R$, the candidate steady state implies a distribution of wages $\hat{F}(w, R)$. The next step is to identify the value of $R$ such that given the market distribution of wages $\hat{F}$, then $R$ is indeed the worker's reservation wage.

Mortensen and Neumann (1988) have shown that the reservation wage of workers, given $F$, is defined by

$$R = b + \frac{\lambda_0 - \lambda_1}{\lambda_1} \int_R^{\overline{w}} \frac{\lambda_1 [1 - F(w')]}{r + \delta + \lambda_1 [1 - F(w')]} \, dw' \tag{9}$$

We define a candidate search equilibrium as a candidate steady state satisfying (M1)–(M2), where $R$ also satisfies the fixed point condition (9) with $F = \hat{F}(w, R)$. For now assume a candidate search equilibrium exists.

### 1.3. *Step* 3: *A Market Equilibrium*

Given its beliefs on the state of the market, each firm announces updated wage $w'$ conditional on its current firm size $n \in [\underline{n}, \overline{n}]$ and current wage $w$. A worker in contact with this firm observes $(w', n)$. This worker then chooses whether to continue (or accept) employment based on this current wage offer $(w', n)$ and on any outside offers. Those stationary (Markov) quit decisions generate net turnover of the form $dn(w', n)$ and employment becomes $n + dn(w', n)$. Most importantly, as all workers use stationary quit strategies, a Markov wage setting strategy is optimal for the firm. Further, given all firms use a Markov wage setting strategy, then $(w', n)$ is a sufficient statistic for each worker to predict future wages at any given firm, and so a stationary quit strategy is privately optimal for each worker.

Also note that a price deviation by a single firm is unobserved by other firms (because of search frictions) while the change in quit decisions at a deviating single firm has no effect on turnover at other firms as the deviating firm has zero measure. Hence a unilateral price deviation by a single firm, and the corresponding turnover responses of workers who contact this deviating firm, does not affect the optimal pricing decision of any other firm. Hence all firms and workers take the (steady state) market outcome as given.

The outcome of the constructed market equilibrium corresponds to a candidate search equilibrium. The equilibrium distribution of firms sizes $G$ is $G(\cdot) = \hat{G}(\cdot, R)$, and the collective wage strategies imply wage distribution $F(\cdot) = \hat{F}(\cdot, R)$. Of course $F, G$ also describes each agent's belief on the state of the market.

Each firm uses the following wage strategy. Given $(w, n)$ and $n \geq \underline{n}$:

(P1) $w' = w^*(n)$ if $w = w^*(n)$;

(P2) $w' = w^*(n)$ if $w > w^*(n)$;

(P3) $w' = R$ if $w < w^*(n)$;

while for $n < \underline{n}$ the firm sets $w' = R$.

Along the equilibrium path (P1), a firm always announces wage $w^*(n)$. Consistency with a candidate search equilibrium requires $w^*(\cdot) = \hat{w}(\cdot, R)$, and steady state [which is guaranteed by (M2)] then implies $n$ never changes over time. Hence along the equilibrium path no firm ever changes wage. Of course should a firm deviate by announcing $w < w^*(n)$, (P3) will imply the firm announces wage $R$ in the entire future, and (M1) will ensure this is an optimal strategy in that subgame.

LEMMA 3. *If* $R < \pi$ *and* (i) *all firms use pricing strategies* (P1)–(P3), (ii) $w^*, F, G$ *are consistent with a candidate search equilibrium, then for* $\Delta$ *small enough a market equilibrium exists only if* $G$ *contains no mass points.*

*Proof.* The proof is in Appendix A.

The argument is the same as that used by B & M—if a mass point exists, a firm in that mass point can profitably deviate by offering a slightly higher wage and so attract employees from the other firms in the mass point.[5] No mass points in $G$ implies a unique candidate search equilibrium.

LEMMA 4. *If* $\hat{G}$ *has no mass points, the candidate steady state defined in Step* 1 *exists, is unique, and is given by*:

$$\hat{w}(n, R) = R + \frac{(\lambda_1 + \delta)(\pi - R)}{r + \lambda_1 + \delta}\left[\frac{n - \underline{n}}{n}\right] \tag{10}$$

$$\hat{F}(w, R) = \frac{\lambda_1 + \delta}{\lambda_1}\left[1 - \sqrt{1 - \frac{r + \lambda_1 + \delta}{\lambda_1 + \delta}\frac{w - R}{\pi - R}}\right] \tag{11}$$

$$\hat{G}(n) = \frac{\lambda_1 + \delta}{\lambda_1}\left[1 - \sqrt{\frac{\underline{n}}{n}}\right] \tag{12}$$

[5] The conclusion shows that market equilibria with mass points exist if firms adopt a different wage setting strategy.

*and the supports of* $\hat{F}$ *and* $\hat{G}$ *are connected where*:

(a) $\underline{w} = R$, $\overline{w} = R + (\pi - R)[\lambda_1(\lambda_1 + 2\delta)/[(\lambda_1 + \delta)(r + \lambda_1 + \delta)]]$

(b) $\underline{n} = \alpha/(\lambda_1 + \delta)$, $\overline{n} = \alpha(\lambda_1 + \delta)/\delta^2$.

*Proof.* The proof is in Appendix A.

Given this closed form solution for $\hat{F}$, it is now straightforward to solve for $R$ defined by (9) and so identify the (unique) candidate search equilibrium.

LEMMA 5. *For* $b < \pi$ *and* $\hat{F}$ *defined by* (11), *a candidate search equilibrium exists and is unique*. *Furthermore* $R < \pi$.

*Proof.* The proof is in Appendix A.

When $G$ has no mass points, Lemmas 4 and 5 imply a unique candidate search equilibrium. The following now constructs a market equilibrium.

Fix $\Delta > 0$. Given wage strategy (P1)–(P3) and a candidate search equilibrium, the proof of Theorem 1 below describes the optimal turnover responses of workers and shows that those strategies describe a perfect equilibrium for $\Delta$ small enough. The limiting turnover strategies (as $\Delta \to 0$) correspond to:

(T1) if the firm announces $w = w^*(n)$:

(i) Unemployed workers and employed workers with no outside offers stay,

(ii) Workers with outside offer $(n', w^*(n'))$ accept that offer if and only if $w^*(n') > w$

(T2) If the firm offers $w > w^*(n)$:

(i) Unemployed workers and employed workers with no outside offers stay,

(ii) Workers with outside offer $(n', w^*(n'))$ accept that offer if and only if $n' > n$

(T3) If the firm offers $w \in [R, w^*(n))$:

(i) Unemployed workers and employed workers with no outside offers stay,

(ii) Workers with outside offer $(n', w^*(n'))$ accept that offer if and only if $w^*(n') > R$.

(T4) If the firm offers $w < R$ all workers quit.

Unemployed workers and employees with no outside offers stay if and only if $w \geq R$. Such workers use a reservation wage strategy consistent with (A1). The turnover strategy of workers holding outside offers depends on the value of their outside offer and on whether they expect the firm to price $w = R$ or $w = w^*(n)$ in the future, which is consistent with (A2).

THEOREM 1. *For* (P1)–(P3) *and in the limit as* $\Delta \to 0$, *a market equilibrium exists where* $w^*, F, G, R$ *are consistent with the candidate search equilibrium described by Lemmas* 4 *and* 5, *and workers use the* (*limiting*) *turnover strategies* (T1)–(T4).

*Proof.* (P1) and (T1) describe the equilibrium path, which by construction coincides with the candidate search equilibrium obtained in Steps 1 and 2 (with no mass points). No firm ever changes size and so always announces the same wage. Given (P1) and (T1), the turnover response (T1) is optimal for each worker. But to verify that these strategies describe a perfect equilibrium, we have to describe equilibrium payoffs when a firm deviates from its equilibrium path. Those details are provided in Appendix B.

### 1.4. *The Correspondence with B & M*

The comparative statics as $\lambda_0, \lambda_1$ vary are the same as in B & M. The main difference is that the above results describe a dynamic equilibrium for $r > 0$, while B & M assume $r = 0$. It is therefore interesting to consider the comparative static as $r$ decreases—all agents become arbitrarily patient. It follows that the market becomes less monopsonistic; the distribution of wages shifts to the right. This occurs for two reinforcing reasons. As $r$ (and $R$) changes, the equilibrium distribution of firm sizes described in Theorem 1 does not change (reflecting Lemma 2). For given $R$ and $n$, (10) implies that a decrease in $r$ causes $\hat{w}(n, \cdot)$ to rise. Lower discounting reduces (relatively) the return to announce $w = R$ (to extract the search rents of employees in the short run) and so the equilibrium wage increases. Furthermore, given higher wages and lower discounting, the workers' reservation wage $R$ also rises. This rise in $R$ leads to a further increase in firm wages. The unemployed are unambiguously better off and the distribution of wages with a low $r$ first order stochastically dominates one with a high $r$. As $r \to 0$, the market equilibrium in fact converges to the dispersed wage equilibrium described by B & M.

## 2. NON-STEADY-STATE DISPERSED PRICE EQUILIBRIA

The above structure can be easily extended to non-steady-state equilibria. To do this, assume $\lambda_0 = \lambda_1 = \lambda$. In the previous section this restriction and (9) imply $R = b$ independent of $F$. In this section the distribution of market wages will change over time, so that the value of being unemployed will be time varying. The optimal job acceptance strategy compares the value of remaining unemployed to the value of taking the job. But as the current wage at the firm cannot describe the whole path of future prices there, a reservation wage strategy will not be optimal in general. Constructing an equilibrium in the general case is complex.

In order to proceed, we have to impose some structure on the worker turnover strategies. We shall only consider turnover strategies which have two particular properties:

**(A1)** Each unemployed worker's job search strategy has the reservation wage property, and that reservation wage $R$ always equals $b$.

Although (A1) need not be true in general, this strategy will be optimal in the equilibrium described below. Also assume:

**(A2)** Along the equilibrium path, workers always quit from a small firm (offering a low current wage) to a large firm (offering a high current wage).

In the equilibrium described below, larger firms always announce higher wages. (A2) guarantees that large firms offering high wages will continue to be large and hence will continue to offer high wages in the future. This implies that (A2) will describe individually rational behavior.

As before, the aim is to construct a perfect equilibrium in Markov strategies. Obviously, these strategies must be consistent with (A1) and (A2). The final equilibrium will be similar to the one described earlier, where at any time $t$, firm $i$ is indifferent to announcing some wage $w_{it} > b$, or deviating by announcing the workers' reservation wage $b$.

At $t = 0$, let $G(n, 0)$ denote the (given) initial distribution of firm sizes which is assumed to be common knowledge. As before, search frictions imply nobody observes how $F, G$ evolve over time. However, as all are small they each take these distributions as given. In equilibrium, each holds beliefs on $F(w, t), G(n, t)$ for all $t \geq 0$, and of course rational expectations requires that those beliefs must be consistent with the equilibrium outcome.

Following the methodology of the first section, we construct a dynamic market equilibrium in two steps. As before, the first step ignores the issue of perfectness and constructs a candidate wage and employment trajectory for each firm given the assumed turnover behavior of workers. That trajectory will satisfy a condition analogous to (M1). The second step shows that a perfect equilibrium exists with strategies analogous to (P1)–(P3) and (T1)–(T4).

### 2.1. *Step* 1: *A Candidate Wage Trajectory*

In this step, assume *firms can precommit to a wage trajectory*. Consider a representative firm which has $n(0)$ employees at $t = 0$. Let $\mu = G(n(0), 0) \in [0, 1]$, and let $\hat{w}(\mu, t)$ denote the wage path posted by this firm and $\hat{n}(\mu, t)$ denote its corresponding employment path. The wage trajectory $\hat{w}(\mu, t)$ is required to satisfy two conditions:

**(M1′)** for all $\mu \in [0, 1]$ and $t \geq 0$, a firm with $\hat{n}(\mu, t)$ workers is indifferent to posting the wage trajectory "$w = \hat{w}(\mu, t')$ for all $t' \geq t$" or posting "$w = b$ for all $t' \geq t$."

(M1′) corresponds to (M1) in the previous section. Although in this section the firm is not allowed to switch to the wage trajectory "$w = b$ in the entire future," the trajectory $\hat{w}(\mu, t)$ has to have the property that the firm would be indifferent to doing this if it were allowed to do so at any stage. Of course, this property will allow us to use a construction similar to (P1)–(P3).

As in the previous section, it will never be optimal to announce a wage below $R$, and so anticipating this, we require the wage trajectory is always bounded below by $b$. We also require $\hat{w}$ is strictly increasing in $\mu$—large firms announce higher wages—in order to guarantee that (A2) describes an equilibrium turnover response. Hence the trajectory must also satisfy:

**(M2′)** for all $t \geq 0$ and $\mu \in [0, 1]$, $\hat{w}(\mu, t) \geq b$ and is strictly increasing in $\mu$.

Finally, if $G(n, t)$ denotes the distribution of firm sizes at time $t$, assume

**(M3′)** $G(n, t)$ contains no mass points and has a connected support for all $t \geq 0$.

For now, only equilibria with no mass points are considered. The remainder of this section constructs the (unique) wage trajectory $\hat{w}(\mu, t)$ which satisfies (M1′)–(M3′), given A1, A2, and $G(n, 0) \in C^0$.

An immediate implication of (A2) and (M2′) is that $G(\hat{n}(\mu, t), t) = \mu$ for all $t \geq 0$; a firm's ranking in the distribution of firm sizes does not change over time. Connectedness of $G$ by (M3′) implies $\hat{n}$ must be continuous in $\mu$ for all $t$, while no mass points implies it must be strictly increasing with $\mu$ for all $t$. But also note that given any such solution for $\hat{n}$, $G(n, t)$ must satisfy $\hat{n}(G, t) = n$. Similarly if $F(w, t)$ denotes the distribution of market wages at time $t$, then $F$ is given by $\hat{w}(F, t) = w$. The constructed trajectories $\hat{n}, \hat{w}$ define implicit functions for $G$ and $F$.

Consider firm $\mu \in [0, 1]$. Define $N(\mu, t)$ as the number of employees who at time $t$ are employed at firms smaller than firm $\mu$. With no mass points, $N$ can be defined as

$$N(\mu, t) = \int_0^{\mu} \hat{n}(\mu', t)\, d\mu'. \tag{13}$$

and so $\hat{n}(\mu, t) = \partial N(\mu, t)/\partial\mu$. Continuity of $\hat{n}$ implies $N$ is continuously differentiable with respect to $\mu$.

As all firms announce a wage greater than $b$, all unemployed workers who contact a firm will accept employment there. Let $\alpha(t)$ denote the rate at which each firm is contacted by unemployed workers. (A2), (M2′), and no mass points imply:

$$\frac{\partial N}{\partial t}(\mu, t) = \mu\alpha(t) - N(\mu, t)[\delta + \lambda(1 - \mu)]. \tag{14}$$

$\mu\alpha(t)$ is the flow of unemployed workers into these smaller firms,[6] while $\delta N$ is the flow out of employees who are exiting the labor market and $\lambda N(1 - \mu)$ is the flow out who are quitting to take better paid jobs in firms larger than firm $\mu$.

As $\partial\hat{n}/\partial t = \partial^2 N/\partial t\,\partial\mu$, (14) implies

$$\frac{\partial \hat{n}}{\partial t}(\mu, t) = \alpha(t) + \lambda N(\mu, t) - \hat{n}(\mu, t)[\delta + \lambda(1 - \mu)]$$

The second term is the flow of workers from smaller firms paying lower wages who contact our representative firm and accept employment, while the third is the loss of workers who either exit the labor market or quit to larger firms paying higher wages.

LEMMA 6. *Given a deterministic path* $[\alpha(t)]_{t=0}^{\infty}$ *and* $\mu \in [0, 1]$, *then* (A1), (A2), (M2′), *and* (M3′) *imply*

$$N(\mu, t) = N(\mu, 0)e^{-[\delta + \lambda(1-\mu)]t} + \int_0^t \mu\alpha(s)e^{-[\delta + \lambda(1-\mu)](t-s)}\, ds \tag{15}$$

*and* $\hat{n}(\mu, t) = \partial N(\mu, t)/\partial\mu$, *where* $G(n, 0) \in C^0$ *determines* $N(\mu, 0) \in C^1$.

[6] Recall there is a unit mass of firms.

*Proof.* The solution for $N$ follows directly from (14).

Clearly the above solution is uniquely determined. The first term in (15) describes the number of workers who at time zero were employed at firms smaller than $\mu$ and who have not left this set of firms by time $t$, while the second term describes the subsequent entry of new workers who also have not left by time $t$. Given these solutions, we now consider (M1′).

THEOREM 2. *Given* (A1), (A2), $G(n,0) \in C^0$ *and any path* $[\alpha(t)]_{t=0}^{\infty}$, *there is a unique wage trajectory satisfying* (M1′) *which is given by*

$$\hat{w}(\mu, t) = b + \frac{\lambda(\pi - b)}{r + \delta + \lambda}\left[\mu + \frac{N(\mu, t)}{\hat{n}(\mu, t)}\right] \tag{16}$$

*Proof.* The proof is in Appendix A.

This is an important result as it describes wages generally, given the current distribution of firms sizes $G(n,t)$ (which describes $\hat{n}$ and $N$). Clearly the wages derived in the steady-state case also satisfy this condition (with $b$ replaced by $R$). To understand why this equilibrium wage equation is static, notice it can be rearranged as

$$n[\hat{w} - b] = \frac{(\pi - b)}{r + \delta + \lambda}[\lambda\mu n + \lambda N]$$

where $n = \hat{n}$. Condition (M1′) requires that the firm is always indifferent to switching to $w = b$. Suppose rather than switch to $b$ at time $t$, the firm switches to $b$ the following period. By switching at $t$ and paying a lower wage, the LHS describes the one period increase in the search rents that are extracted from its current employees. However, the loss is that the firm's employment level is lower. By switching to $b$ rather than paying $\hat{w}$, $\lambda\mu n$ of the firm's current employees quit (a flow) to smaller firms who are paying wages greater than $b$. Previously these workers would have stayed. Similarly $\lambda N$ workers in smaller firms no longer quit to take work in this deviating firm. Hence $\lambda\mu n + \lambda N$ is the (flow) reduction in employment by cutting wage today, and $(\pi - b)/(r + \delta + \lambda)$ is the expected discounted return to employing those workers at wage $b$. The RHS is therefore the expected loss through reduced employment by cutting wages now rather than in the next period. By setting the two sides equal to each other for all $t$, the firm is always indifferent to cutting wages to $b$, and (M1′) is satisfied.

Theorem 2 implies $\hat{w}(0,t) = b$ for all $t$.[7] The smallest firm always announces the workers' reservation wage $b$. Also Theorem 2 and (M3′) imply $\hat{w}$ is a continuous function of $\mu$ for all $t$.

Now consider (M2′)—that $\hat{w}$ is strictly increasing in $\mu$ for all $t$. Assuming $\hat{n}$ is differentiable in $\mu$ (i.e., assume $N(\mu,0)$ is twice differentiable), Theorem 2 implies a necessary and sufficient condition for (M2′) to be satisfied is that

$$2\left[\frac{\partial N}{\partial \mu}\right]^2 - N\frac{\partial^2 N}{\partial \mu^2} > 0 \tag{17}$$

for all $t \geq 0$, $\mu \in [0,1]$, where $N = N(\mu,t)$.[8] Unfortunately, even though $N(\mu,t)$ evolves over time in a relatively simple way, the link between those dynamics and the curvature of $N$ with respect to $\mu$ is complicated. Although it is possible to state a sufficient set of restrictions on $G(n,0)$, $\lambda$ and $\delta$ to guarantee existence of a candidate wage trajectory, the conditions are quite restrictive and not particularly interesting.[9] Equation (17) essentially requires that $\hat{n}$ does not increase too quickly with $\mu$. If $\hat{n}$ grows too quickly, then for very large firms, their level of employment $\hat{n}(\mu,t)$ is large relative to $N(\mu,t)$. These large firms then have a greater incentive to cut wages to extract their current employees' search rents and so $\hat{w}$ has to be low to compensate, which contradicts (M2′). Although this implies an equilibrium of the form we are constructing does not exist, it does not necessarily imply there is no equilibrium.

Assuming $\hat{w}$ defined by Lemma 6 and Theorem 2 satisfies (M2′), it is easy to show that the dynamics are stable and converge to the steady state described in the previous section. To illustrate those dynamics, we consider a simple example. Suppose this industry starts life at $t = 0$ with a unit mass of firms entering the market, each with no employees, i.e., assume $G(n,0) = 1$ for all $n > 0$.[10] We abstract from the previous model by assuming there are no unemployment dynamics, and hence assume $\alpha(t) = \alpha$ for all

[7]As $\hat{n}$ is a positive and strictly increasing function of $\mu$, then by definition of $N$, $N(\mu,t)/\hat{n}(\mu,t) \to 0$ as $\mu \to 0$.

[8]Differentiate (16) w.r.t. $\mu$ and use $\hat{n} = \partial N/\partial \mu$.

[9]A set of sufficient conditions places several curvature restrictions on $N(\mu,0)$, requires sufficiently high unemployment at $t = 0$ and $\delta > \lambda$. The latter two restrictions imply $\alpha(t)$ is high relative to $\lambda N$, so that large firms do not grow disproportionately quickly.

[10]Although formally inconsistent with (M3′), this assumption is not inconsistent with the previous analysis. Solving the equations given in Lemma 6 implies $\hat{n}(\mu,t)$ is strictly increasing in $\mu$ for all $t > 0$ and so (M3′) only fails at $t = 0$. Generalizing the above analysis by allowing mass points at $t = 0$ (but not for $t > 0$) is straightforward, as the following example demonstrates.

$t$.[11] In that case, solving (15) implies

$$\hat{n}(\mu, t) = \frac{\alpha(\delta + \lambda)}{[\delta + \lambda(1 - \mu)]^2}[1 - \exp(-[\delta + \lambda(1 - \mu)]t)] - \frac{\lambda\alpha\mu t}{\delta + \lambda(1 - \mu)}\exp(-[\delta + \lambda(1 - \mu)]t) \tag{18}$$

$$N(\mu, t) = \frac{\alpha\mu}{\delta + \lambda(1 - \mu)}[1 - \exp(-[\delta + \lambda(1 - \mu)]t)] \tag{19}$$

as $N(\mu, 0) = \hat{n}(\mu, 0) = 0$ for all $\mu$. Given this solution, (16) implies $\hat{w}(\mu, t)$.

THEOREM 3. *Given* $G(n, 0) = 1$ *for all* $n > 0$ *and* $\alpha(t) = \alpha$ *for all* $t \geq 0$, *trajectories* $\hat{w}(\mu, t)$, $\hat{n}(\mu, t)$ *defined by* (16), (18), (19) *satisfy* (M1′)–(M2′). *Also*:

(i) $\hat{n}(\mu, t)$ *is strictly increasing in* $\mu$ *for all* $t > 0$ [*no mass points*]

(ii) $\hat{n}(\mu, t)$ *is strictly increasing in* $t$, *for all* $t \geq 0$, $\mu \in [0, 1]$,

(iii) $\hat{w}(\mu, t)$ *is strictly decreasing in* $t$, *for all* $t \geq 0$, $\mu \in (0, 1]$,

(iv) *as* $t \to 0$, $\hat{w}(\mu, t) \to b + 2\mu\lambda(\pi - b)/[r + \delta + \lambda]$, *and* $\hat{n}(\mu, t) \to 0$

*Proof.* The proof is in Appendix A.

(M1′) implies that at $t = 0$, each path generates the same expected discounted profits. Hence at $t = 0$, assume each firm randomizes and chooses $\mu \in [0, 1]$ according to the uniform distribution. Given their choice of $\mu$, each firm then posts the wage trajectory $\hat{w}(\mu, t)$ defined by (16), (18), and (19). By construction, (M1′) and (M2′) are satisfied, and (i) implies there are no mass points for $t > 0$. (iv) implies the initial wage distribution is in fact uniform.[12]

By construction, low $\mu$ firms pay relatively low wages for all $t$. Although (ii) implies all firms grow in size over time, low $\mu$ firms do not grow as quickly as high $\mu$ firms. As a result, the distribution of firm sizes gradually

[11] It can be shown that (A1) and (M2′) imply reduced form dynamics $\alpha(t) = \alpha_0 + \alpha_1 e^{-(\delta+\lambda)t}$. It is easy to solve (15) in this case, and hence obtain $\hat{w}$, but the added dynamic contributes no interesting insights and only serves to complicate the algebra.

[12] This construction can be used for any initial distribution of firm sizes with mass points. If there is a mass of firms $m(n)$ of size $n$ in the initial distribution, assume these firms choose $\mu \in [G(n, 0) - m(n), G(n, 0)]$ according to the uniform distribution. Given $\mu$, Lemma 3 then goes through for $t > 0$ as those dynamics imply $\hat{n}$ is strictly increasing in $\mu$ for $t > 0$. (16) implies that wages are uniformly distributed across these firms at $t = 0$.

becomes more disperse over time. In the short run, low $\mu$ firms make greater flow profit. Their employment level is almost the same as the high $\mu$ firms, but they are paying lower wages. Of course, as the economy converges to the steady state, they make less profit as they have fewer workers (as described in the previous section).

Part (iii) implies that high $\mu$ firms offer relatively high wages at first, but gradually lower the premium paid over time. The distribution of firm wages becomes less disperse over time. Mathematically, this occurs because high $\mu$ firms always grow more quickly than low $\mu$ firms (as they attract workers away from those smaller firms), and so for any $\mu$, $N(\mu, t)/n(\mu, t)$ gradually decreases over time. More intuitively, as each firm grows it has a greater incentive to announce $\underline{w} = \underline{b}$ to extract the search rents of its current employees, and so the equilibrium wage has to fall over time to prevent them from extracting those rents.

### 2.2. *Step* 2: *A Market Equilibrium*

Given $G(n,0) \in C^0$ and $[\alpha(t)]_{t=0}^{\infty}$, consider the trajectories $\{\hat{w}(\mu, t), \hat{n}(\mu, t), N(\mu, t)\}$ defined by Lemma 6 and Theorem 2, and assume $N$ satisfies (17). Define $\underline{n}(t) = \hat{n}(0, t)$ and $\overline{n}(t) = \hat{n}(1, t)$. As $\hat{n}$ is continuous and strictly increasing in $\mu$ for all $t \geq 0$, its inverse function $\mu = \hat{n}^{-1}(n, t)$ is well-defined, continuous and strictly increasing in $n$. This inverse function gives the distribution of firm sizes; $G(n, t) = \hat{n}^{-1}(n, t)$ which has no mass points and connected support denoted $[\underline{n}(t), \overline{n}(t)]$. Similarly we can obtain the wage distribution $F(w, t)$ which also has no mass points and has connected support.

Equilibrium requires that all believe the market evolves over time according to $F, G$ as defined above. Consider a firm at time $t$ which has employment level $n \in [\underline{n}(t), \overline{n}(t)]$. Given its belief $G$, this firm now believes its ranking is $\mu = G(n, t)$, as do all workers who contact this firm. If $n \in [\underline{n}(t), \overline{n}(t)]$ and given its current wage $w$, the firm announces updated wage $w'$ according to the following strategy

(P1′) $w' = \hat{w}(\mu, t)$ if $w = \hat{w}(\mu, t - \Delta)$,

(P2′) $w' = w(\mu, t)$ if $w > \hat{w}(\mu, t - \Delta)$,

(P3′) $w' = b$ if $w_{t-\Delta} < \hat{w}(\mu, t - \Delta)$,

where $\mu = G(n, t)$, while if $n < \underline{n}(t)$, the firm announces $w' = b$.[13]

The structure of this strategy is the same as before. (P1′) describes the equilibrium path, which is now time varying. In the limit as $\Delta \to 0$, the

[13] Note that $n > \overline{n}(t)$ is not feasible.

equilibrium turnover strategies of workers at time $t$ are as follows. Given $n' \in [\underline{n}(t), \bar{n}(t)]$ and $\mu' = G(n', t)$, then

(T1′) if the firm announces $w' = \hat{w}(\mu, t)$:

(i) Unemployed workers and employed workers with no outside offers stay,

(ii) Workers with an outside offer $(\mu', \hat{w}(\mu', t))$ accept that offer if and only if $\hat{w}(\mu', t) > w$

(T2′) If the firm offers $w' > \hat{w}(\mu, t))$:

(i) Unemployed workers and employed workers with no outside offers stay,

(ii) Workers with an outside offer $(\mu', \hat{w}(\mu', t))$ accept that offer if and only if $\mu' > \mu$

(T3′) If the firm offers $w' \in [b, \hat{w}(\mu, t))$:

(i) Unemployed workers and employed workers with no outside offers stay,

(ii) Workers with an outside offer $(\mu', \hat{w}(\mu', t))$ accept that offer if and only if $\hat{w}(\mu', t) > b$.

(T4′) If the firm offers $w' < b$ all workers quit.

Notice these strategies imply that unemployed workers and employees with no outside offers stay if and only if $w \geq b$, i.e., workers use a reservation wage strategy $b$, consistent with (A1). Furthermore, turnover response (T1) is consistent with (A2)—along the equilibrium path, workers always switch to larger firms (paying higher wages). This is an equilibrium response as such firms are expected to announce higher wages in the entire future.

(P1′) and (T1′) describe the equilibrium path, which by construction is described by step 1. Given (P1′) and (T1′), the turnover response (T1′) is optimal for each worker. As before we need to verify that these strategies also form best responses off the equilibrium path. However, as this proof is essentially the same as for Theorem 1, we only sketch its details.

Suppose given $n$ and associated $\mu = G(n, t)$, the firm deviates with $w' > \hat{w}(\mu, t)$. (P2′) implies workers expect the firm to return to the equilibrium path next period. For the same reason as given in the proof of Theorem 1, this price is dominated by announcing $w = \hat{w}(\mu, t)$; the deviating strategy attracts more workers which increases expected discounted profits by $0((w' - \hat{w}(\mu, t))\Delta^2)$ but the additional wage bill is $n[w' - \hat{w}(\mu, t))]\Delta$, and this wage effect dominates for $\Delta$ small enough.

If the firm deviates by announcing $w' \in [b, \hat{w}(\mu, t))$ then workers expect the firm to announce $w = b$ in the entire future [for $\Delta$ small enough]. As before, announcing $w' = b$ dominates any other price in this region, and by construction, the firm is indifferent to announcing $w' = b$ or returning to the equilibrium path and so this describes an optimal pricing strategy in the subgame.

Finally, if the firm announces $w' < b$, all workers except the firm to announce $w = b$ in the entire future. As the rate at which workers contact other firms is independent of being employed or unemployed, all workers choose to quit as the flow payoff to being unemployed $b$ dominates this firm's current and expected future wages. Hence a reservation wage strategy $R = b$ is optimal. Clearly offering a wage $w' < b$ is dominated by offering $w' = b$.

Obviously as all firms always stick to their equilibrium wage strategy $\hat{w}(\mu, t)$, which by construction implies employment path $\hat{n}(\mu, t)$, then $F$ and $G$ indeed evolve according to the beliefs of the agents, and so beliefs are rational and we obtain a market equilibrium.

## 3. CONCLUSION

This paper has extended the approach of B & M by allowing firms to change wage over time. It has not only shown that the B & M approach is robust to this extension (their equilibrium coincides with the limiting case as $r \to 0$), but it has also characterized the non-steady-state market dynamics which converge to the B & M steady state.

An important feature of this extension is it shows explicitly that quit turnover is driven by worker expectations on future wages. An interesting extension would be to allow firm death, where some firms are more likely to go out of business than others. Workers would then be willing to quit to firms paying lower wages should those firms have a greater survival probability. The insight obtained here, that paying higher wages today is a (costly) investment into a larger workforce in the future, suggests that in equilibrium firms with lower survival probabilities will pay lower wages (they discount future returns more heavily). Further, a dynamic model where firms might receive survival shocks over time, could result in firms cutting wages whenever bad news about future survival is received, and their employee quit rate increasing significantly. The future prospects of a firm then has a direct impact on quit turnover and optimal wage setting.

Of course whenever there is repeated interaction between agents, many Nash equilibria may exist. The B & M structure is not immune to this problem. For example, candidate search equilibria with mass points exist, a simple case being that all firms are of equal size $n = \alpha/\delta$, all announce wage $w^* = R + \lambda_1(\pi - R)/(r + \lambda_1 + \delta)$ and

$$R = \frac{(r + \delta + \lambda_1)^2 b + (\lambda_0 - \lambda_1)\lambda_1 \pi}{(r + \delta + \lambda_1)^2 + (\lambda_0 - \lambda_1)\lambda_1}.$$

Such a candidate search equilibrium was ruled out as a market equilibrium by Lemma 3—in that case a firm would slightly raise wage and so attract workers away from the other firms. But a market equilibrium with mass points is possible if we trap firms in those mass points. In particular, we could replace (P2) with (P2′) $w' = R$ if $w > w^*(n)$.

(P2′) implies workers except any firm which raises its wage above its equilibrium level will then post their reservation wage in the future. Rather than attract more workers, this subgame ensures that workers will quit this firm at the first suitable opportunity. As no firm can attract more workers by raising its wage, each is "trapped" in the mass point. But note that unlike the standard folk theorems, workers are not using trigger strategies to punish firm deviations. Indeed, workers use a stationary quit strategy and should the firm return to the equilibrium path, its previous pricing history is "forgotten." Instead, the multiplicity here arises because multiple candidate search equilibria with mass points exist, and a non-stationary wage setting strategy for firms can be constructed which traps firms in those mass points.

## APPENDIX A

*Proof of Lemma* 1. In the limit as $\Delta \to 0$, (2) implies

$$\dot{n} = \alpha - \delta n - \lambda_1(1 - m)n$$

and (3) becomes

$$r\hat{V}(n) - \hat{V}'(n)\dot{n} = n[\pi - R]$$

Clearly as $t \to \infty$, $n(t) \to n_0 = \alpha/(\delta + \lambda_1(1 - m))$. As $\dot{n} = 0$ at $n = n_0$, then $\hat{V}(n_0) = n_0[\pi - R]/r$. Substituting out $\dot{n}$ in the value function equation implies the first order differential equation

$$r\hat{V}(n) - \hat{V}'(n)[\alpha - \delta n - \lambda_1(1 - m)n] = n[\pi - R].$$

Integrating this differential equation using an appropriate integrating factor is standard, where the solution for $\hat{V}(n_0)$ above provides the required boundary condition. As (M1) requires $V(n) = \hat{V}(n)$, this implies

(5) (where direct inspection shows this solution satisfies the above differential equation) and (1) then implies $\hat{w}(n)$.

*Proof of Lemma* 3. Proof by contradiction. Suppose such a market equilibrium exists where $G$ has a mass point $m > 0$ at some $n \geq \underline{n} > 0$. In any such equilibrium, each firm in this mass point announces wage $w = w^*(n)$, makes expected discounted profit $V(n)$ and $dn = 0$.

Suppose one of these firms deviates by announcing $w' = w^*(n) + \varepsilon$, where $\varepsilon = \lambda_1 m V'(n)/2 > 0$. As search frictions imply $dn$ is $0(\Delta)$, this implies $w' > w^*(n + dn)$ for $\Delta$ small enough, and (P2) then implies all who contact this firm expect wage $w^*(n + dn)$ in the entire future.

Now consider any worker who is at one of these mass point firms. Given $w' > w^*(n)$, those workers will now quit to this deviating firm [given contact] as not only is today's offered wage greater than $w^*(n)$, but (P2) and $dn \geq 0$ implies higher wages in the entire future. On-the-job search by these workers and the change in their quit strategy implies $dn$ is at least $mn[\lambda_1 \Delta] > 0$. This price deviation therefore raises future expected discounted profits by an amount $V'(n)dn \geq V'(n)\lambda_1 mn\Delta$. Of course, the cost to this deviation is the increase in this period's wage bill, which is $\varepsilon n \Delta = V'(n)\lambda_1 mn\Delta/2$. Hence this wage deviation is strictly profit increasing for $\Delta$ small enough (given $m > 0$), which is the required contradiction.

*Proof of Lemma* 4. As $\hat{w}$ defined by (4) is a strictly increasing function, then $\hat{G}(n) = \hat{F}(\hat{w}(n))$ for all $n \in [\underline{n}, \overline{n}]$. Note that no mass points in $\hat{G}$ implies no mass points in $\hat{F}$. We now show that no mass points in $\hat{G}$ also implies that the supports of $\hat{F}$ and $\hat{G}$ are connected.

LEMMA A1. *If a candidate steady state exists, then the supports of $\hat{F}$ and $\hat{G}$ are connected.*

*Proof.* By contradiction. Suppose the support of $\hat{G}$ is not connected, i.e., there exists $n_0, n_1 \in S$ where $n_1 > n_0$ and $\hat{G}(n_1) = \hat{G}(n_0)$. Lemma 1 implies $\hat{w}(n_1) > \hat{w}(n_0)$, and as $\hat{G}(n) = \hat{F}(\hat{w}(n))$ for all $n \in [\underline{n}, \overline{n}]$, this implies $d\hat{F}(w) = 0$ for all $w \in (\hat{w}(n_0), \hat{w}(n_1))$. But no mass points in $\hat{F}$ and Lemma 2 now imply $\hat{n}(\hat{w}(n_1)) = \hat{n}(\hat{w}(n_0))$. As $n_0, n_1 \in S$ this implies $n_1 = n_0$ which is the required contradiction.

We now construct the candidate steady state. Lemma A1 implies that $\hat{n}(\hat{w}(n)) = n$ for all $n \in [\underline{n}, \overline{n}]$ which is possible if and only if $\hat{n}(w) = \hat{w}^{-1}(w)$ for all $w \in [\underline{w}, \overline{w}]$. Inverting $\hat{w}$ in Theorem 1 implies

$$n = \hat{w}^{-1}(w) = \frac{\underline{n}(\lambda_1 + \delta)(\pi - R)}{(\lambda_1 + \delta)(\pi - R) - (r + \lambda_1 + \delta)(w - R)}$$

while no mass points and Lemma 2 require

$$n = \hat{n}(w) = \frac{\alpha(\delta + \lambda_1)}{\left[\delta + \lambda_1(1 - \hat{F}(w))\right]^2}$$

Equating these two expressions implies a unique candidate solution for $\hat{F}$, whose solution is given in the Theorem. $\hat{G}(n)$ is given by $\hat{G}(n) = \hat{F}(\hat{w}(n))$ and the supports are identified by solving $\hat{F}(\underline{w}) = \hat{G}(\underline{n}) = 0$ and $\hat{F}(\overline{w}) = \hat{G}(\overline{n}) = 1$, while noting that no mass points implies $n_0 = \alpha/(\delta + \lambda_1)$ and Lemma 3 implies $\underline{n} = \hat{n}(\underline{w}) = n_0$. This completes the proof of Lemma 4.

*Proof of Lemma* 5. Substituting $\hat{F}$ for $F$ in (9) implies $R$ is defined as the solution to

$$R = b + \frac{\lambda_0 - \lambda_1}{\lambda_1}\int_R^{\overline{w}} \frac{\lambda_1\left[1 - \hat{F}(w', R)\right]}{r + \delta + \lambda_1\left[1 - \hat{F}(w', R)\right]}\, dw' \tag{20}$$

Using $\hat{F}$ defined in Lemma 4, it is possible to show that

$$\int_R^{\overline{w}} \frac{\lambda_1\left[1 - \hat{F}(w', R)\right]}{r + \delta + \lambda_1\left[1 - \hat{F}(w', R)\right]}\, dw'$$

$$= \frac{\pi - R}{(\lambda_1 + \delta)(r + \lambda_1 + \delta)}\left[\lambda_1^2 - 2r\lambda_1 + 2r(r + \delta)\log\frac{r + \lambda_1 + \delta}{r + \delta}\right]$$

Using this to substitute out the integral in (20) implies a solution for $R$ exists and must be unique, being a weighted average of $b$ and $\pi$. A little more algebra suffices to show that $R < \pi$ if and only if $b < \pi$.

*Proof of Theorem* 2. Let $\hat{V}(n, t)$ denote the value of announcing $w = b$ for all $t$, given $n$ employees at time $t$. (M2′) implies that any employee who contacts a firm offering a wage greater than $b$ will quit. Together (M2′) and (M3′) implies that this firm's employees will quit at rate $\lambda$ to take better paid jobs. Hence, for $\Delta$ arbitrarily small, $\hat{V}$ satisfies

$$[1 + r\Delta]\hat{V}(n, t)$$

$$= n[\pi - b]\Delta + \hat{V}\big(n + [\alpha(t) - (\lambda + \delta)n]\Delta, t + \Delta\big) + 0(\Delta^2)$$

Rearranging and letting $\Delta \to 0$ implies $\hat{V}$ must satisfy the differential equation:

$$r\hat{V}(n, t) = n[\pi - b] + [\alpha(t) - (\lambda + \delta)n]\frac{\partial \hat{V}}{\partial n}(n, t) + \frac{\partial \hat{V}}{\partial t}(n, t)$$

Inspection establishes that

$$\hat{V}(n,t) = \frac{n[\pi - b]}{r + \delta + \lambda} + \phi(t)$$

where $\phi$ is given by

$$r\phi(t) - \phi'(t) = \frac{\alpha(t)[\pi - b]}{r + \delta + \lambda}$$

satisfies this differential equation. Note that $n[\pi - b]/(r + \lambda + \delta)$ is the expected discounted profit by paying $b$ to $n$ employees who leave at rate $\lambda + \delta$. $\phi(t)$ describes the expected profit by hiring future employees at wage $b$, who arrive at rate $\alpha(t)$ over time.

Now (M1′) requires that $\hat{w}$ satisfies

$$\hat{V}(n,t) = \frac{1}{1 + r\Delta}\Big[n[\pi - \hat{w}]\Delta + \hat{V}\big(n + \big(\alpha(t) + \lambda N - [\delta + \lambda(1-\mu)]n\big)\Delta, t + \Delta\big)\Big]$$

where $n = \hat{n}(\mu, t)$. This condition ensures that the firm is indifferent to setting $w = b$ in the entire future and getting $\hat{V}$, or posting $w = \hat{w}$ today (in which case today's net turnover is given by $(\alpha(t) + \lambda N - [\delta + \lambda(1 - \mu)]n)\Delta$. Of course, the same must be true tomorrow and so we can use $\hat{V}$ to calculate that continuation value. Again, rearranging and letting $\Delta \to 0$ implies $\hat{w}$ must satisfy

$$n[\pi - \hat{w}] = r\hat{V} - \frac{\partial \hat{V}}{\partial t} - \big[\alpha(t) + \lambda N - [\delta + \lambda(1-\mu)n]\big]\frac{\partial \hat{V}}{\partial n}$$

where $n = \hat{n}(\mu, t)$. Using the above solution for $\hat{V}$ implies the Theorem.

*Proof of Theorem* 3. By construction these trajectories satisfy (M1′) for $t \geq 0$ and (M3′) for $t > 0$. Establishing that (M2′) is also satisfied requires showing that the candidate functions satisfy (17) for all $t \geq 0$. Let $x = [\delta + \lambda(1 - \mu)]t$. Using (19), it can be shown that (17) is satisfied for all $t > 0$ if and only if

$$2(\lambda + \delta)(\delta + \lambda(1-\mu))[1 - e^{-x}]^2 - 2\mu\lambda(\lambda + \delta)xe^{-x}(1 - e^{-x}) + \mu^2\lambda^2x^2e^{-x}(1 + e^{-x}) > 0$$

for all $x > 0$. Given any $x > 0$ and $\mu \in [0,1]$, it is straightforward to show this function is strictly increasing in $\delta$. Hence a lower bound is established by putting $\delta = 0$. With $\delta = 0$ and any $x > 0$, it is then straightforward to show this function is strictly decreasing in $\mu$ for $\mu \leq 1$. With $\delta = 0$ and $\mu = 1$, it follows this function is strictly positive for all $x > 0$. Hence it is strictly positive for all feasible parameter values, and so (M2′) is satisfied for $t > 0$. Further, as $t \to 0$ l'Hopital's rule implies $\lim_{t \to 0} N(\mu, t)/\hat{n}(\mu, t) = \lim_{t \to 0}[\partial N/\partial t]/[\partial \hat{n}/\partial t] = \mu$. (16) now establishes (iv) and that (M2′) is satisfied.

Differentiation of (18) directly establishes that $\partial \hat{n}/\partial t = \alpha[1 + \mu\lambda t]e^{-[\delta + \lambda(1-\mu)]t} > 0$.

Theorem 2 implies $\hat{w}$ is strictly decreasing in $t$ if and only if $(\mu N_\mu + N)/N_\mu$ is strictly decreasing in $t$. Hence $\hat{w}$ is strictly decreasing in $t$ if and only if $N_\mu N_t - N N_{\mu t} < 0$. As the solution for $N$ satisfies $N \in C^\infty$, (14) implies expressions for $N_t, N_{\mu t}$ which then imply $N_\mu N_t - N N_{\mu t} < 0$ if and only if $\alpha \mu N_\mu - \alpha N - \lambda N^2 < 0$. Substituting in the solution for $N$ given in (19) implies the result.

## APPENDIX B: PROOF OF THEOREM 1

Fix $\Delta > 0$ (small). We consider equilibrium behavior should a firm with $n \geq \underline{n}$ employees deviate and set $w' \neq w^*(n)$.

(a) Suppose the firm deviates by announcing $w' > w^*(n)$ and let $dn' \geq 0$ denote the additional number of workers attracted by this price deviation. Such a price deviation raises an employee's expected discounted utility by an amount

$$dW = (w' - w^*(n))\Delta + 0\left(\frac{dw^*(n)}{dn} dn'\right),$$

where $dW$ not only includes the increase in today's wage, but also the expected discounted value of higher wages (through increased employment at this firm) in the future.

On-the-job search implies that the number of workers who observe this wage offer and also hold an outside offer is $0(\Delta)$. For those workers, the added surplus $dW$ changes their quit decision if their outside offer $w^*(n') \in [w^*(n), w^*(n) + 0(dW))$. For $\Delta$ small enough, no mass points in $F$ (Lemma 4) implies $dn' = 0[\Delta dW]$. Using the above to substitute out $dW$ and rearranging implies $dn' = 0((w' - w(n))\Delta^2))$. As $V'(n)$ is finite, this increase in employment raises expected future discounted profits by an amount $0((w' - w(n))\Delta^2))$. But this price deviation increases today's wage bill by $n(w' - w(n))\Delta$. As $n \geq \underline{n} > 0$, this wage deviation is profit decreasing for $\Delta$ small enough. Hence announcing any wage $w > w^*(n)$ is dominated by announcing $w = w^*(n)$ for $\Delta$ small enough. Given (P2), the turnover response (T2) is optimal in the limit as $\Delta \to 0$.

(b) Suppose the firm deviates and announces $w' \in [R, w^*(n))$. Further suppose the resulting turnover $dn' \leq 0$ implies $w' < w^*(n + dn')$. In this case, the firm's pricing strategy implies the firm will announce $w = R$ in the entire future [as $n > \underline{n}$ in the entire future and by construction $R < w^*(n)$ for such $\underline{n}$]. The argument used to establish (a) now implies announcing $w = R$ dominates announcing this deviating wage; the latter wage attracts an additional number of employees [relative to wage $R$] by an amount which is $0((w' - R)\Delta^2)$ but the increased wage bill is $n(w' - R)\Delta$ and this cost dominates for $\Delta$ small enough.

Instead suppose the resulting turnover $dn' \leq 0$ implies $w' \geq w^*(n + dn')$ (which is possible for $dn'$ sufficiently negative) and so (P2) implies the firm returns to its equilibrium path [it announces wage $w^*(n + dn')$ in the entire future]. Now consider a firm with initial employment level $n_0 = n + dn'$. By announcing $w = w^*(n_0)$ its expected discounted profit is greater than that of the deviating firm [their expected future discounted profits are the same, while $w^*(n_0) \leq w'$ implies today's profit is greater]. Hence (M1) (which implies $V$ is strictly increasing in $n$) and $n_0 < n$ implies announcing $w = w^*(n)$, which generates discounted payoff $V(n)$, strictly dominates this deviating wage $w'$.

Hence for $\Delta$ small enough, announcing $w' \in [R, w^*(n))$ is dominated by announcing $w' = R$. Further, the (limiting) turnover strategy (T3) is optimal as in the limit as $\Delta \to 0$, any deviation $w' < w^*(n)$ implies the firm announces $w = R$ in the entire future.

(c) Finally if the firm announces $w' < R$, everyone expects a wage $w = R$ in the entire future. By construction of $R$, the value of search is strictly greater than the value of staying at this firm and so everyone quits. In the subgame, the firm always offers $w = R$ and only unemployed workers accept employment at this firm. Its employment level $n$ gradually rises and converges to $\underline{n}$. As before, offering a higher wage does not attract sufficiently many more workers to make such a deviation profitable (as they expect the wage increase to last for an arbitrarily short period of time), and this price response is therefore optimal in this subgame.

By construction, no wage deviation dominates announcing $w = w^*(n)$ or $R$ for $\Delta$ small enough, and hence these strategies describe a perfect equilibrium whose equilibrium path corresponds to the search equilibrium characterized above. This completes the proof of the theorem.

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