2. The Overlapping Generations Model of Fiat Money

We will now set up a general model in which fiat money may turn out to have value in equilibrium, the *Overlapping Generations* Model. This model was first developed by Samuelson (1958) and has been used to study not only fiat money, but also a range of issues in public finance and policy. We will consider fairly simple versions of the model.

I. describing economic "environments"

II. considering the types of "outcomes" (*i.e.* equilibria) that can arise in these environments.

I. *The Environment*

By the "environment" we mean those aspects of the model that are exogenously specified and which *agents* in the model must take as given.

1. Time

The overlapping generations (OLG) model is *dynamic* (as opposed to *static*) in that it contains an explicit time dimension. We model time as *discrete* as opposed to *continuous.* While this might be viewed as an unnatural assumption, it has the advantage of making analysis simpler. We label the first time period "period 1", and index time with the letter t, where $t = 1, \ldots, \infty$.

2. Agents

The economy is populated by a large number of individuals who live for (at least) two periods each. Agents in their first period of life are called "young" agents and those in their second period of life are called "old". In each period $t \geq 1$, a generation of comprised of N_t agents is "born". These agents are referred to as generation t At each point in time, t, the *population* of the economy is equal to the total number of agents, both young and old, alive in that period: the population at time $t = N_t + N_{t-1}$. Note that at any point in time there are agents of all generations alive.

3. Goods

To begin with there is one good *per period* in the economy. This good cannot be stored from one period to the next.

4. Endowments

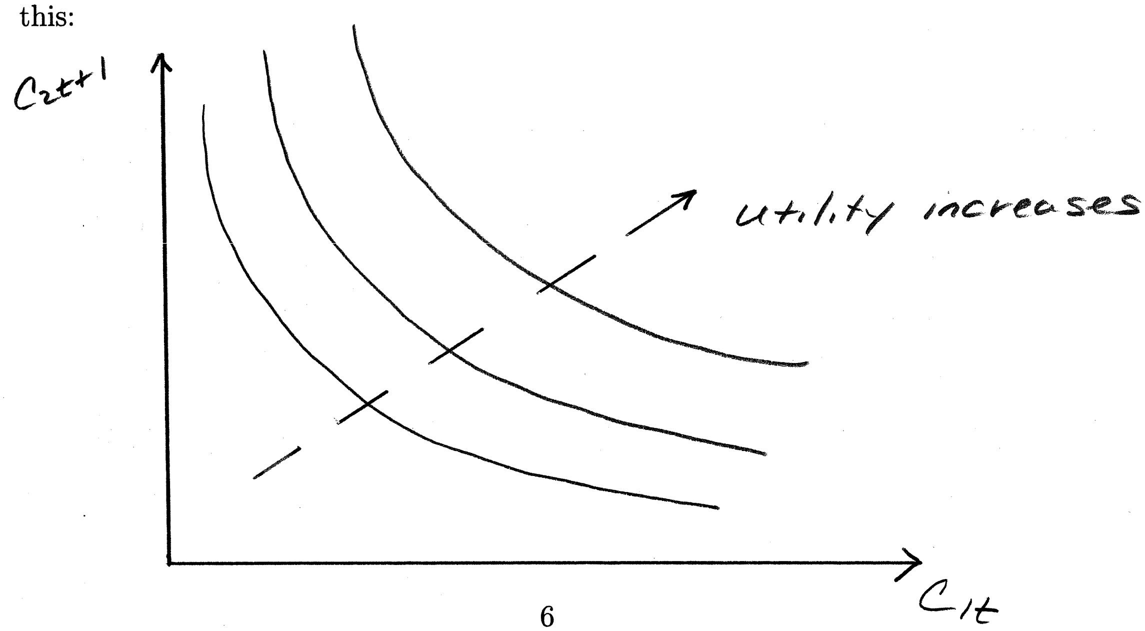
To begin with we assume that there is no production. The single good simply appears at the beginning of each period. Each young agent of generation t is *endowed* with y units of the time t good. Old agents are endowed with nothing. That is, agents of generation t are endowed with y units of good t and 0 units of good $t+1$.

5. Preferences

Each agent derives utility from consuming the two goods that exist during their lifetime, *i.e* goods t and $t+1$. The consumption allocation for a generation t agent is written, (c_{1t}, c_{2t+1}). Typically we assume that agents' preferences are described by a utility function, $u(c_{1t}, c_{2t+1})$ where this function has certain standard properties:

a. strict monotonicity: $\frac{\partial u}{\partial c_{ij}} > 0$ for $i = 1, 2$, $j = t, t+1$.

b. strict convexity: loosely speaking, this is a complementarity assumption. Agents prefer an even consumption profile over the two periods of their life to consume a lot when young and very little when old etc. It means that indifference curves look like this:



We now consider the properties of different *allocations*. In this environment (economy) an allocation is a complete list of the consumption profiles of all agents. For simplicity, we begin under the assumption that all agents of a particular generation are identical, and we focus on *symmetric* allocations, by which we mean allocations in which all agents of a given generation receive the same consumption. In many circumstances we will also restrict attention to allocations which are symmetric also in that members of every generation have the same lifetime consumption pattern, *i.e.* $c_{1t} = c_1$ and $c_{2t+1} = c_2$ for all t, $t+1$.

6. Feasible allocations

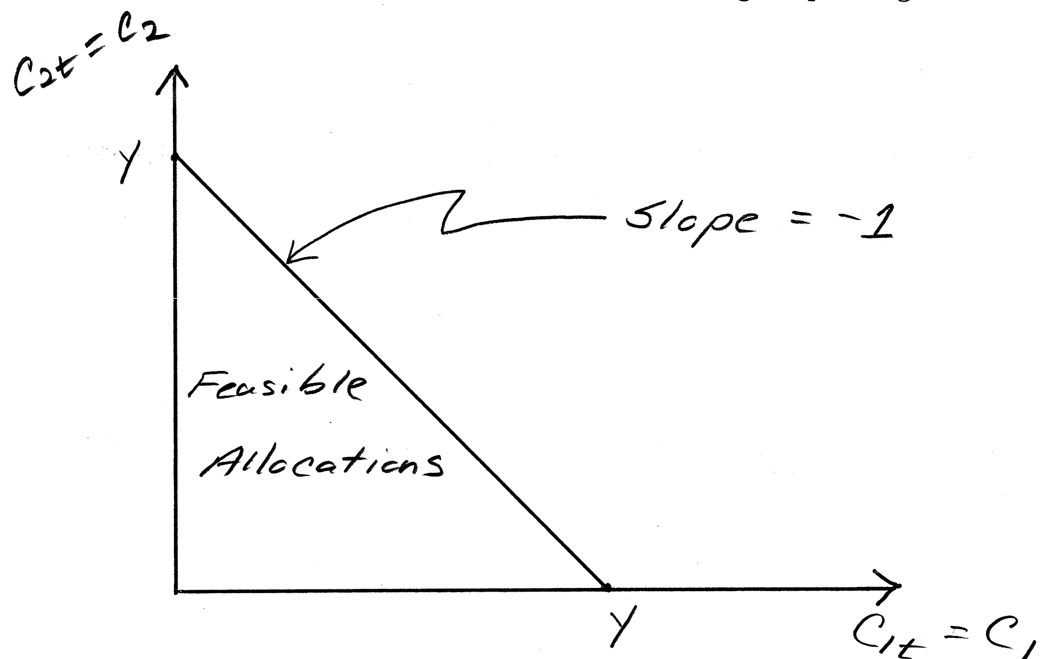
An allocation is *feasible* if total consumption is no greater than the total quantity of available consumption goods. In this environment feasible allocations satisfy the following inequality:

$$N_t c_{1t} + N_{t-1} c_{2t} \leq N_t y \tag{2.1}$$

If the population is constant, $N_t = N_{t+1}$ for all t and (2.1) becomes,

$$c_{1t} + c_{2t} \leq y \tag{2.2}$$

For the case of stationary symmetric allocations in an economy with constant population, we can represent the feasible allocations with the following simple diagram:



Now, consider the case where the population grows at a constant gross rate n so that:

$$N_{t+1} = nN_t \tag{2.3}$$

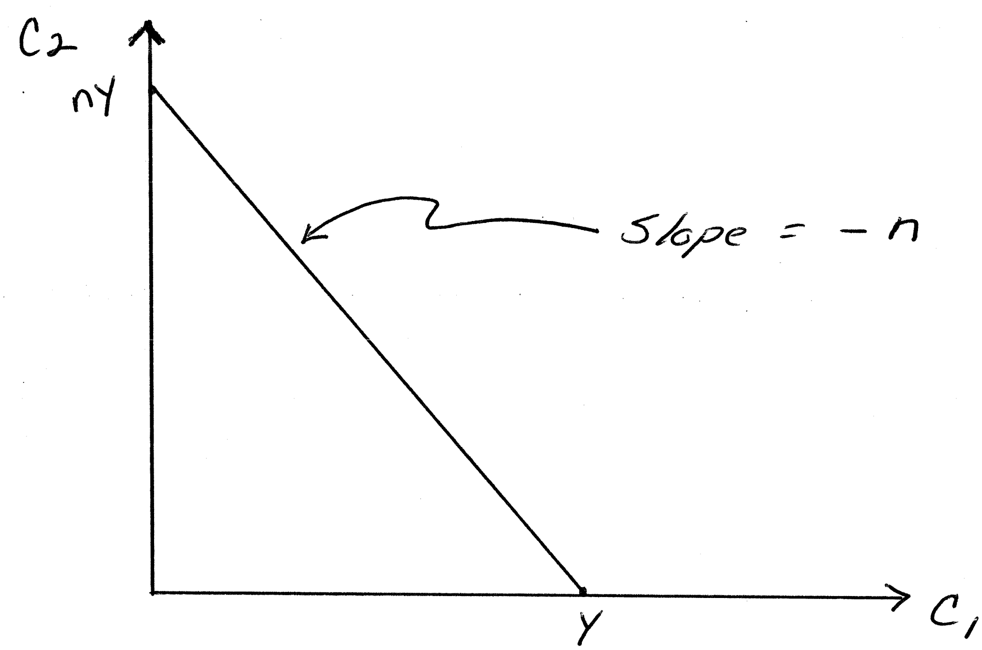
In this case the inequality satisfied by feasible allocations is:

$$N_t c_{1t} + N_{t-1} c_{2t} \leq N_t y \tag{2.4}$$

or, in the case of stationary allocations

$$c_1 + \frac{c_2}{n} \leq y. \tag{2.5}$$

In this case feasible allocations can be represented with the following diagram:



7. Pareto efficient allocations

An allocation is said to be Pareto efficient if it is:

i. feasible

ii. there is no other feasible allocation in which all agents are at least as well off and at least one agent is strictly better off.

Somewhat more formally... Consider two symmetric allocations in an economy with a constant population. We will abbreviate the allocations as A and B, where

$A = \left(c_{1t}^A, c_{2t}^A\right)$ and $B = \left(c_{1t}^B, c_{2t}^B\right)$, for all t. Suppose that allocation A is feasible, meaning that

$$c_{1t}^A + c_{2t}^A \leq y. \qquad \forall t \tag{2.6}$$

Allocation B is said to be *Pareto superior* to A if and only if:

i. B is feasible: $c_{1t}^B + c_{2t}^B \leq y$

ii. All agents are as least as well off at B as at A:

$$u(c_{1t}^B, c_{2t+1}^B) \geq u(c_{1t}^A, c_{2t+1}) \qquad \forall t \tag{2.7}$$

iii. *Some* agent is strictly better off at B than at A:

$$u(c_{1t}^B, c_{2t+1}^B) > u(c_{1t}^A, c_{2t+1}) \qquad \text{for some } t \tag{2.8}$$

Allocation A is then *Pareto efficient* if and only if there exists no allocation that is Pareto superior to it.

Pareto Efficiency is actually a fairly subtle concept, especially in an OLG model. Consider what is required for an allocation to be Pareto efficient.

i. The allocation must be on the boundary of the set of feasible allocations:

$$N_t c_{1t}^A + N_{t-1} c_{2t}^A = N_t y. \tag{2.9}$$

It should be clear why any allocation that does not satisfy (2.9) cannot be Pareto efficient.

ii. All agents must have equal marginal rates of substitution at that allocation. Recall that the marginal rate of substitution (MRS) for a generation t agent is given by:

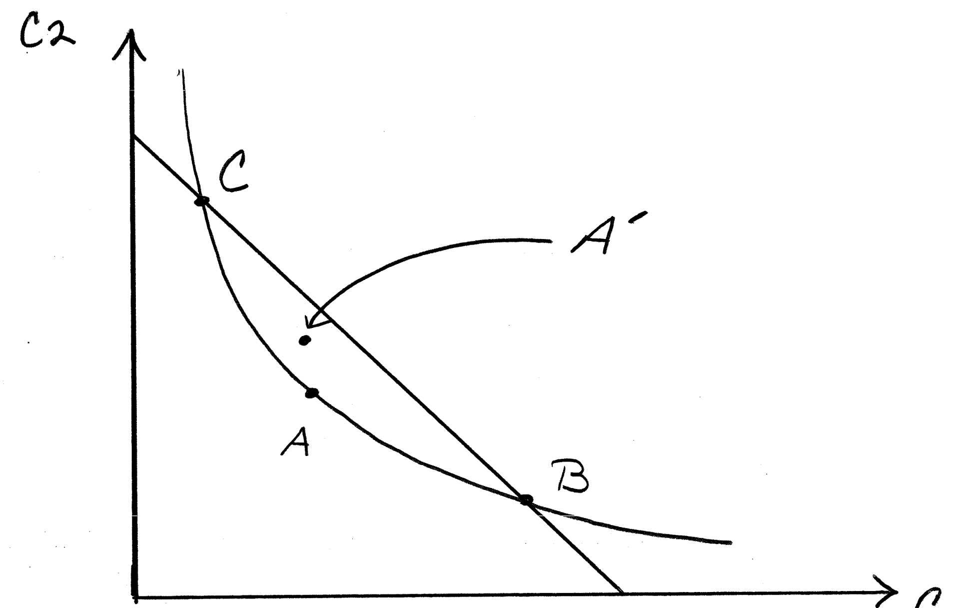
$$MRS_t = \frac{\frac{\partial u}{\partial c_{1t}}}{\frac{\partial u}{\partial c_{2t+1}}} \tag{2.10}$$

and is equal to the slope of the indifference curve. Clearly, it only makes sense to talk about this being equal for two agents who consume the same goods, *i.e.* two agents of the same generation. Why is equality of the MRS's required for Pareto efficiency? Note that in a symmetric allocation, since all agents of a given generation have the same consumption profile, they all automatically have equal marginal rates of substitution.

Note: In many environments without frictions, i and ii above are both necessary *and sufficient* for an allocation to be Pareto efficient. This is not true in an OLG model. In these environments the following additional requirement is needed:

iii. The marginal rates of substitution must be "high enough".

Consider the diagram which depicts stationary symmetric allocations for an economy with no population growth. Strictly speaking, the diagram only depicts the consumption of agents of generation $t = 1$ onward. Assume that the time 1 old consume the same amount as old agents in all subsequent generations: $c_{21} = c_{2t}$ for all t.



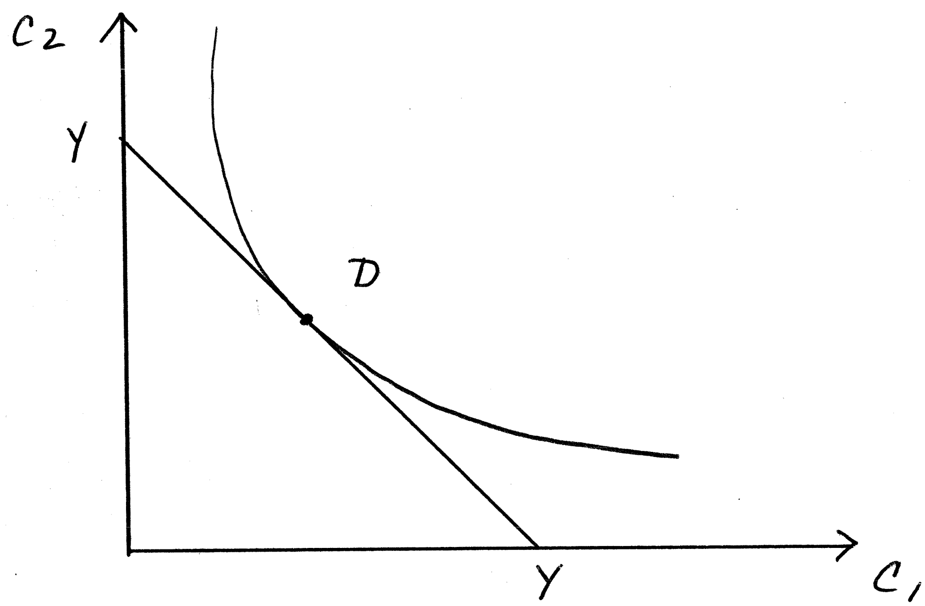
In this diagram, allocations **A** and **B** are not Pareto efficient, but allocation **C** is.

It should be immediately clear why allocation **A** is not Pareto efficient. Because **A** is not on the boundary of the feasible set, it is easy to find an allocation, like $\mathbf{A}'$, that is Pareto superior to it. Note that at $\mathbf{A}'$ each young agent consumes the same amount as they would at allocation **A**. Each old agent, however, consumes strictly more at $\mathbf{A}'$ than at **A**. Even though $\mathbf{A}'$ is Pareto superior to **A**, it is not itself Pareto efficient, because like **A** it lies in the interior of the set of feasible allocations.

More subtle reasoning is required to show that allocation **B** is not Pareto efficient. We show that allocation **C** Pareto dominates allocation **B**. This requires us to make use of the dynamic nature of the economy. All agents born at time $t = 1$ or later are indifferent between **B** and **C**. The time 1 old, however, strictly prefer **C** to **B** because they get higher consumption at **C**. This is sufficient to show that **B** is not Pareto efficient. At this point we can be clear about what it means to say that the marginal rate of substitution must be "high enough" for an allocation to be Pareto efficient. Note that any allocation **B** is typical of any allocation on the boundary of the feasible set at which the indifference curve cuts the boundary from below. In any such case there is an allocation like **C** on the same indifference curve for all generations born at $t \geq 1$ but which gives strictly higher utility to the time 1 old. Allocations like **C**, where the indifference curve cuts the boundary from above are Pareto efficient. Any movement downward and to the right along the boundary makes the initial old worse off, while any movement upward and to the left makes members of all generations born at $t = 1$ or later worse off.

8. The "Golden Rule" allocation

Consider allocation **D** in the following diagram:



At **D** the utility of all agents born at $t = 1$ and later is maximized. Moving in either direction along the boundary from **D** makes all generations born at $t \geq 1$ strictly worse off. Note, however, that allocation **D** does not maximize the utility of the time 1 old. These agents would most prefer to be at the upper left corner of the set of feasible stationary symmetric allocations. The Golden Rule allocation also helps us describe the entire set of Pareto efficient allocations for this economy. These allocations are those along the boundary of the feasible set beginning with the Golden Rule and continuing upward and to the left.

II. Competitive equilibrium

A *competitive equilibrium* is a consumption allocation and sequence of prices, $\{p_t\}_{t=1}^{\infty}$, such that:

i. Taking prices as given each agent chooses a consumption profile to maximize utility.

ii. The supply of and demand for each good are equal (*i.e.* markets "clear").

Mathematically, the market clearing conditions can be written:

$$N_t c_{1t} + N_{t-1} c_{2t} = N_t y. \qquad \forall t \tag{2.11}$$

In the absence of "money", the competitive equilibrium of an economy of the type we have been considering is trivial. There is no way for trade to take place in this economy, and thus each agent consumes his/her endowment in each period of life. That is, they consume y units when young and 0 units when old. We call this an "autarkic" equilibrium. Note that in this environment the competitive equilibrium is not Pareto efficient.

• Competitive equilibrium with "money"

We now add to the economy a fixed number of units of an intrinsically worthless yet perfectly storable object we call *fiat money.* We assume that fiat money can be costlessly produced by a "government" whose only other function is to prevent any other agent from producing fiat money on their own. Because fiat money cannot be directly consumed, it can only have "value" if agents find it useful in acquiring consumption goods. The existence of money opens up this possibility: A young agent can now sell some of his/her endowment to an old agent for fiat money in the expectation that in the next period a young agent will be willing to make a similar trade of goods for money. An equilibrium in which such trades take place is an equilibrium with *valued* fiat money.

Let p_t denote the price of a time t consumption good in units of money. Note that if money has no value, then p_t is undefined. Let $v_t = 1/p_t$ denote the price of a unit of money at time t in units of time t consumption good. Because trades are now possible, agents no longer have trivial utility maximization problems:

Budget when young

$$c_{1t} + v_t m_t \leq y \tag{2.12}$$

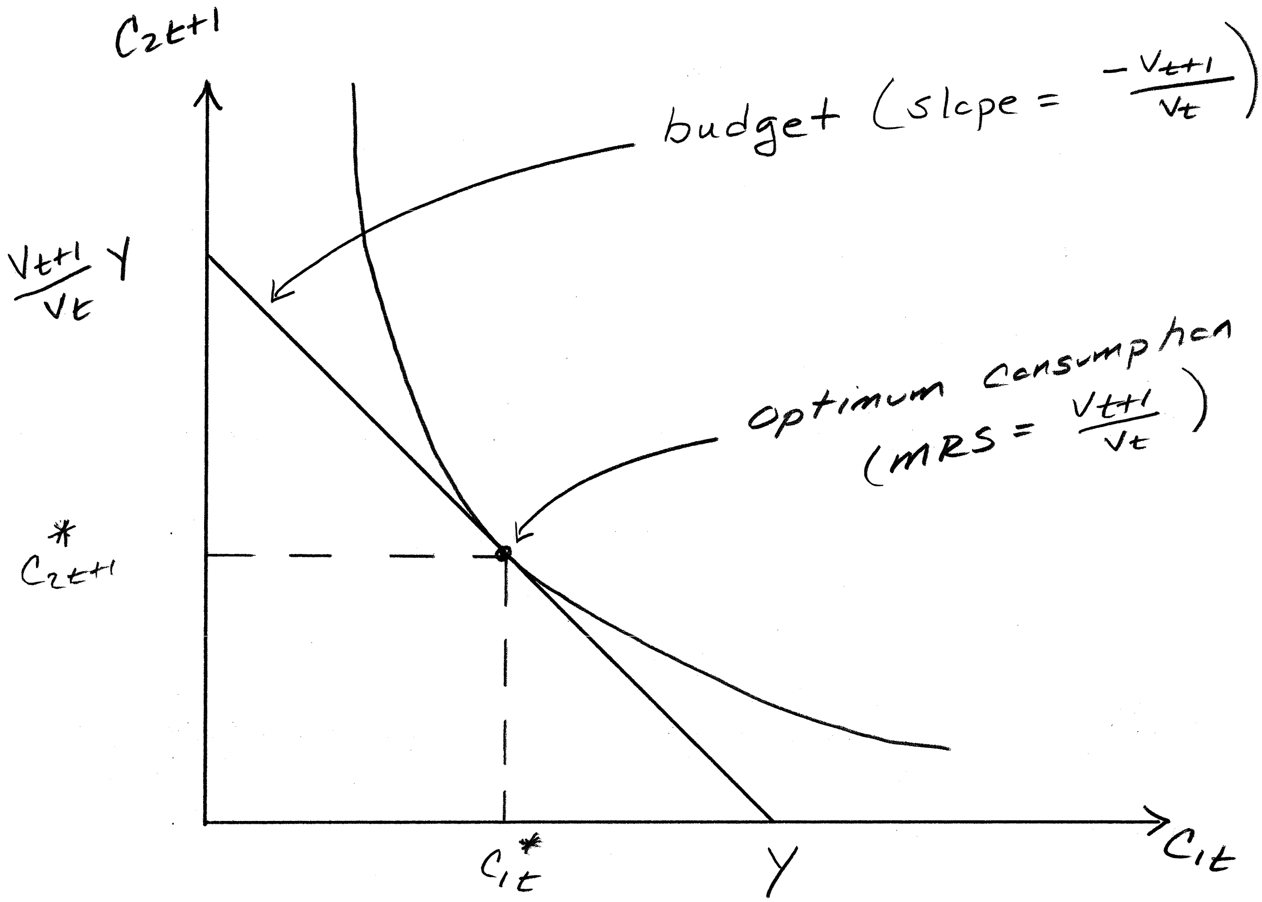
Budget when old

$$c_{2t+1} = v_{t+1} m_t \tag{2.13}$$

Solving (2.13) for m_t and plugging into (2.12) we have the agent's lifetime budget constraint:

$$c_{1t} + \frac{v_t}{v_{t+1}} c_{2t+1} \leq y. \tag{2.14}$$

Here v_{t+1}/v_t is the gross real rate of return on fiat money, because it tells us the amount of time $t+1$ consumption good and agent can get by exchanging one unit of time t consumption good for fiat money and then holding the money from period t to period $t+1$. The following diagram depicts utility maximization by an individual agent:



Now, how do we find the real rate of return on fiat money? Note that this is the key determinant of the competitive equilibrium allocation. In the case where there is no fiat money, its real return is zero. In this case the agents budget is horizontal (and coincident with the c_{1t} axis) and thus the only possible competitive equilibrium is the autarkic one discussed above. When we consider utility maximization in a world with two actual "goods", we tend to appeal to a notion of excess demand to find the prices. The problem here is that the demand for fiat money is not driven directly by utility. Rather, fiat money is only demanded in one period by agents who *believe* that they will be able to use it in the second period. What do we suppose these beliefs would be like and where do they come from?

We assume that beliefs (or "expectations") about the future value of fiat money are the same for members of all generations. This is perhaps reasonable as we have assumed that generations are identical in all other respects. This is consistent with the notion of a *stationary* equilibrium, which is simply a competitive equilibrium in which the allocation is a stationary allocation.

We also typically assume that agents' expectations of the future are *rational.* By this we mean that they do not systematically ignore relevant information and instead weight possible future outcomes by their actual probabilities of occurring. Our economy, however, has no uncertainty about the *fundamentals* (preferences, endowments, etc.). There are no "shocks", and the only thing that changes over time is the population, and it does so in a deterministic manner. Our agents simply must form a belief about the return to fiat money and act is if they are 100% certain that they will be correct. We may consider this a form of *perfect foresight*, but it is perhaps better to view it in a game-theoretic sense. Agents choose their best response (in terms of holding money or consuming) to the rate of return on fiat money that they expect to prevail. Their behaviour (all of them, together) in turn affects the return on fiat money. An equilibrium is a rate of return (that is, a belief) and an allocation such that given agents' belief, the allocation maximizes their utility, and the actual rate of return on money is that which agents expected.

A stationary equilibrium in which money has value

At this point we restrict attention to cases in which the equilibrium allocation is stationary. That is,

$$c_{1t} = c_1 \quad \text{and} \quad c_{2t} = c_2 \qquad \forall t. \tag{2.15}$$

We proceed to characterize a monetary equilibrium as follows:

1. We assume that there exists a stationary monetary equilibrium.
2. Under this assumption we work out what the real rate of return on money, $\frac{v_{t+1}}{v_t}$ must be for such an equilibrium to exist.
3. Using this rate of return on money, we calculate optimal c_1 and c_2.

Then, if all households believe that the rate of return on money calculated at step 2 will prevail, there is a stationary monetary equilibrium with that rate of return and allocation given by the consumption levels calculated in step 3.

To find the return on fiat money consistent with the existence of a stationary monetary equilibrium, we consider "clearing" of the market for money. That is, we equate the demand for and supply of fiat money:

i. The demand for fiat money:

The quantity of goods given up in exchange for money by young agents. From (2.12) we have

$$v_t m_t = y - c_{1t}. \tag{2.16}$$

Aggregating over all the young generation t agents, (2.16) becomes

$$N_t v_t m_t = N_t(y - c_{1t}) \tag{2.17}$$

ii. The supply of fiat money:

Let M_t denote the exogenous supply of fiat money. Measured in goods the supply of fiat money is $v_t M_t$.

Equating (2.17) to the supply of money and solving for v_t we have

$$v_t = \frac{N_t(y - c_{1t})}{M_t} \qquad \forall t. \tag{2.18}$$

Since (2.18) holds for all t, we can push it forward one period to get an expression for the gross real return on money:

$$\frac{v_{t+1}}{v_t} = \frac{N_{t+1}(y - c_{1t+1})M_t}{N_t(y - c_{1t})M_{t+1}}. \tag{2.19}$$

In a stationary equilibrium, $c_{1t} = c_1$ for all t, so (2.19) collapses to:

$$\frac{v_{t+1}}{v_t} = \frac{N_{t+1}M_t}{N_t M_{t+1}}. \tag{2.20}$$

If we further assume that there is no population growth and that the stock of money is constant, (2.20) simply becomes:

$$\frac{v_{t+1}}{v_t} = 1 \quad \text{or} \quad v_{t+1} = v_t. \tag{2.21}$$

What have we shown by deriving (2.21)? We have shown that if this environment has a stationary monetary equilibrium in the case of a constant money stock and no population growth, then it must be the case that the rate of return is one. Note that this does *not* say that this is the unique monetary equilibrium. There will in general be non-stationary monetary equilibria (recall that a non-stationary equilibrium is simply one in which consumption profiles differ across generations). It is always the case, however, that the stationary equilibrium yields higher welfare than any of the non-stationary equilibria.

Return now to the agents' maximization problem:

$$\max_{c_1, c_2} \; u(c_1, c_2)$$

subject to:

$$c_1 + \frac{v_t}{v_{t+1}} c_2 = y$$

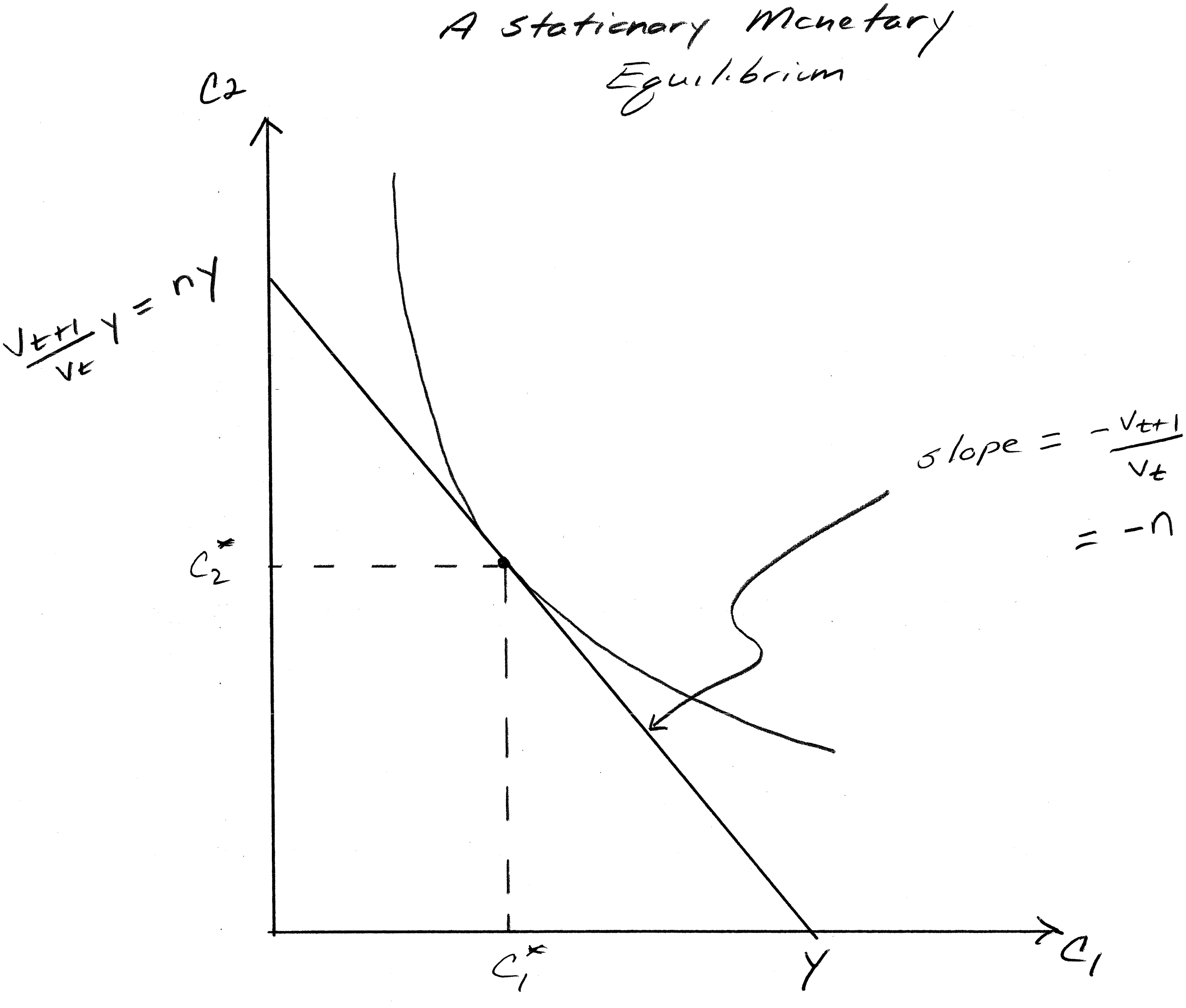
Recall the standard conditions for utility maximization:

i. The MRS must equal the price ratio:

$$MRS = \frac{\frac{\partial u}{c_1}}{\frac{\partial u}{c_2}} = \frac{v_{t+1}}{v_t} = 1 \qquad (2.22)$$

ii. $c_1 + c_2 = y$

The following diagram depicts the utility maximizing choice for all generations born at $t \geq 1$. Note that the equilibrium allocation *is* the golden rule allocation.



Now we show that this result continues to hold with a growing population. Returning to (2.20), with a growing population and a constant money stock, we have

$$\frac{v_{t+1}}{v_t} = \frac{N_{t+1}}{N_t} = n \tag{2.23}$$

The maximization problem for an individual agent is unchanged, so we have

$$MRS = \frac{\frac{\partial u}{c_1}}{\frac{\partial u}{c_2}} = \frac{v_{t+1}}{v_t} = \frac{N_{t+1}}{N_t} = n \tag{2.24}$$

In this case, what is happening to the *price level*? Recall that the price of goods in units of money (our normal notion of price) satisfies $p_t = 1/v_t$ for all t. Since v_t is *growing* at rate n. It must be that p_t is falling over time. Intuitively, this makes sense. Over time the quantity of goods in this economy grows without bound. The quantity of money available to purchase them, however, stays constant. If the money stock continues to purchase the same share of goods from each young generation, then the quantity of goods exchanged for each unit of money must rise. Thus we have constant *deflation* at the rate of growth of the economy.

The Quantity Theory of Money

The simple "quantity theory" of money predicts that the price level is exactly proportional to the quantity of money in the economy. Using (2.18) we have

$$p_t = \frac{1}{v_t} = \frac{M_t}{N_t(y - c_{1t})} \qquad \forall t. \tag{2.25}$$

Clearly, a doubling (for example) of the money stock leads to a doubling of the price level and the simple quantity theory holds for our economy.

Also, note that the size of the stock of money has no effect on the equilibrium consumption allocation. This is known as the *neutrality* of money. Note, however, that here we are only talking about neutrality with respect to the nominal size of a *constant* stock of money.

Concluding Comments

We close this section by summing up the role of fiat money in the OLG model. The environment we consider is one with a severe friction: Agents of different generations have no way to trade with one another. One thing to keep in mind is that it is not important that we have assumed that the old have *no* endowment. The important thing is that the competitive equilibrium in the absence of money (or some other institution) is not Pareto efficient. Money enables trade between generations and is able not only to lead to a Pareto efficient equilibrium allocation but can actually attain the "Golden Rule" allocation, *i.e.* the allocation that maximizes lifetime utility for all generations born at $t \geq 1$. Finally, while the introduction of money at time 1 does not maximize the utility of generation 0 (the initial old), in general it will raise their utility relative to what they achieve in the autarkic non-monetary equilibrium.

Also, note that the monetary equilibrium is "tenuous" (to use Neil Wallace's term). It rests on beliefs or expectations. Right from the start, the young agents at time 1 must believe that money will have value at time 2, or they will have no incentive to accept it in exchange for goods with the initial old. We do not consider them here, but in addition to the stationary monetary equilibrium that achieves the golden rule, there are also many *non-stationary* monetary equilibria in which the value of money (and the price level) change over time purely in response to changes in agents' expectations. We will return to these issues when we consider banking panics and hyperinflations later in the course.

Note also that there are *two* stationary equilibria here. The autarkic equilibrium in which all agents consume only their endowment and never trade is also an equilibrium in the economy. It is the equilibrium associated with the belief that the rate of return on money is zero. In this case, no young agent will give up consumption goods for money, and so its value will equal zero.

Finally, note that fiat money is not the only "institution" that can achieve the Golden Rule in this environment. To see this, consider the following interpretation of the environment. Suppose that agents' endowments are actually labour income. Young agents are able to