Answers to Assignment 3

1a. In order to write down the agent’s maximization problem, we must first derive his/her lifetime budget. Supposing that agents use fiat money to finance consumption when middle-aged and capital to finance consumption when old, their budgets when young, middle-aged, and old are:

\[ c_{1,t} + v_t m_t + k_t = 3 \]  
(1a.1)
\[ c_{2,t+1} = v_{t+1} m_t \]  
(1a.2)
\[ c_{3,t+2} = 1.5 k_t \]  
(1a.3)

Substituting for \( m_t \) in (1a.1) using (1a.2) and for \( k_t \) using (1a.3), we have

\[ c_{1,t} + \left[ \frac{v_t}{v_{t+1}} \right] c_{2,t+1} + \left[ \frac{1}{1.5} \right] c_{3,t+2} = 3. \]  
(1a.4)

The maximization problem of a generation \( t \) agent can then be written:

\[
\max_{c_{1,t},c_{2,t+1},c_{3,t+2}} \ln c_{1,t} + \ln c_{2,t+1} + \ln c_{3,t+2}
\]

subject to:

(1a.4).

1b. In order to answer this, we must first derive the return on fiat money and compare it to the rate of return on capital. In this economy, agents must hold fiat money if they want to consume anything when middle-aged. And they do, because the marginal utility of middle-aged consumption goes to infinity as \( c_{2,t+1} \) goes to zero. It is not immediately obvious, however, that agents will want to hold capital. They could use fiat money to finance consumption both when middle-aged and old (above we simply assumed that they would invest in capital). We derive the rate of return on fiat money in the usual way, using the first period budget:

\[ v_t = \frac{3 - c_{1,t} - k_t}{m_t} \quad \text{and} \quad v_{t+1} = \frac{3 - c_{1,t+1} - k_{t+1}}{m_{t+1}} \]  
(1b.1)

Imposing market clearing in the stationary monetary equilibrium (SME) and combining we have

\[ \frac{v_{t+1}}{v_t} = \frac{N_{t+1}(3 - c_1 - k)M_t}{N_t(3 - c_1 - k)M_{t+1}} = \frac{N M_t}{N M_{t+1}} = \frac{1}{z} = \frac{1}{1.25}. \]  
(1b.2)

The two-period rate of return on fiat money is \((1/1.25)^2 = 16/25\), which is less than one, and therefore less than \( x = 1.5 \). Since the two-period rate of return on capital is higher than that on fiat money, agents will want to hold capital in the SME.

Agents will want to hold both assets, capital and fiat money, in the SME. They use fiat money to finance consumption when middle-aged because there is no alternative, and capital to finance consumption when old because it pays the highest rate of return.
1c. In 1b. the rate of return on fiat money was derived. Using this, write a generation $t$ agent’s maximization problem in the SME as:

$$
\max_{c_1, c_2, c_3} \ln c_1 + \ln c_2 + \ln c_3
$$

subject to:

$$
c_1 + zc_2 + \frac{c_3}{x} = 3 \quad \text{or} \quad c_1 + 1.25c_2 + \frac{c_3}{1.5} = 3
$$

The Lagrangian is

$$
L = \ln c_1 + \ln c_2 + \ln c_3 - \lambda [c_1 + 1.25c_2 + \frac{c_3}{1.5} - 3].
$$

The first-order conditions are

$$
\frac{1}{c_1} = \lambda, \quad \frac{1}{c_2} = 1.25\lambda, \quad \frac{1}{c_3} = \frac{\lambda}{1.5}
$$

Combining the first order conditions,

$$
c_2 = \frac{c_1}{1.25}, \quad c_3 = 1.5c_1.
$$

Substituting for $c_2$ and $c_3$ in (1c.2) and solving for $c_1$, and then using (1c.6) we have the SME consumption allocation:

$$
c_1 = 1, \quad c_2 = \frac{1}{1.25} = 4/5, \quad c_3 = 1.5.
$$

The SME allocation is not Pareto efficient, for two separate reasons. First, the government is printing money and using it to collect seignorage. This is a distorting tax on the use of the medium of exchange and it raises the price of consumption when middle-aged ($c_2$) relative to consumption when young ($c_1$) and causes agents to substitute the latter for the former. As we argued earlier, inflation which results in seignorage is always inefficient in the sense that the same government revenue could be raised via lump-sum taxes and cause less welfare loss.

A second reason for inefficiency emanates from the informational friction that prohibits agents from using private loans to finance consumption when middle-aged and forces them to use fiat money. Consider the following feasible scheme which generates an allocation in which all agents are better off than at the SME allocation, $c_1 = 1; c_2 = 4/5; c_3 = 1.5$:

i. Each young agent consumes one unit of there endowment ($c_1 = 1$), and invests two units in capital ($k = 2$).

ii. Each old agent takes their three units of income ($xk = 1.5 \times 2 = 3$), consumes half of it ($c_3 = 1.5$) and gives the other half to a middle-aged agent ($c_2 = 1.5$).

Since the population is constant, this allocation is feasible. Moreover, it results in a consumption allocation of: $c_1 = 1; c_2 = 1.5; c_3 = 1.5$. This certainly leads to higher utility than
the SME allocation. The government can recover its seignorage revenue, which in the SME was
\[ g = \left[ 1 - \frac{1}{z} \right] v_t m_t = (1/5) \times 2 = .4, \]  
by levying a lump-sum tax of .4 units on each middle-aged agent. This will leave them with net consumption, \( c_2 = 1.1 \), and they will still be better off than in the SME.

1d. If there are banks that can commit to repaying loans, then agents will have the option of making deposits in banks when young and using them to finance middle-aged consumption. They will see this as superior to holding fiat money, because deposits will pay a positive return. In fact, since there are a large number of competitive banks, we can assume that the rate of return on deposits will be forced to the one-period return on capital, \( r^* = \sqrt{1.5} \). Let \( \hat{h} \) denote the level of deposits that an agent makes when young, and assume that agents use deposits only to finance consumption when middle-aged, financing consumption when old with capital that they have invested themselves. Here variables in the equilibrium with banks will be distinguished by a “\( \hat{\} \)”. Then, given that agents will not have any demand for fiat money in the stationary equilibrium, their budgets when young, middle-aged, and old are:

\[ \hat{c}_1 + \hat{h} + \hat{k} = 3 \]  
\[ \hat{c}_2 = r^* \hat{h} = \sqrt{1.5} \hat{h} \]  
\[ \hat{c}_3 = \hat{x} \hat{k} = 1.5 \hat{k} \]

and the lifetime budget is
\[ \hat{c}_1 + \frac{\hat{c}_2}{\sqrt{1.5}} + \frac{\hat{c}_3}{1.5} = 3. \]

Replacing constraint (1c.2) with (1d.4) in the generation \( t \) agent’s maximization problem (see (1c.1) and (1c.2) above), the Lagrangian and first order conditions in this case can be written:

\[ L = \ln \hat{c}_1 + \ln \hat{c}_2 + \ln \hat{c}_3 - \lambda [\hat{c}_1 + \frac{\hat{c}_2}{\sqrt{1.5}} + \frac{\hat{c}_3}{1.5} - 3]. \]

\[ \frac{1}{\hat{c}_1} = \lambda \]  
\[ \frac{1}{\hat{c}_2} = \frac{\lambda}{\sqrt{1.5}} \]  
\[ \frac{1}{\hat{c}_3} = \frac{\lambda}{1.5} \]

Combining the first order conditions we have
\[ \hat{c}_2 = \sqrt{1.5} \hat{c}_1 \quad \hat{c}_3 = 1.5 \hat{c}_1. \]

Solving as in part 1c. we have the consumption allocation in the stationary equilibrium
\[ \hat{c}_1 = 1 \quad \hat{c}_2 = \sqrt{1.5} = \frac{\sqrt{3}}{\sqrt{2}} \quad \hat{c}_3 = 1.5. \]
1e. The capital stock will be higher in the equilibrium with banks. In the economy with no banks, some of agents’ savings was held in fiat money so as to finance consumption when middle-aged. This portion of savings could therefore not be invested in capital. When an agent makes a deposit with a bank, however, the bank does invest in capital. This is why the bank is able to pay a return on deposits. In the economy with banks total savings is invested in capital, rather than just that part of savings which is used to finance consumption when old. Since total savings \((3 - c_1 \text{ or } 3 - \hat{c}_1)\) is the same in the two economies \((c_1 = \hat{c}_1 = 1)\), it therefore must be the case that there is more capital in the economy with banks. The increase in the capital stock accounts for the difference in the consumption allocations. Comparing (1d.10) with (1c.7) we see that middle-aged consumption is higher in the economy with banks:

\[
\hat{c}_2 = \frac{\sqrt{3}}{\sqrt{2}} \approx \frac{1.732}{1.414} = 1.225 > \frac{4}{5} = c_2.
\]  

(1e.1)

Higher consumption when middle-aged is only possible because there is higher investment, and thus higher output in the economy with banks.