

Technological Progress in the Solow Model

In the basic Solow model, growth occurs only as a result of factor accumulation.

There are two factors, labour and capital

1. *Labour* grows *exogenously* through population growth.
2. *Capital* is accumulated as a result of savings behaviour.

Because the technology has the neoclassical form (diminishing returns to *per capita* capital), capital accumulation cannot raise *per capita* income forever.

This does not depend on the assumption of a constant savings rate. It will happen even if $s = 1$, that is, if people save *all* of their income.

In order to generate *sustained* economic growth, some assumptions must be abandoned. Here are a couple of possibilities:

1. Constant returns to scale (diminishing returns to $k = K/L$).
2. Static technology. (*i.e.* a constant production function)

Because we are still studying the *Solow* model, we will maintain assumption #1, and allow for *technological progress*. By this we mean shifts in the production function over time.

There are many ways for the production function to “shift” over time. That is, there are many different types of technological progress. In an attempt to be consistent with two observations:

- i. The return to capital is roughly constant over time.
- ii. Capital's share of income is roughly constant over time.

we will assume that technological progress is *labour augmenting*...

$$Y = F(K, AL) = K^\alpha (AL)^{1-\alpha} \quad (30)$$

where A represents the level of technology at the current time.

Labour augmenting technological change is sometimes called “Harrod neutral” and is associated with a **constant capital-output ratio** (K/Y) in a “steady-state”.

Other possibilities:

1. “Hicks neutral” (constant K/L): $Y = AF(K, L)$
2. Capital augmenting (constant L/Y): $Y = F(AK, L)$

Note: The data suggests we are experiencing *labour augmenting* technological change. Why?

The return to capital is proportional to the ration of output to capital:

$$r = \alpha \frac{Y}{K}: \text{ If } Y/K \text{ is constant, so is } r.$$

Under the assumption of the Cobb-Douglas technology, labour augmenting technological change is also Hicks neutral.

Technological change:

$$A(t) = A(0)e^{gt} \quad (31)$$

$$\ln A(t) = \ln A(0) + gt \quad (32)$$

$$\frac{\dot{A}}{A} = \frac{d \ln A(t)}{dA(t)} = g. \quad (33)$$

Here g is a *parameter* denoting the *exogenous* rate of technological progress.

Technological progress is exogenous here (like population growth) because it is determined outside the model, not as a consequence of agents' actions.

Recall the capital accumulation equation in the basic Solow model:

$$\frac{\dot{k}}{k} = s \frac{y}{k} - (d + n)$$

So, for the growth rate of *per capita* capital to be constant, it must be the case that $\frac{y}{k} = \frac{Y}{K}$ must be constant as well.

Recall that this is the distinguishing characteristic of Harrod neutral or labour augmenting technological change.

Now, we want to derive the steady-state, or *balanced growth path*, of the Solow model with technological progress:

$$Y = K^\alpha (AL)^{1-\alpha} \quad (34)$$

Divide through by AL : the number of *effective* labour units.

$$\frac{Y}{AL} = \left(\frac{K}{AL} \right)^\alpha \left(\frac{AL}{AL} \right)^{1-\alpha} \quad (35)$$

or,

$$\tilde{y} = \tilde{k}^\alpha \quad PF' \quad (36)$$

In this way we represent the production in variables *per unit of effective labour*, whereas before, PF was in *per capita* units. Next, we need to do the same thing with the capital accumulation equation...

$$\begin{aligned}
\frac{\dot{\tilde{k}}}{\tilde{k}} &= \frac{d \ln(K/AL)}{dt} \\
&= \frac{d \ln K}{dt} - \frac{d \ln A}{dt} - \frac{d \ln L}{dt} \\
&= \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} \\
&= \frac{\dot{K}}{K} - g - n.
\end{aligned} \tag{37}$$

Now, substitute using:

$$\frac{\dot{K}}{K} = s \frac{Y}{K} - d$$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = s \frac{Y}{K} - d - g - n$$

Now since,

$$\frac{Y}{K} = \frac{Y}{AL} \frac{AL}{K} \equiv \frac{\tilde{y}}{\tilde{k}}$$

we have,

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = s \frac{\tilde{y}}{\tilde{k}} - (d + g + n)$$

or,

$$\dot{\tilde{k}} = s\tilde{y} - (d + g + n)\tilde{k} \quad \text{CA}' \quad (38)$$

Now we can represent the Solow model with technological progress by combining PF' and CA' :

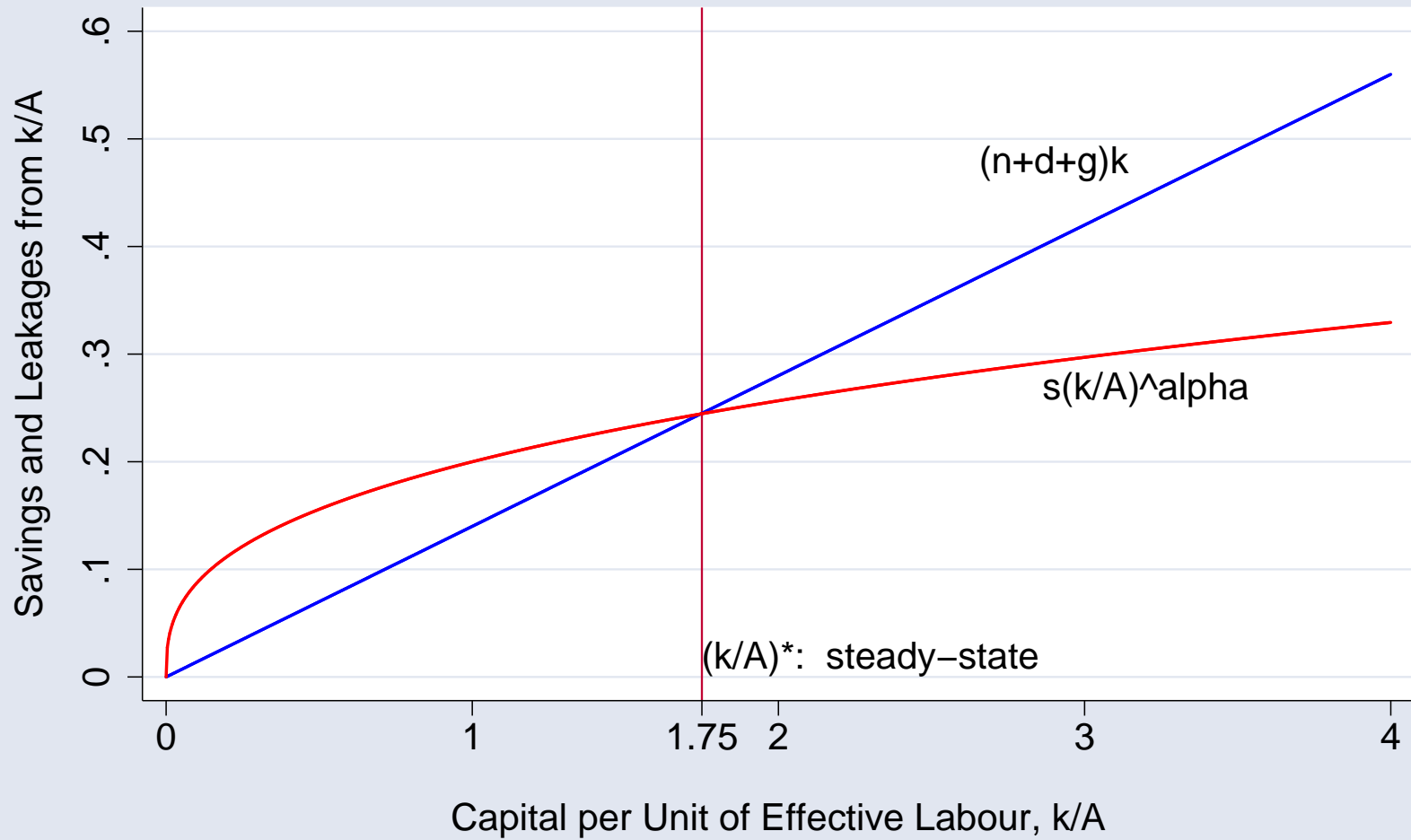
$$\dot{\tilde{k}} = s\tilde{k}^\alpha - (d + n + g)\tilde{k} \quad (39)$$

We can depict this in a diagram which looks *very* similar to the diagram for the basic Solow model, except that it depicts capital *per unit of effective labour* rather than *per capita*.

As before, we can set $\dot{\tilde{k}} = 0$ and solve for the steady-state levels of capital and output *per unit of effective labour*:

$$\tilde{k}^* = \left[\frac{s}{d + n + g} \right]^{\frac{1}{1-\alpha}} \quad \tilde{y}^* = \left[\frac{s}{d + n + g} \right]^{\frac{\alpha}{1-\alpha}} \quad (40)$$

The Solow Model with Exogenous Technological Progress



— Pop. growth, depreciation, and tech. progress — Savings

In the steady-state of this model, what happens to *per capita* capital and output?

$$\tilde{k} = \frac{K}{AL} \quad \text{and} \quad k = \frac{K}{L} \quad \longrightarrow \quad k = A\tilde{k}$$

In the steady-state,

$$\tilde{k} = \tilde{k}^* = \left[\frac{s}{d + n + g} \right]^{\frac{1}{1-\alpha}},$$

$$k(t) = A(t)\tilde{k}^*$$

$$\ln k(t) = \ln A(t) + \ln \tilde{k}^*$$

$$\frac{\dot{k}}{k} = \frac{\dot{A}}{A}, \tag{41}$$

Capital (and income) *per capita* grow at the rate of exogenous technological progress.

Now we can do the same type of exercises that we did before:

1. A change in the savings rate, s :

This changes the savings curve in the diagram (just as before) and therefore changes the *level* of \tilde{k}^* .

2. A change in the rate of population growth, n , or depreciation, d

This rotates the straight line (again just as before), and therefore also changes \tilde{k}^* .

Changes in any of these parameters affect the *growth rate* of the *per capita* capital and income only outside of the steady-state.

3. A change in the exogenous rate of technological progress...

- i. changes the diagram in a manner similar to a change in n or d and therefore changes the *level* of \tilde{k}^* .
- ii. changes the growth rate of *per capita* income and capital in the steady-state, or as we say, along the *balanced growth path*.

Can this or that policy be “good” for economic growth? (lower taxes, etc.)

What can policy influence?

1. The savings rate
2. The population growth rate?
3. The rate of technological progress?

According to the Solow model, only if policy can influence the rate of technological progress can it affect economic growth in the long-run.

Some issues raised by the Solow model:

I. Total Factor Productivity:

With technological progress, both capital and labour can become more productive over time:

$$\begin{aligned}y &= K^\alpha (AL)^{1-\alpha} \\ &= A^{1-\alpha} K^\alpha L^{1-\alpha} \\ &= BK^\alpha L^{1-\alpha}\end{aligned}\tag{42}$$

Where we may refer to $B \equiv A^{1-\alpha}$ as *total factor productivity*.

$$B = \frac{Y}{K^\alpha L^{1-\alpha}}\tag{43}$$

B is output per unit of all factors, suitably combined.

Over time, if total income, Y , grows it must be because of growth in B , K or L :

$$\ln Y(t) = \ln B(t) + \alpha \ln K(t) + (1 - \alpha) \ln L(t)$$

$$\frac{\dot{Y}}{Y} = \frac{\dot{B}}{B} + \alpha \frac{\dot{K}}{K} + (1 - \alpha) \frac{\dot{L}}{L}$$

Now, Y , K , L , and α are all quantities that we can *observe*.

B we cannot. So consider:

$$\frac{\dot{B}}{B} = \frac{\dot{Y}}{Y} - \alpha \frac{\dot{K}}{K} - (1 - \alpha) \frac{\dot{L}}{L}. \quad (44)$$

This is sometimes called the “*Solow Residual*” and it is a measure of total factor productivity (and labour-augmenting technological change).

The Solow residual can be used to assess the extent to which a country's growth can be explained by factor accumulation.

It is therefore a measure of the “unexplained” or at least not easily measured components of economic growth.

Note that the measure of total factor productivity depends on the exact form of the technology.

II. Optimal Savings

So far, we have specified the savings rate, s , exogenously.

Presumably, however, s is determined by people's behaviour. In neoclassical economics, we typically assume that people behave as they do in order to optimize relative to some objective.

1. Higher s leads to higher k and y . But does it lead to higher *welfare*?
2. If differences in s account for differences in k and y , then we might want to know what determines s . That is, which features of economies account for differences in savings rates.

Before taking on optimal savings directly, return to the Solow diagram. Note that consumption (per unit of effective labour) is given by:

$$\tilde{c} = (1 - s)\tilde{y} \quad (45)$$

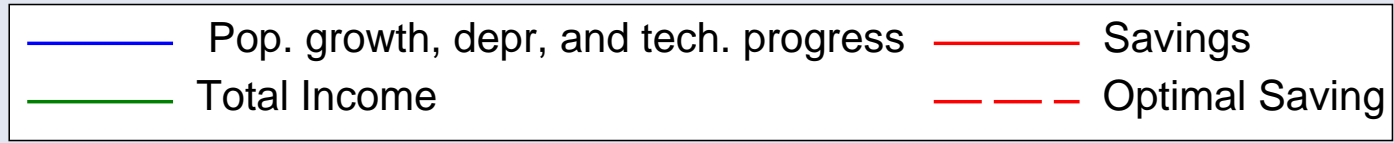
In the steady-state this satisfies:

$$\begin{aligned} \tilde{c}^* &= f(\tilde{k}^*) - (n + d + g)\tilde{k}^* \\ &= (\tilde{k}^*)^\alpha - (n + d + g)\tilde{k}^* \end{aligned} \quad (46)$$

Consumption, \tilde{c}^* , is maximized where:

$$\begin{aligned} f'(\tilde{k}^*) - (n + d + g) &= 0 \\ \alpha(\tilde{k}^*)^{\alpha-1} &= n + d + g \end{aligned} \quad (47)$$

The Golden Rule



Solving, we get

$$\tilde{k}_g \equiv \tilde{k}^* = \left[\frac{\alpha}{n + d + g} \right]^{\frac{1}{1-\alpha}} \quad (48)$$

and we refer to \tilde{k}_g as the “golden rule” capital stock (per unit of effective labour). We then say if:

1. $\tilde{k}^* > \tilde{k}_g$: The economy is *dynamically inefficient*.
2. $\tilde{k}^* \leq \tilde{k}_g$: The economy is *dynamically efficient*.

To understand the “golden rule” language, interpret the economy this way:

Suppose that each period there is a new “generation”, They inherit the capital stock from the previous generation, and then consume and save, passing the capital stock on to the generation that follows.

- I. If $\tilde{k}(t) > \tilde{k}_g$, then it is possible for the current generation and *all* future generations to have higher consumption (and higher utility) simply by reducing savings.

This reduction in savings will not harm past generations, but will make every else (current and future) better off. *Therefore, it is inefficient to continue at a sufficient savings rate to remain on the current path for the capital stock, $\tilde{k}(t)$.*

- II. If $\tilde{k}(t) \leq \tilde{k}_g$, then a reduction of savings makes *only* the current generation better off and harms future generations.

Thus these growth paths are *efficient* because no person can be made better off without making someone else worse off.

The “golden rule” path, associated with \tilde{k}_g in the steady-state is the level of capital per unit of effective labour that maximizes the consumption of all future generations.

Would competitive agents (firms and consumers) act in such a way as to choose the savings rate associated with the golden rule?

If not, would they choose a savings rate that leads the economy to be dynamically efficient?

We cannot ask these questions in a setting where the savings rate is assumed to be constant and determined outside of the model.

To this end, we now relax the assumption of a fixed and exogenous savings rate, and consider an extension of the Solow model with optimizing consumers.