ECON 815
Macroeconomic Theory

1 Growth through expanding variety of products

Consider the version of the model of Romer (1990) without physical capital described in class. Agents have logarithmic utility:

$$U = \int_0^\infty e^{-\rho t} \ln C(t)\, dt$$

The households' budget constraint is:

$$\dot{A} = rA + wL - C$$

The number of varieties evolves according to (L stands for the size of the constant labor force, while L_y is the amount of workers employed in the final good sector):

$$\frac{\dot{N}}{N} = \delta (L - L_y)$$

The profits for the final good sector (competitive) firms are (x_j stands for the quantity of intermediate good j):

$$\pi = L_y^{1-\alpha} \int_0^N x_j^\alpha dj - wL_y - \int_0^N p_j x_j dj$$

while the intermediate good sector (monopolistic) producer of the single intermediate good j maximizes:

$$\pi_j = x_j p_j - x_j$$

(a) Characterize the social optimum of this economy. Find, in particular, the growth rate of the economy.

(b) Contrast the above results with those of a laissez-faire (decentralized) equilibrium. Provide an interpretation for these differences.

(c) Using the same model, assume that the government introduces a subsidy to the use of variable inputs in the production of intermediate goods (financed by lump sum taxation). This means that intermediate firms will only have to pay $\frac{1}{1+\tau_s}$ for each unit of final good purchased to produce intermediate goods. What is the rationale of this policy? Does this policy enhance welfare? Does it enhance growth?

(d) Now, imagine that the government taxes proportionally the use of labor in final production (and rebates the proceedings lump sum). This means that final firms will have to pay $1 + \tau_w$ for each unit of labor used to produce final goods. What are the effects on the growth rates? Is this policy sufficient to restore the first-best Pareto optimum? If not, can you suggest a policy or a menu of policies which would implement it?

2 Answers

Let's denote with $h^{pl}, h^{de}, h^{des}, h^{det}$ the optimal/equilibrium values of variable h in the Social planner problem (point **a**), in the decentralized economy (point **b**), in the decentralized economy with a subsidy (point **c**) and in the decentralized economy with a labor tax (point **d**), respectively.

(a) To solve the social planner problem we need to consider two steps.

- We build the current value Hamiltonian, considering only the dynamic constraint for the state variable (N), with the related costate variable ν.

$$H = \ln(C) + \nu\left[\delta\left(L - L_y\right)N\right]$$

- Recall that in each moment of time there is an aggregate feasibility constraint that must be satisfied:$Y = Nx^{\alpha}L_y^{1-\alpha} = Nx + C$. This represents an additional constraint to the problem that must be included in the solution. We have to build a Lagrangean function, where μ is a (dynamic) Lagrange multiplier:

$$\ell = H + \mu\left(Nx^{\alpha}L_y^{1-\alpha} - Nx - C\right)$$

The FOC's become:

$$\begin{cases} \dfrac{\partial \ell}{\partial C} = 0 \Rightarrow \frac{1}{C} = \mu \quad \text{log-differentiate to get} \quad \frac{\dot{C}}{C} = -\frac{\dot{\mu}}{\mu} \\ \dfrac{\partial \ell}{\partial x} = 0 \Rightarrow \alpha N x^{\alpha-1}L_y^{1-\alpha} = N \Rightarrow x = \alpha^{\frac{1}{1-\alpha}}L_y \\ \dfrac{\partial \ell}{\partial L_y} = 0 \Rightarrow \mu\left(1-\alpha\right)Nx^{\alpha}L_y^{-\alpha} = \nu\delta N \quad \text{using } x \quad \mu = \delta\left(1-\alpha\right)^{-1}\alpha^{\frac{\alpha}{\alpha-1}}\nu \\ \dfrac{\partial \ell}{\partial N} = \rho\nu - \dot{\nu} \Rightarrow \mu\left(x^{\alpha}L_y^{1-\alpha} - x\right) + \nu\delta\left(L - L_y\right) = \rho\nu - \dot{\nu} \end{cases}$$

note that log-differentiating μ we have: $\frac{\dot{\mu}}{\mu} = \frac{\dot{\nu}}{\nu}$; using the expressions for x and μ already found yields:

$$-\frac{\dot{\nu}}{\nu} = \delta\left(1-\alpha\right)^{-1}\left[L_y - \alpha L_y\right] + \delta\left(L - L_y\right) - \rho = \delta L - \rho = \frac{\dot{C}}{C} \equiv \gamma^{pl}$$

Using x in the production function we have:

$$Y = Nx^{\alpha}L_y^{1-\alpha} = N\alpha^{\frac{\alpha}{1-\alpha}}L_y \quad \text{log-differentiating} \quad \gamma_Y = \frac{\dot{Y}}{Y} = \frac{\dot{N}}{N}$$

Consider the aggregate feasibility constraint:

$$Y = Nx + C \rightarrow \frac{d\log Y}{dt} = \frac{d\log\left(Nx + C\right)}{dt} \rightarrow \frac{\dot{Y}}{Y} = \frac{\dot{N}x + N\dot{x} + \dot{C}}{Nx + C}$$

or

$$\begin{aligned} \frac{\dot{Y}}{Y} &= \frac{Nx}{Nx+C}\frac{\dot{N}}{N} + \frac{Nx}{Nx+C}\frac{\dot{x}}{x} + \frac{C}{Nx+C}\frac{\dot{C}}{C} = \\ &= \frac{Nx}{Nx+C}\frac{\dot{N}}{N} + \frac{C}{Nx+C}\frac{\dot{C}}{C} = \eta\frac{\dot{N}}{N} + \left(1-\eta\right)\frac{\dot{C}}{C} \end{aligned}$$

where we have used the fact that $\frac{\dot{x}}{x} = \frac{\dot{L}_y}{L_y} = 0$ and we have defined a positive weight $0 < \eta < 1$. Since $\dfrac{\dot{Y}}{Y} = \dfrac{\dot{N}}{N}$ (from the production function), it follows also that $\frac{\dot{C}}{C} = \dfrac{\dot{N}}{N}$.

We know that $\gamma^{pl} = \frac{\dot{C}}{C} = \dfrac{\dot{Y}}{Y} = \dfrac{\dot{N}}{N}$; therefore, from $\dfrac{\dot{N}}{N} = \delta\left(L - L_y\right) = \delta L - \rho$, we immediately get $L_y^{pl} = \dfrac{\rho}{\delta}$. Also, the transversality condition $\lim_{t\to\infty} e^{-\rho t}\nu\left(t\right)N\left(t\right)$ reduces to $\rho > 0$, since $\dfrac{\dot{N}}{N} = -\frac{\dot{\nu}}{\nu}$.

(b) Households. The current value Hamiltonian for an agent is $H = \ln C + \nu\,[rA + wL - C]$, which yields the by now familiar Euler equation $\frac{\dot{C}}{C} = r - \rho = \gamma_c^{de}$. The transversality condition is $r > \gamma_c^{de}$ (i.e. $\rho > 0$).

Final good sector (competitive). Firms maximize $\pi = L_y^{1-\alpha}\int_0^N x_j^\alpha dj - wL_y - \int_0^N p_j x_j dj$; FOC's:

$$p_j = \alpha L_y^{1-\alpha} x_j^{\alpha-1} \quad \text{and} \quad w = (1-\alpha)\,L_y^{-\alpha}\int_0^N x_j^\alpha dj$$

Intermediate good sector (monopolistic). The producer of a single intermediate good j maximizes:

$$\pi_j = x_j p_j - x_j = \alpha L_y^{1-\alpha} x_j^\alpha - x_j$$

the FOC is

$$\alpha^2 L_y^{1-\alpha} x_j^{\alpha-1} = 1 \Rightarrow x_j^{de} = \alpha^{\frac{2}{1-\alpha}} L_y$$

Therefore, the solutions for quantity, prices and profits in the intermediate good market are:

$$\begin{aligned} x^{de} &= x_j^{de} = \alpha^{\frac{2}{1-\alpha}} L_y \\ p^{de} &= p_j^{de} = \frac{1}{\alpha} \\ \pi^{de} &= \pi_j^{de} = \left(\frac{1-\alpha}{\alpha}\right)\alpha^{\frac{2}{1-\alpha}} L_y \end{aligned}$$

Replacing x in the production function yields $Y^{de} = N\alpha^{\frac{2\alpha}{1-\alpha}} L_y$ and by log-differentiation we see that $\gamma_Y^{de} = \frac{\dot{Y}}{Y} = \frac{\dot{N}}{N}$; further, the wage rate becomes:

$$w^{de} = (1-\alpha)\,\alpha^{\frac{2\alpha}{1-\alpha}} N$$

New firms enter the intermediate sector as long as the present discounted value of future profits covers the cost of innovations, represented by wages paid to researchers:

$$\frac{\pi^{de}}{r} = \frac{w^{de}}{\delta N} \Rightarrow \frac{\left(\frac{1-\alpha}{\alpha}\right)\alpha^{\frac{2}{1-\alpha}} L_y}{r} = \frac{(1-\alpha)\,\alpha^{\frac{2\alpha}{1-\alpha}} N}{\delta N} \Rightarrow r^{de} = \alpha\delta L_y \tag{1}$$

We know that the solution will have $\gamma^{de} = \frac{\dot{C}}{C} = \frac{\dot{Y}}{Y} = \frac{\dot{N}}{N}$; therefore, using the interest rate in the Euler equation for consumption and setting it equal to the R&D equation $\gamma^{de} = \delta\,(L - L_y)$, we get:

$$r^{de} - \rho = \alpha\delta L_y - \rho = \delta\,(L - L_y) \Rightarrow L_y^{de} = \frac{\delta L + \rho}{\delta\,(1+\alpha)} \tag{2}$$

Use (2) into the R&D equation $\gamma^{de} = \delta\left(L - L_y^{de}\right)$ to find the growth rate:

$$\gamma^{de} = \delta L - \delta\frac{\delta L + \rho}{\delta\,(1+\alpha)} = \frac{\alpha\delta L - \rho}{(1+\alpha)} \tag{3}$$

Now compare the social planner equilibrium with the decentralized equilibrium.

$$\gamma_Y^{pl} > \gamma_Y^{de} \quad \text{as} \quad \delta L - \rho > \frac{\alpha\delta L - \rho}{(1+\alpha)}$$

$$L_y^{pl} < L_y^{de} \quad \text{as} \quad \frac{\rho}{\delta} < \frac{\delta L + \rho}{\delta\,(1+\alpha)}$$

$\gamma_Y^{pl} > \gamma_Y^{de} \Rightarrow$ the decentralized economy equilibrium is dynamically inefficient. In the decentralized economy too little resources are used in research, as $\left(1 - L_y^{de}\right) < (1 - L_y^{pl})$.

$$Y^{pl} \gtreqless Y^{de} \quad \text{for} \quad N\alpha^{\frac{\alpha}{1-\alpha}} L_y^{pl} \gtreqless N\alpha^{\frac{2\alpha}{1-\alpha}} L_y^{de}$$

We get ambiguous results for the production of the final good, since $\frac{\rho\,(1+\alpha)}{\delta L + \rho} \gtreqless \alpha^{\frac{\alpha}{1-\alpha}}$.

There are 2 inefficiencies in the decentralized economy:

1. The monopoly power of the producers of intermediate goods means that they produce too little of each variety. This is a static inefficiency, in the form of too little production every period, captured by the term $\alpha^{\frac{\alpha}{1-\alpha}} > \alpha^{\frac{2\alpha}{1-\alpha}}$.

2. In the decentralized economy too high a proportion of the labor force is assigned into producing the final good and not into research. This is a dynamic inefficiency in the allocation of labor. This happens because researchers do not internalize the knowledge spillovers and is reflected in the term $L_y^{pl} < L_y^{de}$.

(c) A subsidy in the production of intermediates. Now the problem for an individual firm is to maximize:

$$\pi_j = x_j p_j - \frac{x_j}{1+\tau_s} = \alpha L_y^{1-\alpha} x_j^{\alpha} - \frac{x_j}{1+\tau_s}$$

The FOC is

$$\alpha^2 L_y^{1-\alpha} x_j^{\alpha-1} = \frac{1}{1+\tau_s} \Rightarrow x_j^{des} = (1+\tau_s)^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} L_y = x^{des}$$

and the solutions for prices and profits become:

$$p^{des} = p_j^{des} = \frac{1}{\alpha(1+\tau_s)}$$

$$\pi^{des} = \pi_j^{des} = \frac{1-\alpha}{\alpha(1+\tau_s)} (1+\tau_s)^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} L_y = (1-\alpha)(1+\tau_s)^{\frac{\alpha}{1-\alpha}} \alpha^{\frac{1+\alpha}{1-\alpha}} L_y$$

Output is

$$Y^{des} = N x^{\alpha} L_y^{1-\alpha} = (1+\tau_s)^{\frac{\alpha}{1-\alpha}} \alpha^{\frac{2\alpha}{1-\alpha}} L_y N$$

Wages become:

$$w^{des} = (1-\alpha) N x^{\alpha} L_y^{-\alpha} = (1-\alpha)(1+\tau_s)^{\frac{\alpha}{1-\alpha}} \alpha^{\frac{2\alpha}{1-\alpha}} N$$

Free entry in the intermediate sector implies the usual arbitrage equation:

$$\frac{\pi^{des}}{r} = \frac{w^{des}}{\delta N} \Rightarrow \frac{\alpha^{\frac{1+\alpha}{1-\alpha}} L_y}{r} = \frac{\alpha^{\frac{2\alpha}{1-\alpha}} N}{\delta N} \Rightarrow r^{des} = \alpha \delta L_y$$

Note that the interest rate is unaffected by the subsidy, or $r^{des} = r^{de}$. Therefore, the growth rate is unchanged. The effect of the subsidy is then an increase in x and in Y, a reduction of p, while r, L_y and γ_Y remain unchanged.

The idea of the subsidy is to correct the inefficiency caused by the monopoly power in the production of intermediates. If the subsidy is set such that $\alpha = (1+\tau_s)^{-1}$, then you induce firms to sell at a price equal to marginal cost, $p = [\alpha(1+\tau_s)]^{-1} = 1$. You get then $x^{de} = x^{pl}$. But when x^{de} increases, the marginal product of labor increases as x is a complementary input, and so do wages. Therefore the effect of the subsidy cancels out in the equation $\frac{\pi}{r} = \frac{w}{\delta N}$, and it does not affect the interest rate. This subsidy can remove the static inefficiency, but not the dynamic one. Growth is not enhanced, but welfare rises as the level of consumption increases with production.

(d) Tax on labor in final production. Firms in the final good sector now maximizes:

$$\pi = L_y^{1-\alpha} \int_0^N x_j^{\alpha} dj - (1+\tau_w) w L_y - \int_0^N p_j x_j dj$$

We immediately see that only the FOC for labor is affected, giving:

$$w^{det} = \frac{(1-\alpha)}{(1+\tau_w)} L_y^{-\alpha} \int_0^N x_j^{\alpha} dj$$

The problem faced by firms in the intermediate sector is basically unchanged; the only difference now is in the arbitrage equation because wages are affected by the tax. Using the old result for x, we find the following expression for wages:

$$w^{det} = \frac{(1-\alpha)}{(1+\tau_w)} \alpha^{\frac{2\alpha}{1-\alpha}} N \tag{4}$$

The free entry condition in the intermediate good markets becomes

$$\frac{\pi}{r} = \frac{w^{det}}{\delta N} \Rightarrow \frac{\left(\frac{1-\alpha}{\alpha}\right)\alpha^{\frac{2}{1-\alpha}} L_y}{r} = \frac{\frac{(1-\alpha)}{(1+\tau_w)}\alpha^{\frac{2\alpha}{1-\alpha}} N}{\delta N} \Rightarrow r^{det} = \alpha\delta\left(1+\tau_w\right) L_y \tag{5}$$

Now the wage is lower and the interest rate is higher. Using the new interest rate in the Euler equation for consumption (unchanged) and using the R&D equation $\gamma = \delta\left(L - L_y\right)$ we get:

$$\alpha\delta\left(1+\tau_w\right) L_y - \rho = \delta\left(L - L_y\right) \Rightarrow L_y^{det} = \frac{\delta L + \rho}{\delta\left[1 + \alpha\left(1+\tau_w\right)\right]}$$

and the growth rate becomes

$$\gamma^{det} = \delta\left(L - L_y\right) = \delta\left(L - \frac{\delta L + \rho}{\delta\left[1 + \alpha\left(1+\tau_w\right)\right]}\right) = \frac{\delta\alpha\left(1+\tau_w\right) L - \rho}{\left[1 + \alpha\left(1+\tau_w\right)\right]}$$

The effect of the tax on the use of labor in the final sector is to reduce the workers employed in the production of the final good and thus increase the workers employed in research. This increase in $(L - L_y)$ enhances growth and hence this policy can correct the dynamic inefficiency. But this policy alone is not sufficient to restore the first-best outcome as it does not solve the monopoly distortion. The first-best Pareto optimum could be implemented by combining this policy with the one suggested in part **(c)**. Then both inefficiencies (static and dynamic) could be eliminated at the same time.