ECON 815
Macroeconomic Theory
Winter Term 2008/09

Practice Questions for the Midterm Exam - *ANSWER KEY*

1 Endogenous growth with transitional dynamics

(a) The elasticity of substitution is a measure of curvature of the isoquants. The slope of an isoquant (TRS) for the CES technology is:

$$TRS = \frac{dL}{dK} = -\frac{f_K}{f_L} = -\frac{\eta}{1-\eta}\left(\frac{K}{L}\right)^{\sigma-1} \tag{1}$$

$$\varepsilon_{K,L} = -\frac{(f_K/f_L)}{(K/L)}\frac{d(K/L)}{d(f_K/f_L)} = \left[-\frac{d\ln(f_K/f_L)}{d\ln(K/L)}\right]^{-1}$$

By taking logs of (1), we obtain:

$$\ln(f_K/f_L) = \text{constant} + (\sigma-1)\ln(K/L)$$

Hence,

$$\varepsilon_{K,L} = \left[-\frac{d\ln(f_K/f_L)}{d\ln(K/L)}\right]^{-1} = \frac{1}{1-\sigma}$$

As $\sigma \to 1$, $\varepsilon_{K,L} \to \infty$ as in a linear technology where factors are perfect substitutes. As $\sigma \to 0$, $\varepsilon_{K,L} \to 1$ as in the Cobb-Douglas technology[1], and as $\sigma \to -\infty$, $\varepsilon_{K,L} \to 0$ as in the Leontief case where production occurs in fixed proportions and there is no substitutability at all.

(b) The CES production function is:

$$Y = A\left\{\eta K^\sigma + (1-\eta)L^\sigma\right\}^{\frac{1}{\sigma}} = LA\left\{\eta k^\sigma + (1-\eta)\right\}^{\frac{1}{\sigma}}$$

Replacing the production function into $sY = \delta K + \dot{K}$ we find: $\delta K + \dot{K} = sLA\left\{\eta k^\sigma + (1-\eta)\right\}$. Dividing by L yields

$$\delta k + \frac{\dot{K}}{L} = sA\left\{\eta k^\sigma + (1-\eta)\right\}^{\frac{1}{\sigma}} \tag{2}$$

We know that

$$K = kL \Rightarrow \dot{K} = \dot{k}L + k\dot{L} \Rightarrow \frac{\dot{K}}{L} = \dot{k} + nk \tag{3}$$

Use (2) into (3) and rearrange

$$\dot{k} = sA\left\{\eta k^\sigma + (1-\eta)\right\}^{\frac{1}{\sigma}} - (\delta+n)k \tag{4}$$

The first term on the right hand side of (4) is per-capita saving and the second term is the effective per-capita depreciation of capital.

[1] Let's prove that the CES production function corresponds to the Cobb-Douglas case when $\sigma \to 0$. Take logs of the production function: $\log Y = \log A + \frac{1}{\sigma}\log\left[\eta K^\sigma + (1-\eta)L^\sigma\right]$. Consider the $\lim_{\sigma\to 0}\log Y = \log A + \lim_{\sigma\to 0}\frac{1}{\sigma}\log\left[\eta K^\sigma + (1-\eta)L^\sigma\right]$.
Now, $\lim_{\sigma\to 0}\frac{1}{\sigma}\log\left[\eta K^\sigma + (1-\eta)L^\sigma\right] = \frac{\lim_{\sigma\to 0}\log[\eta K^\sigma + (1-\eta)L^\sigma]}{\lim_{\sigma\to 0}\sigma} = \frac{\log[\eta+(1-\eta)]}{\lim_{\sigma\to 0}\sigma} = \left[\frac{0}{0}\right]$.
Hence, we can apply De L'Hopital rule to solve the limit: $\lim_{\sigma\to 0}\frac{1}{\sigma}\log\left[\eta K^\sigma + (1-\eta)L^\sigma\right] = \lim_{\sigma\to 0}\frac{d}{d\sigma}\left(\log\left[\eta K^\sigma + (1-\eta)L^\sigma\right]\right) = \lim_{\sigma\to 0}\frac{1}{[\eta K^\sigma + (1-\eta)L^\sigma]}\left[\eta K^\sigma\log K + (1-\eta)L^\sigma\log L\right] = \eta\log K + (1-\eta)\log L$.
Hence $\log Y = \log A + \eta\log K + (1-\eta)\log L$ or $Y = AK^\eta L^{1-\eta}$.

(c) The growth rate of per capita capital is $\gamma_k = \frac{\dot{k}}{k}$

$$\gamma_k = sA\left\{k^{-\sigma}\left[\eta k^{\sigma} + (1-\eta)\right]\right\}^{\frac{1}{\sigma}} - (\delta+n) = sA\left\{\eta + (1-\eta)\,k^{-\sigma}\right\}^{\frac{1}{\sigma}} - (\delta+n) \quad (5)$$

Since γ_k depends on $k^{-\sigma}$, we can get only an asymptotic steady-state, i.e. a constant rate of growth of k and K as $k \to \infty$; only in this case the term $k^{-\sigma}$ vanishes, given that.$0 < \sigma < 1$. For finite k there is no constant rate of growth of k and K. We obtain the steady-state growth of k by taking the limit of (5) for $k \to \infty$:

$$\lim_{k\to\infty} \gamma_k = \gamma_k^* = sA\eta^{\frac{1}{\sigma}} - (\delta+n) \quad (6)$$

The asymptotic growth rate is just asymptotic saving (per unit of k) minus effective depreciation. Note that, since $0 < \eta < 1$ and $0 < \sigma < 1$, the growth rate is maximized as $\sigma \to 1$ because technology approaches the linear one and decreasing returns on capital disappear.

To find the growth rate of K, log-differentiate $k = K/L$ to get $\frac{d}{dt}\ln(k) = \frac{\dot{k}}{k} = \frac{d}{dt}\left[\ln(K) - \ln(L)\right] = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \Rightarrow \gamma_K = \gamma_k + n$. From (6), the steady state growth of K is

$$\gamma_K^* = \gamma_k^* + n = sA\eta^{\frac{1}{\sigma}} - \delta$$

(d) Find the economy's growth rate of GDP per capita as a function of parameters and the capital level.

From the production function, output per capita is

$$y = A\left\{\eta k^{\sigma} + (1-\eta)\right\}^{\frac{1}{\sigma}}$$

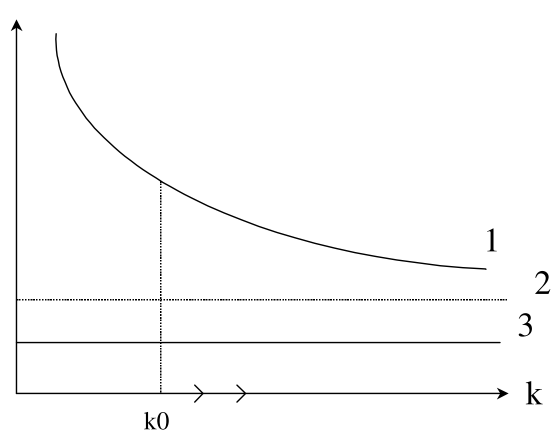
Its time derivative is

$$\dot{y} = A\frac{1}{\sigma}\left\{\eta k^{\sigma} + (1-\eta)\right\}^{\frac{1}{\sigma}-1}\eta\sigma k^{\sigma-1}\dot{k} = A\eta\left\{\eta + (1-\eta)\,k^{-\sigma}\right\}^{\frac{1-\sigma}{\sigma}}\dot{k}$$

The growth rate of GDP per capita then is

$$\begin{aligned}\gamma_y &= \frac{\dot{y}}{y} = \frac{A\eta\left\{\eta + (1-\eta)\,k^{-\sigma}\right\}^{\frac{1-\sigma}{\sigma}}\dot{k}}{A\left\{\eta k^{\sigma} + (1-\eta)\right\}^{\frac{1}{\sigma}}} = \frac{\eta\left\{\eta + (1-\eta)\,k^{-\sigma}\right\}^{\frac{1-\sigma}{\sigma}}k}{\left\{\eta k^{\sigma} + (1-\eta)\right\}^{\frac{1}{\sigma}}}\frac{\dot{k}}{k} = \\ &= \frac{\eta\left\{k^{-\sigma}\left[\eta k^{\sigma} + (1-\eta)\right]\right\}^{\frac{1-\sigma}{\sigma}}k}{\left\{\eta k^{\sigma} + (1-\eta)\right\}^{\frac{1}{\sigma}}}\gamma_k = \frac{\eta k^{\sigma}\gamma_k}{\eta k^{\sigma} + 1 - \eta} = \frac{\eta k^{\sigma}\left(sA\eta^{\frac{1}{\sigma}} - \delta\right)}{\eta k^{\sigma} + 1 - \eta}\end{aligned}$$

(e) In the following graph the vertical axis represents the growth rate of per capita capital, curve 1 represents the function $s\frac{f(k)}{k}$, the dotted line 2 represents the constant $sA\eta^{1/\sigma}$ and line 3 represents the constant $(n+\delta)$.



Graph 1.

The curve $s\dfrac{f(k)}{k}$ is downward sloping as $\lim\limits_{k\to 0}\dfrac{f(k)}{k}=\infty$ and $\lim\limits_{k\to\infty}\dfrac{f(k)}{k}=A\eta^{1/\sigma}$.

If we start at k_0, the growth rate of k, γ_k, is the vertical distance between 1 and 3. The fact that $\gamma_k>0$ means that k will grow over time. Note that as k grows, its growth rate decreases and approaches it's asymptotic steady-state level. If $sA\eta^{1/\sigma}>n+\delta$ (as we have assumed), then the steady-state growth rate γ_k^* is positive.

There is conditional convergence in this economy as $\dfrac{d\left(\dfrac{f(k)}{k}\right)}{dk}<0$. This means that countries with lower per capita capital, other parameters equal, will grow at a higher rate than countries with a higher k.

(f) If $\sigma<0$, the steady-state growth rate is zero. Remember that $\gamma_k=\frac{sf(k)}{k}-(\delta+n)$

$$\lim_{k\to\infty}\frac{sf(k)}{k}=\lim_{k\to\infty}sA\left\{\eta+(1-\eta)k^{-\sigma}\right\}^{\frac{1}{\sigma}}=0$$

As k grows, the marginal and average product of capital will fall up to the point where $\frac{sf(k)}{k}=(\delta+n)$ and growth stops.

Intuition. With $\sigma<0$, K and L are poor substitutes: as K/L increases, the relative scarcity of labor cannot be compensated and therefore reduces the marginal productivity of capital substantially. In the previous case ($0<\sigma<1$), instead, K and L were good substitutes: since capital could "make up" for labor, an increase of its stock relative to the other factor did not depress its marginal productivity much. Remember that in the limit $\sigma\to 1$ the technology becomes linear.

2 Factor rewards and the golden rule

Consider a neoclassical production function $Y=F(K,L)$. Define $y=\dfrac{Y}{L}$, $k=\dfrac{K}{L}$, then $Y=Ly=Lf(k)$. The marginal product of capital is $\dfrac{\partial Y}{\partial K}=f'(k)$ and the marginal product of labor is

$$\frac{\partial Y}{\partial L}=\frac{\partial Lf(K/L)}{\partial L}=f(K/L)-\frac{Lf'(K/L)K}{L^2}=f(k)-kf'(k)$$

A competitive firm, which takes the interest rate and wage as given, will choose k so as to maximize its profits:

$$\max_k L\left[f(k)-(r+\delta)k-w\right]$$

The first order condition is

$$f'(k)=(r+\delta)$$

In a full market equilibrium wages must be such that profits are zero (otherwise firms will either close or expand to infinite size):

$$w=f(k)-(r+\delta)k=f(k)-f'(k)k$$

Hence, each factor of production earns its marginal product.

The fact that rewarding each factor with its marginal productivity exhausts output is an application of the Euler theorem. Since $Y(K,L)$ is homogeneous of degree 1, $\alpha Y=F(\alpha K,\alpha L)\quad\forall\alpha$. Take the derivative w.r.t. α, set $\alpha=1$ and you will find the result: $Y=F_KK+F_LL$.

We have just seen that per capita income can be decomposed into capital income + labour income, when the technology exhibits constant return to scale:

$$y=\underbrace{f'(k)k}_{Capital}+\underbrace{f(k)-f'(k)k}_{Labour}$$

By assumption, capitalists own the whole capital stock, while workers are the only group of agents supplying labour. Then, if capitalists save their income and workers spend it entirely, it follows that the we can express per capita savings as:

$$\bar{s} = f'(k)k$$

In equilibrium savings must be equal to investments, or $S = I$ ($\bar{s} = \bar{i}$, in per capita terms). Moreover, recall that in a steady-state total investment offsets exactly the effective depreciation of capital, or $\bar{i} = (n + \delta)k$.

Imposing the equilibrium condition gives:

$$f'(k)k = (n + \delta)k \rightarrow f'(k^*) = (n + \delta)$$

where the marginal productivity of capital = rate of effective depreciation. This is exactly the golden rule of capital accumulation. The idea is that one additional unit of investment rises output per capita of its marginal product but brings about a maintenance cost (in a steady-state k must be constant) of $n + \delta$. If you want to maximize the revenue from investment in steady-state, you should invest up to the point where the return on capital offsets exactly its maintenance cost.

3 Dynamics in a Solow-type model

See the scanned pages.

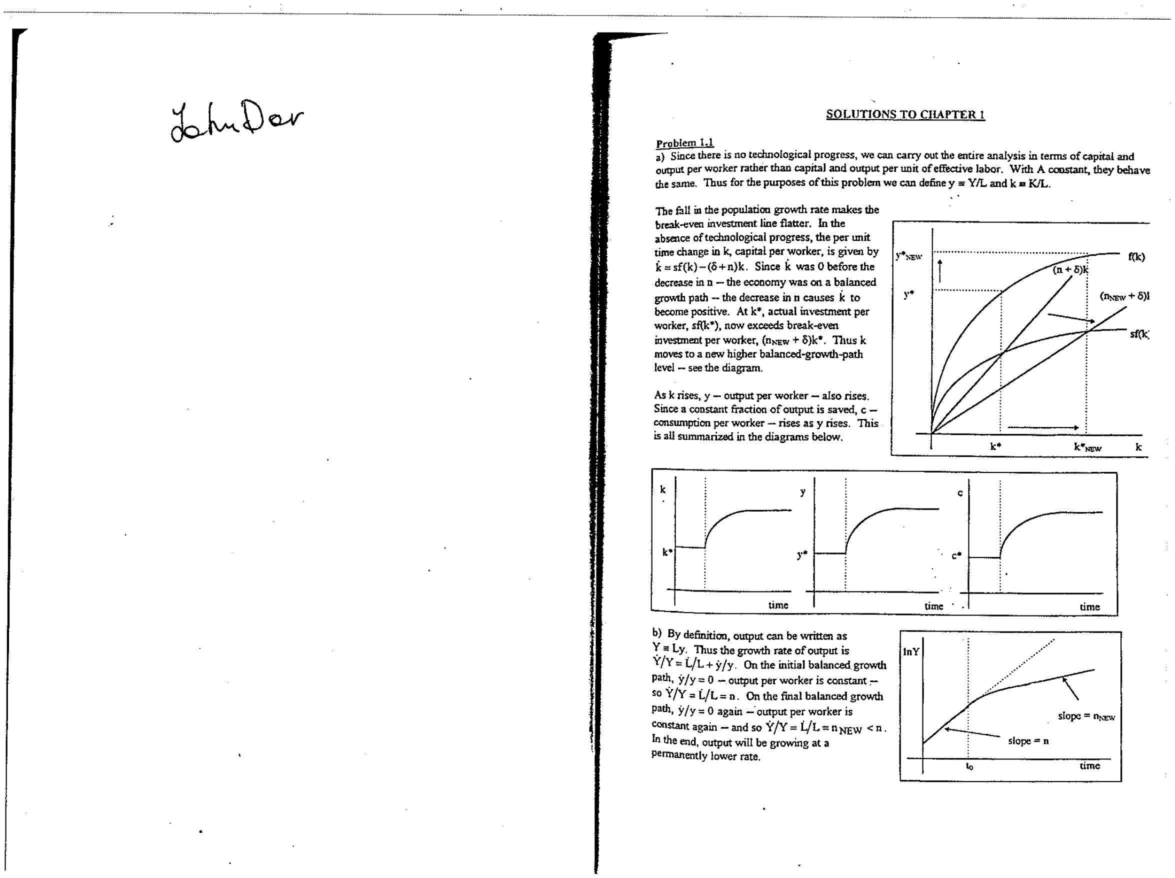
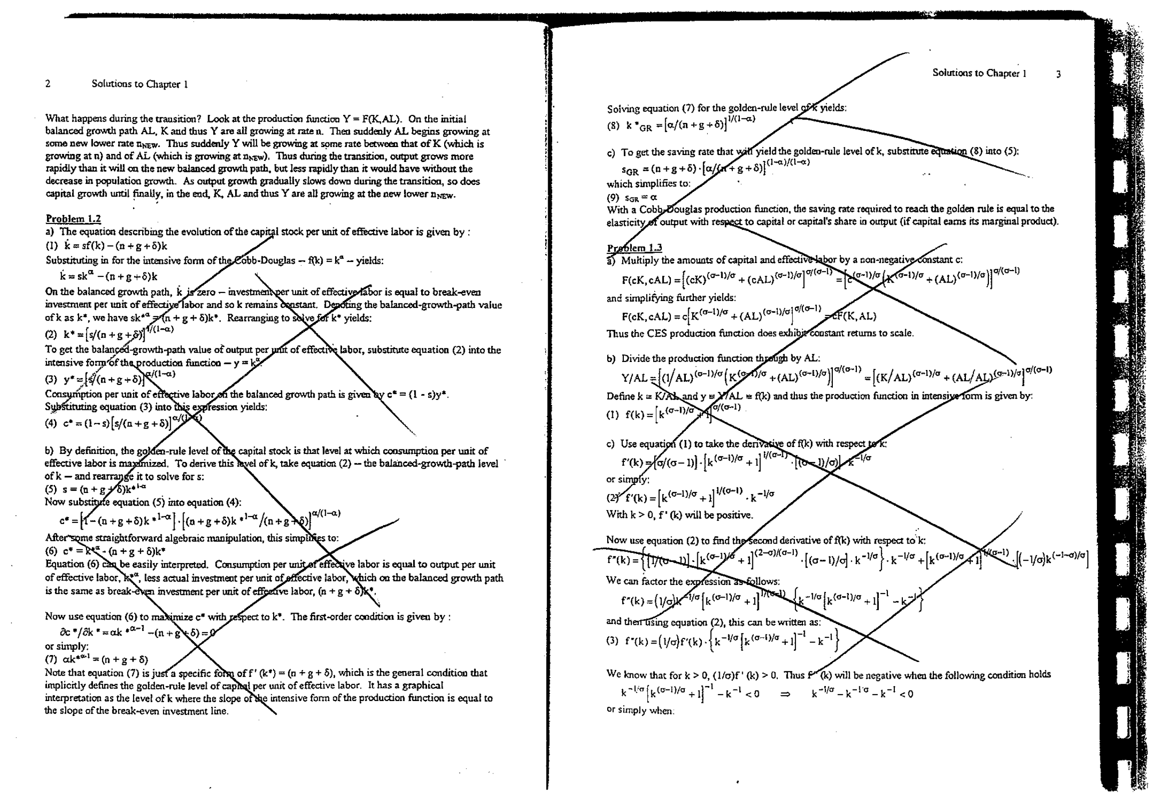


Figure 1:



2 Solutions to Chapter 1

What happens during the transition? Look at the production function $Y = F(K,AL)$. On the initial balanced growth path AL, K and thus Y are all growing at rate n. Then suddenly AL begins growing at some new lower rate n_{NEW}. Thus suddenly Y will be growing at some rate between that of K (which is growing at n) and of AL (which is growing at n_{NEW}). Thus during the transition, output grows more rapidly than it will on the new balanced growth path, but less rapidly than it would have without the decrease in population growth. As output growth gradually slows down during the transition, so does capital growth until finally, in the end, K, AL and thus Y are all growing at the new lower n_{NEW}.

Problem 1.2

a) The equation describing the evolution of the capital stock per unit of effective labor is given by :

(1) $\dot{k} = sf(k) - (n+g+\delta)k$

Substituting in for the intensive form of the Cobb-Douglas -- $f(k) = k^{\alpha}$ -- yields:

$$\dot{k} = sk^{\alpha} - (n+g+\delta)k$$

On the balanced growth path, $\dot{k}$ is zero -- investment per unit of effective labor is equal to break-even investment per unit of effective labor and so k remains constant. Denoting the balanced-growth-path value of k as k^*, we have $sk^{*\alpha} = (n+g+\delta)k^*$. Rearranging to solve for k^* yields:

(2) $k^* = [s/(n+g+\delta)]^{1/(1-\alpha)}$

To get the balanced-growth-path value of output per unit of effective labor, substitute equation (2) into the intensive form of the production function -- $y = k^{\alpha}$:

(3) $y^* = [s/(n+g+\delta)]^{\alpha/(1-\alpha)}$

Consumption per unit of effective labor on the balanced growth path is given by $c^* = (1 - s)y^*$. Substituting equation (3) into this expression yields:

(4) $c^* = (1-s)[s/(n+g+\delta)]^{\alpha/(1-\alpha)}$

b) By definition, the golden-rule level of the capital stock is that level at which consumption per unit of effective labor is maximized. To derive this level of k, take equation (2) -- the balanced-growth-path level of k -- and rearrange it to solve for s:

(5) $s = (n+g+\delta)k^{*1-\alpha}$

Now substitute equation (5) into equation (4):

$$c^* = [1-(n+g+\delta)k^{*1-\alpha}] \cdot [(n+g+\delta)k^{*1-\alpha}/(n+g+\delta)]^{\alpha/(1-\alpha)}$$

After some straightforward algebraic manipulation, this simplifies to:

(6) $c^* = k^{*\alpha} - (n+g+\delta)k^*$

Equation (6) can be easily interpreted. Consumption per unit of effective labor is equal to output per unit of effective labor, $k^{*\alpha}$, less actual investment per unit of effective labor, which on the balanced growth path is the same as break-even investment per unit of effective labor, $(n+g+\delta)k^*$.

Now use equation (6) to maximize c^* with respect to k^*. The first-order condition is given by :

$$\partial c^*/\partial k^* = \alpha k^{*\alpha-1} - (n+g+\delta) = 0$$

or simply:

(7) $\alpha k^{*\alpha-1} = (n+g+\delta)$

Note that equation (7) is just a specific form of $f'(k^*) = (n+g+\delta)$, which is the general condition that implicitly defines the golden-rule level of capital per unit of effective labor. It has a graphical interpretation as the level of k where the slope of the intensive form of the production function is equal to the slope of the break-even investment line.

Solutions to Chapter 1 3

Solving equation (7) for the golden-rule level of k yields:

(8) $k^*_{GR} = [\alpha/(n+g+\delta)]^{1/(1-\alpha)}$

c) To get the saving rate that will yield the golden-rule level of k, substitute equation (8) into (5):

$$s_{GR} = (n+g+\delta) \cdot [\alpha/(n+g+\delta)]^{(1-\alpha)/(1-\alpha)}$$

which simplifies to:

(9) $s_{GR} = \alpha$

With a Cobb-Douglas production function, the saving rate required to reach the golden rule is equal to the elasticity of output with respect to capital or capital's share in output (if capital earns its marginal product).

Problem 1.3

a) Multiply the amounts of capital and effective labor by a non-negative constant c:

$$F(cK, cAL) = [(cK)^{(\sigma-1)/\sigma} + (cAL)^{(\sigma-1)/\sigma}]^{\sigma/(\sigma-1)} = [c^{(\sigma-1)/\sigma}(K^{(\sigma-1)/\sigma} + (AL)^{(\sigma-1)/\sigma})]^{\sigma/(\sigma-1)}$$

and simplifying further yields:

$$F(cK, cAL) = c[K^{(\sigma-1)/\sigma} + (AL)^{(\sigma-1)/\sigma}]^{\sigma/(\sigma-1)} = cF(K, AL)$$

Thus the CES production function does exhibit constant returns to scale.

b) Divide the production function through by AL:

$$Y/AL = [(1/AL)^{(\sigma-1)/\sigma}(K^{(\sigma-1)/\sigma} + (AL)^{(\sigma-1)/\sigma})]^{\sigma/(\sigma-1)} = [(K/AL)^{(\sigma-1)/\sigma} + (AL/AL)^{(\sigma-1)/\sigma}]^{\sigma/(\sigma-1)}$$

Define $k \equiv K/AL$ and $y \equiv Y/AL \equiv f(k)$ and thus the production function in intensive form is given by:

(1) $f(k) = [k^{(\sigma-1)/\sigma} + 1]^{\sigma/(\sigma-1)}$

c) Use equation (1) to take the derivative of f(k) with respect to k:

$$f'(k) = [\sigma/(\sigma-1)] \cdot [k^{(\sigma-1)/\sigma} + 1]^{1/(\sigma-1)} \cdot [(\sigma-1)/\sigma] \cdot k^{-1/\sigma}$$

or simply:

(2) $f'(k) = [k^{(\sigma-1)/\sigma} + 1]^{1/(\sigma-1)} \cdot k^{-1/\sigma}$

With $k > 0$, $f'(k)$ will be positive.

Now use equation (2) to find the second derivative of f(k) with respect to k:

$$f''(k) = \{[1/(\sigma-1)] \cdot [k^{(\sigma-1)/\sigma} + 1]^{(2-\sigma)/(\sigma-1)} \cdot [(\sigma-1)/\sigma] \cdot k^{-1/\sigma}\} \cdot k^{-1/\sigma} + [k^{(\sigma-1)/\sigma} + 1]^{1/(\sigma-1)} \cdot [(-1/\sigma)k^{(-1-\sigma)/\sigma}]$$

We can factor the expression as follows:

$$f''(k) = (1/\sigma)k^{-1/\sigma}[k^{(\sigma-1)/\sigma} + 1]^{1/(\sigma-1)} \{k^{-1/\sigma}[k^{(\sigma-1)/\sigma} + 1]^{-1} - k^{-1}\}$$

and then using equation (2), this can be written as:

(3) $f''(k) = (1/\sigma)f'(k) \cdot \{k^{-1/\sigma}[k^{(\sigma-1)/\sigma} + 1]^{-1} - k^{-1}\}$

We know that for $k > 0$, $(1/\sigma)f'(k) > 0$. Thus $f''(k)$ will be negative when the following condition holds

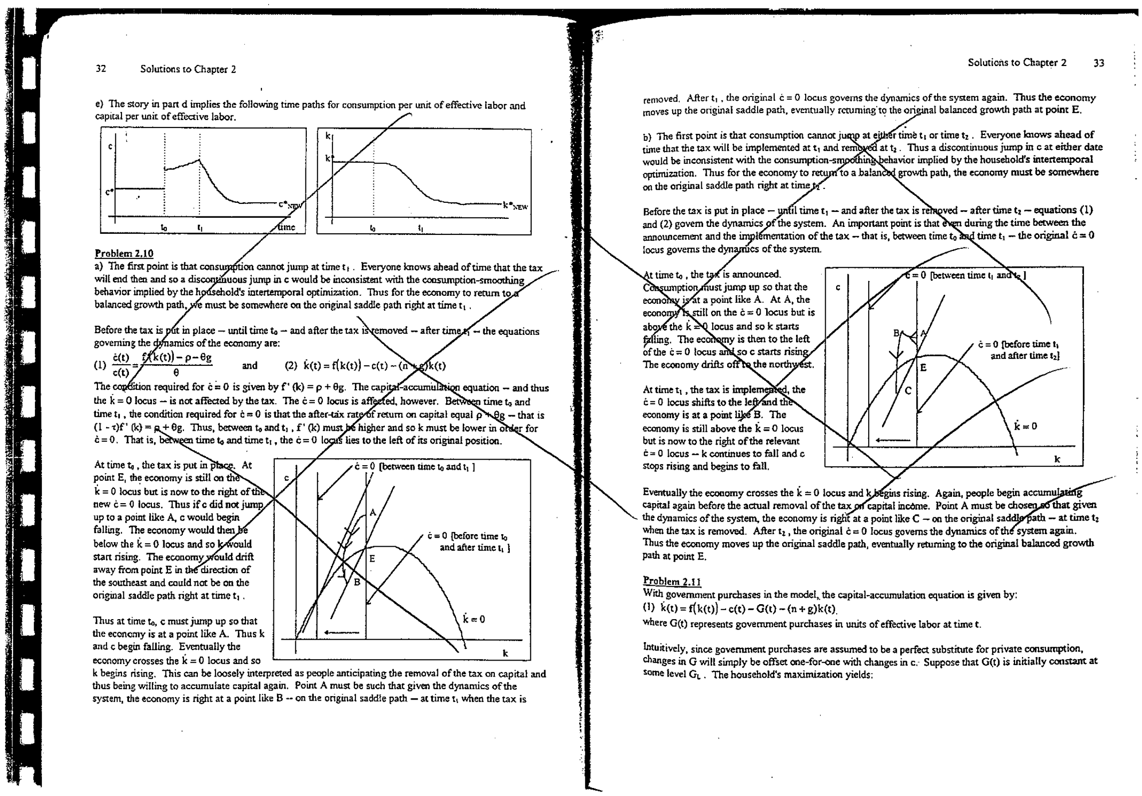
$$k^{-1/\sigma}[k^{(\sigma-1)/\sigma} + 1]^{-1} - k^{-1} < 0 \quad \Rightarrow \quad k^{-1/\sigma} - k^{-1/\sigma} - k^{-1} < 0$$

or simply when:

Figure 2:

4 A Simple Ramsey Model

See the scanned pages.



32 Solutions to Chapter 2

e) The story in part d implies the following time paths for consumption per unit of effective labor and capital per unit of effective labor.

Problem 2.10

a) The first point is that consumption cannot jump at time t_1. Everyone knows ahead of time that the tax will end then and so a discontinuous jump in c would be inconsistent with the consumption-smoothing behavior implied by the household's intertemporal optimization. Thus for the economy to return to a balanced growth path, we must be somewhere on the original saddle path right at time t_1.

Before the tax is put in place – until time t_0 – and after the tax is removed – after time t_1 – the equations governing the dynamics of the economy are:

(1) $\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$ and (2) $\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$

The condition required for $\dot{c} = 0$ is given by $f'(k) = \rho + \theta g$. The capital-accumulation equation – and thus the $\dot{k} = 0$ locus – is not affected by the tax. The $\dot{c} = 0$ locus is affected, however. Between time t_0 and time t_1, the condition required for $\dot{c} = 0$ is that the after-tax rate of return on capital equal $\rho + \theta g$ – that is $(1 - \tau)f'(k) = \rho + \theta g$. Thus, between t_0 and t_1, $f'(k)$ must be higher and so k must be lower in order for $\dot{c} = 0$. That is, between time t_0 and time t_1, the $\dot{c} = 0$ locus lies to the left of its original position.

At time t_0, the tax is put in place. At point E, the economy is still on the $\dot{k} = 0$ locus but is now to the right of the new $\dot{c} = 0$ locus. Thus if c did not jump up to a point like A, c would begin falling. The economy would then be below the $\dot{k} = 0$ locus and so k would start rising. The economy would drift away from point E in the direction of the southeast and could not be on the original saddle path right at time t_1.

Thus at time t_0, c must jump up so that the economy is at a point like A. Thus k and c begin falling. Eventually the economy crosses the $\dot{k} = 0$ locus and so k begins rising. This can be loosely interpreted as people anticipating the removal of the tax on capital and thus being willing to accumulate capital again. Point A must be such that given the dynamics of the system, the economy is right at a point like B -- on the original saddle path – at time t_1 when the tax is

Solutions to Chapter 2 33

removed. After t_1, the original $\dot{c} = 0$ locus governs the dynamics of the system again. Thus the economy moves up the original saddle path, eventually returning to the original balanced growth path at point E.

b) The first point is that consumption cannot jump at either time t_1 or time t_2. Everyone knows ahead of time that the tax will be implemented at t_1 and removed at t_2. Thus a discontinuous jump in c at either date would be inconsistent with the consumption-smoothing behavior implied by the household's intertemporal optimization. Thus for the economy to return to a balanced growth path, the economy must be somewhere on the original saddle path right at time t_2.

Before the tax is put in place – until time t_1 – and after the tax is removed – after time t_2 – equations (1) and (2) govern the dynamics of the system. An important point is that even during the time between the announcement and the implementation of the tax – that is, between time t_0 and time t_1 – the original $\dot{c} = 0$ locus governs the dynamics of the system.

At time t_0, the tax is announced. Consumption must jump up so that the economy is at a point like A. At A, the economy is still on the $\dot{c} = 0$ locus but is above the $\dot{k} = 0$ locus and so k starts falling. The economy is then to the left of the $\dot{c} = 0$ locus and so c starts rising. The economy drifts off to the northwest.

At time t_1, the tax is implemented, the $\dot{c} = 0$ locus shifts to the left and the economy is at a point like B. The economy is still above the $\dot{k} = 0$ locus but is now to the right of the relevant $\dot{c} = 0$ locus – k continues to fall and c stops rising and begins to fall.

Eventually the economy crosses the $\dot{k} = 0$ locus and k begins rising. Again, people begin accumulating capital again before the actual removal of the tax on capital income. Point A must be chosen so that given the dynamics of the system, the economy is right at a point like C – on the original saddle path – at time t_2 when the tax is removed. After t_2, the original $\dot{c} = 0$ locus governs the dynamics of the system again. Thus the economy moves up the original saddle path, eventually returning to the original balanced growth path at point E.

Problem 2.11

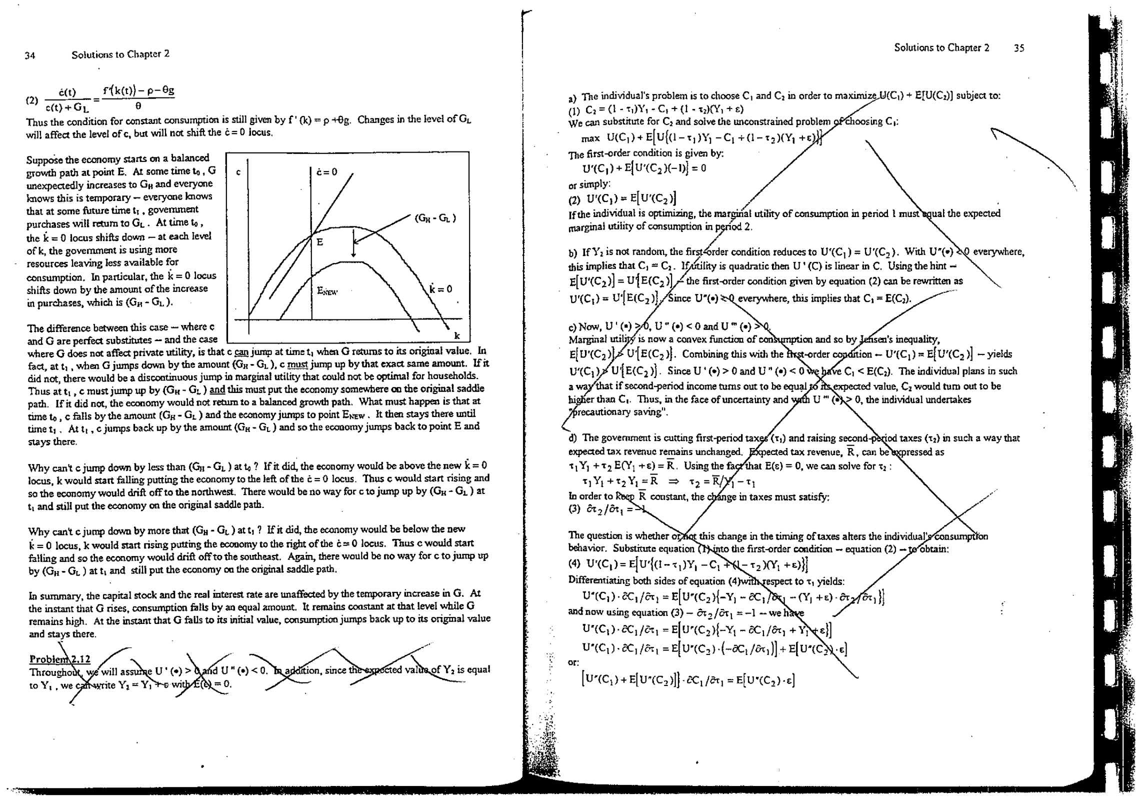
With government purchases in the model, the capital-accumulation equation is given by:

(1) $\dot{k}(t) = f(k(t)) - c(t) - G(t) - (n+g)k(t)$,

where G(t) represents government purchases in units of effective labor at time t.

Intuitively, since government purchases are assumed to be a perfect substitute for private consumption, changes in G will simply be offset one-for-one with changes in c. Suppose that G(t) is initially constant at some level G_L. The household's maximization yields:

Figure 3:



34 Solutions to Chapter 2

(2) $\frac{\dot{c}(t)}{c(t)+G_L} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$

Thus the condition for constant consumption is still given by $f'(k) = \rho + \theta g$. Changes in the level of G_L will affect the level of c, but will not shift the $\dot{c} = 0$ locus.

Suppose the economy starts on a balanced growth path at point E. At some time t_0, G unexpectedly increases to G_H and everyone knows this is temporary -- everyone knows that at some future time t_1, government purchases will return to G_L. At time t_0, the $\dot{k} = 0$ locus shifts down -- at each level of k, the government is using more resources leaving less available for consumption. In particular, the $\dot{k} = 0$ locus shifts down by the amount of the increase in purchases, which is $(G_H - G_L)$.

The difference between this case -- where c and G are perfect substitutes -- and the case where G does not affect private utility, is that c can jump at time t_1 when G returns to its original value. In fact, at t_1, when G jumps down by the amount $(G_H - G_L)$, c must jump up by that exact same amount. If it did not, there would be a discontinuous jump in marginal utility that could not be optimal for households. Thus at t_1, c must jump up by $(G_H - G_L)$ and this must put the economy somewhere on the original saddle path. If it did not, the economy would not return to a balanced growth path. What must happen is that at time t_0, c falls by the amount $(G_H - G_L)$ and the economy jumps to point E_{NEW}. It then stays there until time t_1. At t_1, c jumps back up by the amount $(G_H - G_L)$ and so the economy jumps back to point E and stays there.

Why can't c jump down by less than $(G_H - G_L)$ at t_0? If it did, the economy would be above the new $\dot{k} = 0$ locus, k would start falling putting the economy to the left of the $\dot{c} = 0$ locus. Thus c would start rising and so the economy would drift off to the northwest. There would be no way for c to jump up by $(G_H - G_L)$ at t_1 and still put the economy on the original saddle path.

Why can't c jump down by more that $(G_H - G_L)$ at t_1? If it did, the economy would be below the new $\dot{k} = 0$ locus, k would start rising putting the economy to the right of the $\dot{c} = 0$ locus. Thus c would start falling and so the economy would drift off to the southeast. Again, there would be no way for c to jump up by $(G_H - G_L)$ at t_1 and still put the economy on the original saddle path.

In summary, the capital stock and the real interest rate are unaffected by the temporary increase in G. At the instant that G rises, consumption falls by an equal amount. It remains constant at that level while G remains high. At the instant that G falls to its initial value, consumption jumps back up to its original value and stays there.

Problem 2.12

Throughout, we will assume $U'(\bullet) > 0$ and $U''(\bullet) < 0$. In addition, since the expected value of Y_2 is equal to Y_1, we can write $Y_2 = Y_1 + \varepsilon$ with $E(\varepsilon) = 0$.

Solutions to Chapter 2 35

a) The individual's problem is to choose C_1 and C_2 in order to maximize $U(C_1) + E[U(C_2)]$ subject to:

(1) $C_2 = (1 - \tau_1)Y_1 - C_1 + (1 - \tau_2)(Y_1 + \varepsilon)$

We can substitute for C_2 and solve the unconstrained problem of choosing C_1:

$$\max\ U(C_1) + E\left[U\{(1-\tau_1)Y_1 - C_1 + (1-\tau_2)(Y_1+\varepsilon)\}\right]$$

The first-order condition is given by:

$$U'(C_1) + E\left[U'(C_2)(-1)\right] = 0$$

or simply:

(2) $U'(C_1) = E\left[U'(C_2)\right]$

If the individual is optimizing, the marginal utility of consumption in period 1 must equal the expected marginal utility of consumption in period 2.

b) If Y_2 is not random, the first-order condition reduces to $U'(C_1) = U'(C_2)$. With $U''(\bullet) < 0$ everywhere, this implies that $C_1 = C_2$. If utility is quadratic then $U'(C)$ is linear in C. Using the hint -- $E\left[U'(C_2)\right] = U'\left[E(C_2)\right]$ -- the first-order condition given by equation (2) can be rewritten as $U'(C_1) = U'\left[E(C_2)\right]$. Since $U''(\bullet) < 0$ everywhere, this implies that $C_1 = E(C_2)$.

c) Now, $U'(\bullet) > 0$, $U''(\bullet) < 0$ and $U'''(\bullet) > 0$.

Marginal utility is now a convex function of consumption and so by Jensen's inequality, $E\left[U'(C_2)\right] > U'\left[E(C_2)\right]$. Combining this with the first-order condition -- $U'(C_1) = E\left[U'(C_2)\right]$ -- yields $U'(C_1) > U'\left[E(C_2)\right]$. Since $U'(\bullet) > 0$ and $U''(\bullet) < 0$ we have $C_1 < E(C_2)$. The individual plans in such a way that if second-period income turns out to be equal to its expected value, C_2 would turn out to be higher than C_1. Thus, in the face of uncertainty and with $U'''(\bullet) > 0$, the individual undertakes "precautionary saving".

d) The government is cutting first-period taxes (τ_1) and raising second-period taxes (τ_2) in such a way that expected tax revenue remains unchanged. Expected tax revenue, $\overline{R}$, can be expressed as $\tau_1 Y_1 + \tau_2 E(Y_1 + \varepsilon) = \overline{R}$. Using the fact that $E(\varepsilon) = 0$, we can solve for τ_2:

$$\tau_1 Y_1 + \tau_2 Y_1 = \overline{R} \quad \Rightarrow \quad \tau_2 = \overline{R}/Y_1 - \tau_1$$

In order to keep $\overline{R}$ constant, the change in taxes must satisfy:

(3) $\partial\tau_2/\partial\tau_1 = -1$

The question is whether or not this change in the timing of taxes alters the individual's consumption behavior. Substitute equation (1) into the first-order condition -- equation (2) -- to obtain:

(4) $U'(C_1) = E\left[U'\{(1-\tau_1)Y_1 - C_1 + (1-\tau_2)(Y_1+\varepsilon)\}\right]$

Differentiating both sides of equation (4) with respect to τ_1 yields:

$$U''(C_1)\cdot\partial C_1/\partial\tau_1 = E\left[U''(C_2)\{-Y_1 - \partial C_1/\partial\tau_1 - (Y_1+\varepsilon)\cdot\partial\tau_2/\partial\tau_1\}\right]$$

and now using equation (3) -- $\partial\tau_2/\partial\tau_1 = -1$ -- we have

$$U''(C_1)\cdot\partial C_1/\partial\tau_1 = E\left[U''(C_2)\{-Y_1 - \partial C_1/\partial\tau_1 + Y_1 + \varepsilon\}\right]$$

$$U''(C_1)\cdot\partial C_1/\partial\tau_1 = E\left[U''(C_2)\cdot(-\partial C_1/\partial\tau_1)\right] + E\left[U''(C_2)\cdot\varepsilon\right]$$

or:

$$\left[U''(C_1) + E\left[U''(C_2)\right]\right]\cdot\partial C_1/\partial\tau_1 = E\left[U''(C_2)\cdot\varepsilon\right]$$

Figure 4: