SOLUTIONS TO PROBLEM SET 3

Question 1.

a) The final goods producer chooses labour and inputs in order to maximise profits. His profit function is:

$$\Pi(X_{ij}, L_i) = Y_i - wL_i - \sum_{j=1}^{N} p_j X_{ij}^{\alpha}$$

His problem can be stated as:

$$\max_{L_i, X_{ij}} \Pi(\cdot)$$

The first order conditions for this problem are:

$$\begin{cases} L_i : & (1-\alpha)AL_i^{-\alpha}\sum_{j=1}^{N} X_{ij}^{\alpha} - w = 0 \\ X_{ij} : & \alpha AL_i^{1-\alpha}X_{ij}^{\alpha-1} - p_j = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} (1-\alpha)y = w \\ X_{ij} = \left(\frac{\alpha A}{p_j}\right)^{\frac{1}{1-\alpha}} L \end{cases}$$

The varieties of goods are produced in a competitively monopolistic way, such that each producer has some degree of market power:

$$\max_{p_j} \pi(\cdot) = (p_j - 1)X_{ij}(p_j)$$

The first order conditions for this problem are:

$$X_{ij}(p_j) + X'_{ij}(p_j)p_j - X'_{ij}(p_j) = 0 \Rightarrow X_{ij}(p_j) = (1-p_j)X'_{ij}(p_j)$$

With:

$$X'_{ij}(p_j) = L\left(\frac{\alpha A}{p_j}p_j\right)^{\frac{1}{1-\alpha}-1} \frac{1}{1-\alpha}\left(\frac{-\alpha A}{p_j^2}\right) = -X_{ij}\frac{1}{(1-\alpha)p_j}$$

Hence, the FOC becomes:

$$X_{ij} = (1-p_j)(-X_{ij})\frac{1}{(1-\alpha)p_j} \Rightarrow p_j = \frac{1}{\alpha}$$

Monopolised intermediate products (inputs) are priced according to:

$$p_j^* = \frac{1}{\alpha}$$

And the optimal quantity of X_{ij} is:

$$X_{ij}^* = X^* = (\alpha^2 A)^{\frac{1}{1-\alpha}} L$$

Quantities of the different varieties are therefore identical in equilibrium.

b) The R & D firm's profit per period of time is given by the static profit derived in the previous section:

$$\pi = (p_j - 1)X_{ij} = \left(\frac{1}{\alpha} - 1\right) L(A\alpha^2)^{\frac{1}{1-\alpha}}$$

Because profit does not change with time, the discounted stream of future profits faced by the R & D firm is:

$$V(t) = \int_t^\infty \pi e^{\int_t^s r_v dv} ds$$

The growth in $V(t)$ throughout time is given by[1]:

$$\frac{dV(t)}{dt} = \dot{V}(t) = -\pi + r_t V(t)$$

The free-entry condition imposes that firms will enter this market until the profits of engaging in research activity are brought down to zero, thus equating discounted stream of profits with the costs of engaging in this activity. Thus:

$$V(t) = \eta - T$$

Since this must hold in equilibrium, the discounted value of the stream of profits is constant and, hence, $\dot{V}(t) = 0$. This in turn implies:

$$0 = -\pi + r_t V(t) = -\pi + r_t(\eta - T) \Rightarrow r_t = r = \frac{\pi}{\eta - T}$$

c) The consumer's problem, on the other hand, is:

$$\max_{c_t} \int_0^\infty e^{-\rho t} U(c_t)$$

$$\text{s.t.} \qquad \dot{a} = ra + w - c - t\dot{N}$$

where $t = \frac{T}{L}$We can set up a present value Hamiltonian for this problem:

$$\mathcal{H}(\cdot) = u(c(t))e^{-\rho t} + \lambda(t)[ra(t) + Y - c(t)]$$

F.O.C.:

$$\begin{cases} \frac{\partial \mathcal{H}(\cdot)}{\partial c} = 0 \\ \frac{\partial \mathcal{H}(\cdot)}{\partial a} = -\dot{\lambda}(t) \\ \frac{\partial \mathcal{H}(\cdot)}{\partial \lambda(t)} = \dot{a} \end{cases} \Rightarrow \begin{cases} e^{\rho t} U'(c(t)) = \lambda(t) \\ r\lambda(t) = -\dot{\lambda}(t) \\ \dot{a} = ra + w - c - t\dot{N} \end{cases} \Rightarrow \begin{cases} \dot{c}(t) = \frac{U'(c(t))}{U''(c(t))}(\rho - r) \\ \dot{a}(t) = ra + w - c - t\dot{N} \end{cases}$$

[1]Using the fact that if $I(t) = \int_{a(\theta)}^{b(\theta)} g(x,\theta)dx$, then $\frac{\partial I(\theta)}{\partial \theta} = b'(\theta)g(b(\theta),\theta) - a'(\theta)g(a(\theta),\theta) + \int_{a(\theta)}^{b(\theta)} g_\theta(x,\theta)dx$.

Hence, the growth rate of consumption is given by:

$$g_c = \frac{\dot{c}(t)}{c(t)} = \frac{U'(c(t))}{U''(c(t))}\frac{(\rho - r)}{c(t)}$$

In equilibrium, the total amount of assets (aL) in the economy is given by:

$$aL = (\eta - T)N$$

Growth of these assets is given by:

$$\dot{aL} = (\eta - T)\dot{N}$$

Using the fact that:

$$r(t) = \frac{1}{\eta - T}\frac{1-\alpha}{\alpha}L(A\alpha^2)^{\frac{1}{1-\alpha}} = \frac{(1-\alpha)}{\alpha(\eta - T)}LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}}$$

$$X = A^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}}LN$$

$$Y = AL^{1-\alpha}N^{1-\alpha}X^{\alpha} = AL^{1-\alpha}N^{1-\alpha}A^{\frac{\alpha}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}}L^{\alpha}N^{\alpha} = LNA^{\frac{1}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}}$$

$$r(t) = \frac{\alpha(1-\alpha)}{\eta - T}\frac{Y}{N}$$

the last expression because

$$\alpha\alpha^{\frac{2\alpha}{1-\alpha}} = \frac{1}{\alpha}\alpha^{\frac{2}{1-\alpha}}.$$

From

$$\dot{a} = ra + w - c - t\dot{N}$$

we get

$$\dot{aL} = raL + wL - C - T\dot{N}$$

However,

$$\begin{aligned}
wL + raL &= (1-\alpha)Y + aL\frac{\alpha(1-\alpha)}{\eta - T}\frac{Y}{N} = \\
&= (1-\alpha)Y + (\eta - T)N(\frac{\alpha(1-\alpha)}{\eta - T}\frac{Y}{N}) \\
&= (1-\alpha)Y + \alpha(1-\alpha)Y \\
&= Y - \alpha^2 Y = Y - X
\end{aligned}$$

where we used $X = \alpha^2 Y$ as obtained in the book (eq. 6.12 and 6.13). Finaly we the obtain

$$\dot{aL} = Y - X - C - T\dot{N}$$

or

$$(\eta - T)\dot{N} = Y - X - C - T\dot{N}$$

giving

$$Y = \eta\dot{N} + X + C$$

where we used $\dot{aL} = (\eta - T)\dot{N}$.

With a utility function of the kind:

$$u(c) = \frac{c^{1-\theta}}{1-\theta}$$

The growth rate of consumption becomes:

$$\frac{\dot{c}(t)}{c(t)} = \frac{\dot{C}(t)}{C(t)} = \gamma = \frac{1}{\theta}(r - \rho) = \frac{1}{\theta}\left(\frac{1-\alpha}{\alpha(\eta - T)} LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}} - \rho\right)$$

The whole exercise is to find an equilibrium with a constant growth rate. In fact all variables grow at the same rate γ. Indeed, the level of consumption is obtained from

$$Y = \eta\dot{N} + X + C$$

so

$$\begin{aligned} C &= Y - \eta\dot{N} - X \\ &= Y - X - \eta\gamma N \end{aligned}$$

where we assumed that γ is also the growth rate of N. Then

$$\begin{aligned} C &= Y - X - \eta\gamma N \\ &= LNA^{\frac{1}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}} - A^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}}LN - \eta\gamma N \\ &= N\left[LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}} - A^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}}L - \eta\gamma\right] \end{aligned}$$

which compatible with the assumption that C and N grow at the same rate. Since now we know that C and N grow at the same rate γ from

$$Y = LNA^{\frac{1}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}}$$

we get that Y grows at rate γ.

Consequently, with a suitable T the growth rate of the variables can be made as high as the one obtained for the Social Planner (see page 297-298 in Economic Growth, Barro and Sala-i-Martin). However, for given N, the "static" efficiency is not reached because of a too low X and the level of Y is lower than the Y obtained by the central planner. Indeed,

$$Y_{\text{central planner}} = Y = LNA^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}$$

while here we get

$$Y = LNA^{\frac{1}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}}$$

For more discussion see Economic Growth, Barro and Sala-i-Martin, 2nd ed, page 300, for a discussion.