ECON 815
Macroeconomic Theory
Winter Term 2012/13

Assignment 3 - ANSWER KEY

1 Government and growth

The current value Hamiltonian for the representative household is

$$H = \log c_t + \mu_t \left[(1 - \xi_t) r_t a_t + w_t - c_t - \tau_t\right]$$

The necessary conditions for an optimum are

$$\begin{cases} H_c = 0 \Leftrightarrow \frac{1}{c_t} = \mu_t \quad \text{log-differentiate} \quad \frac{\dot{\mu}_t}{\mu_t} = -\frac{\dot{c}_t}{c_t} \\ H_a = -\dot{\mu}_t + \rho\mu_t \Leftrightarrow \mu_t (1 - \xi_t) r_t = -\dot{\mu}_t + \rho\mu_t \Leftrightarrow \frac{\dot{\mu}_t}{\mu_t} = \rho - (1 - \xi_t) r_t \\ \dot{a}_t = (1 - \xi_t) r_t a_t + w_t - c_t - \tau_t \end{cases} \tag{1}$$

and the transversality condition: $\lim_{t\to\infty} e^{-\rho t} k_t \mu_t = 0$. Use the first two results to arrive at:

$$\frac{\dot{c}_t}{c_t} = (1 - \xi_t) r_t - \rho$$

Firms maximize profits, and choose k to solve

$$\max_k \pi = L_t \left[k_t^\alpha \left(g_t^I\right)^\beta - r_t k_t - w_t\right]$$

which yields the FOC

$$r_t = \alpha k_t^{\alpha-1} \left(g_t^I\right)^\beta \tag{2}$$

Note that firms take g_t^I as parametric. The economy is characterised by an externality, since its production possibility set depends on public investment, which is not chosen by the firm[1].

Free entry implies $\pi = 0$ so,

$$w_t = k_t^\alpha \left(g_t^I\right)^\beta - r_t k_t = (1 - \alpha) k_t^\alpha \left(g_t^I\right)^\beta \tag{3}$$

Substituting these conditions in the dynamic equations for the representative households and imposing the general equilibrium requirement $a_t = k_t$, we find the two laws of motion for this economy:

$$\frac{\dot{c}_t}{c_t} = (1 - \xi_t) \alpha k_t^{\alpha-1} \left(g_t^I\right)^\beta - \rho \tag{4}$$

$$\dot{k}_t = (1 - \xi_t) r_t k_t + w_t - c_t - \tau_t = (1 - \xi_t \alpha) k_t^\alpha \left(g_t^I\right)^\beta - c_t - \tau_t \tag{5}$$

Also the TVC must be fulfilled: $\lim_{t\to\infty} \left[k_t e^{-\rho t} \mu_t\right] = \lim_{t\to\infty} \left[k_t e^{-\int_0^t (1-\xi_u) r_u du}\right] = 0$

Finally, notice that the aggregate resource constraint can be obtained by considering the private sector dynamic constraint together with the equilibrium conditions for the firm and the government.

Recall that each period the government runs a balanced budget, or that $\tau_t = g_t^C + g_t^I - \xi_t r_t a_t$, for given public expenditure levels $\left(g_t^C, g_t^I\right)$.

Plugging τ_t into the intertemporal budget constraint gets:

[1] It is a well known fact that in the presence of externalities, the Welfare Theorems cease to hold: as a consequence, the decentralised solution of the problem and the social planner one will lead to two different results.

$$\dot{a}_t = (1-\xi_t)\, r_t a_t + w_t - c_t - g_t^C - g_t^I + \xi_t r_t a_t = r_t a_t + w_t - c_t - g_t^C - g_t^I \tag{6}$$

Now substitute the expressions for prices, (2) and (3), into (6):

$$\dot{a}_t = \alpha k_t^{\alpha-1}\left(g_t^I\right)^{\beta} a_t + (1-\alpha)\, k_t^{\alpha}\left(g_t^I\right)^{\beta} - c_t - g_t^C - g_t^I$$

Since in equilibrium the stock of asset has to be equal to the capital stock, we impose the condition $a_t = k_t$.

$$\dot{k}_t = \alpha k_t^{\alpha}\left(g_t^I\right)^{\beta} + (1-\alpha)\, k_t^{\alpha}\left(g_t^I\right)^{\beta} - c_t - g_t^C - g_t^I = y_t - c_t - g_t^C - g_t^I$$

1. Assume that $g_t^C = g^C, g_t^I = g^I$ and $\xi_t = \xi$ are constant over time. The dynamic equations that characterize the equilibrium in this economy is given by (4) and (5)

$$\begin{cases} \dot{c}_t = c_t\left[(1-\xi)\,\alpha k_t^{\alpha-1}\left(g^I\right)^{\beta} - \rho\right] \\ \dot{k}_t = (1-\xi\alpha)\, k_t^{\alpha}\left(g^I\right)^{\beta} - c_t - \tau_t \end{cases}$$

In steady-state the variables are constant. Let an asterisk denote steady-state values. Imposing $\dot{c}_t = 0$ we get

$$k^* = \left[\frac{(1-\xi)\,\alpha\left(g^I\right)^{\beta}}{\rho}\right]^{\frac{1}{1-\alpha}} \tag{7}$$

Notice that, compared to the standard Ramsey model, the steady-state capital level is affected by both the public investment and by the distortionary part of the taxes (ξ), while the lump sum part does not have any effect.

Imposing $\dot{k}_t = 0$ and using (7) we find

$$c^* = (1-\xi\alpha)\left[\frac{\alpha(1-\xi)\left(g^I\right)^{\beta}}{\rho}\right]^{\frac{\alpha}{1-\alpha}}\left(g^I\right)^{\beta} - \tau$$

Phase diagram. First we draw the lines representing constant consumption and capital stock. When $\dot{c}_t = 0$ we have $\alpha(1-\xi)\, k_t^{\alpha-1}\left(g^I\right)^{\beta} = \rho$, i.e., a constant. In the (k, c)-plane, this is a vertical line. When $\dot{k}_t = 0$ we have $c_t = (1-\xi\alpha)\, k_t^{\alpha}\left(g^I\right)^{\beta} - \tau_t$ which is an increasing concave function[2]. Note that the intercept is $-\tau_t$.

Examine what happens off these paths. Imagine to be on the $\dot{c}_t = 0$ line and to let k increase slightly. To study what happens to c, we differentiate $\dot{c}_t$ w r t k_t. Since $\partial\dot{c}_t/\partial k_t < 0$, consumption decrease to the east of $\dot{c}_t = 0$ (see the arrow in the diagram). To the west of $\dot{c}_t = 0$ the situation is the opposite.

Then consider being on the $\dot{k}_t = 0$ line. Since $\partial\dot{k}_t/\partial c_t < 0$, capital is decreasing to the north of that line (left arrow) and increasing to the south. We see that the system is saddle-path stable.

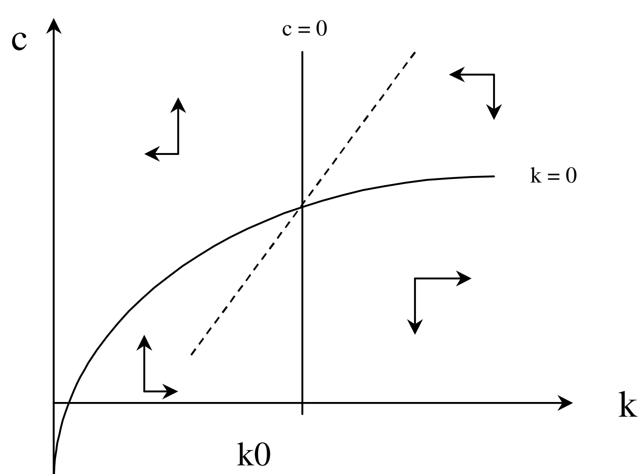

Figure 2.1.

[2] Notice that we got rid of the decreasing part of the locus by assuming $n = \delta = 0$.

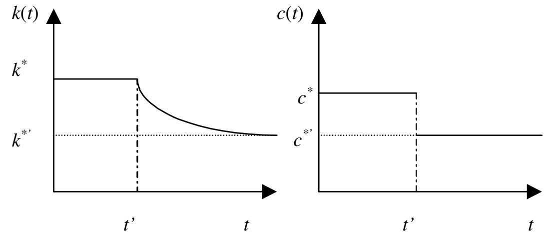

Figure 1:

2. (a) The $\dot{c}_t = 0$ locus is unaffected by the increased tax τ_t. The $\dot{k}_t = 0$ curve, however, shifts down with the increase in lump-sum taxes.

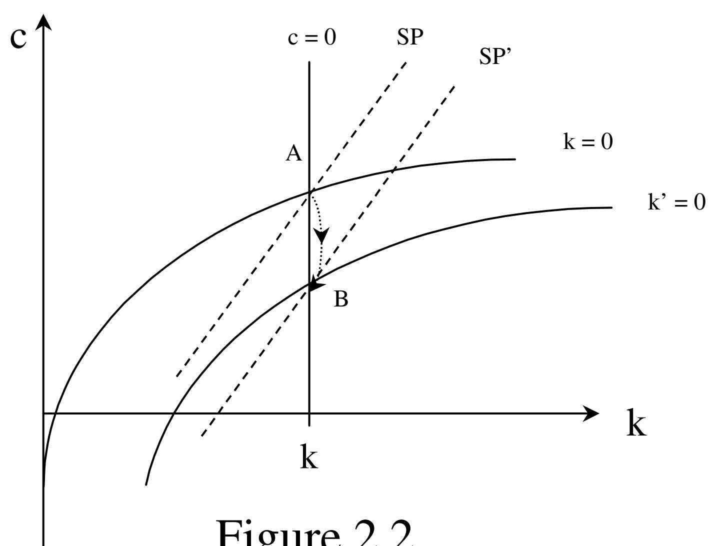

Figure 2.2.

How does the household respond? The agent makes a jump in consumption such that he reaches the saddle path of the new system. If the agent was initially in the steady-state, it means that he jumps directly to the new steady-state, with a lower consumption level.

2. (b) The government announces a reduction of g^C (tax cut of τ_t) in advance. First, consider what happens to the dynamic system. Between t_0 and t' the system looks like in part (1). At t' the lump-sum tax is reduced, which shifts the $\dot{k}_t = 0$ line upwards (and stays there forever after).

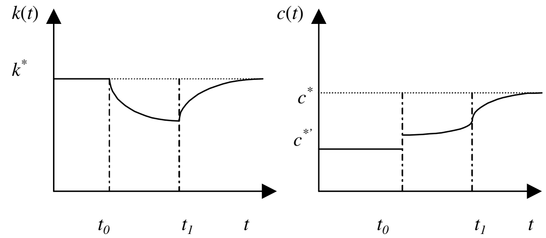

Figure 2:

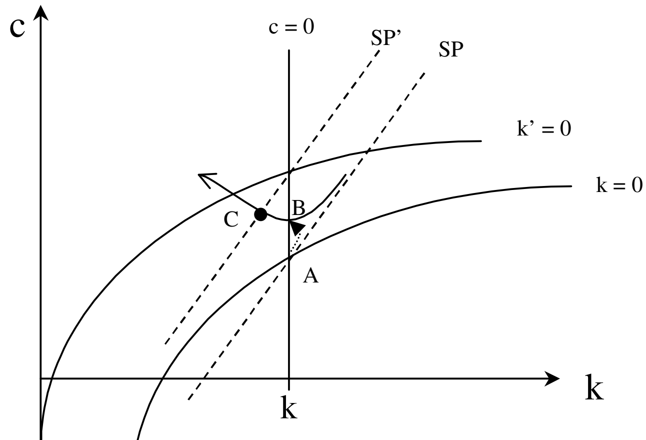

Figure 2.3.

Optimal behavior calls for action as soon as new information is revealed, hence at the time of the announcement t_0. Therefore, before t_0, agents do not change their decisions: they are in the initial steady state (A) and the potential dynamics of the system are governed by the old arrows of motion. Further, agents prefer smooth paths so that only one jump is optimal. At t_0, the optimal behaviour for the household is to jump to a new trajectory (governed by the dynamics of the old system) such that the agent will be on the saddle path of the new system exactly at time t'. Notice that such trajectory is unique. Initially, we are in the steady-state A. At time t_0 it is revealed that the dynamic system will change at a future date. We react by jumping immediately to B, which is on a trajectory that will take us to the saddle path of the new system, point C, exactly at time t'. We stay on the new saddle path and eventually reach the new steady-state.

3. Now assume that $g_t^C = \tau_t = 0$ while $\xi_t = \xi$ is constant over time. The balanced budget constraint for the government now tells us that $g_t^I = \xi r_t k_t$. Factor prices are however as before, since firms take g_t^I as parametric. Using the same procedure as before we obtain the dynamic system.

$$\begin{cases} \dot{c}_t = c_t \left[\alpha (1-\xi) k_t^{\alpha-1} \left(g_t^I\right)^{\beta} - \rho \right] \\ \dot{k}_t = (1-\xi\alpha) k_t^{\alpha} \left(g_t^I\right)^{\beta} - c_t \end{cases}$$

Now consider the level of g_t^I. We have

$$g_t^I = \xi r_t k_t = \xi \alpha k_t^{\alpha} \left(g_t^I\right)^{\beta} \rightarrow g_t^I = (\xi\alpha)^{\frac{1}{1-\beta}} k_t^{\frac{\alpha}{1-\beta}}$$

Use this in the equation for consumption growth and the law of motion for capital, and we get

$$\begin{cases} \frac{\dot{c}_t}{c_t} = \alpha(1-\xi) k_t^{\alpha-1} \left[(\xi\alpha)^{\frac{1}{1-\beta}} k_t^{\frac{\alpha}{1-\beta}} \right]^{\beta} - \rho \\ \dot{k}_t = (1-\xi\alpha) k_t^{\alpha} \left((\xi\alpha)^{\frac{1}{1-\beta}} k_t^{\frac{\alpha}{1-\beta}} \right)^{\beta} - c_t \end{cases}$$

If $\beta < 1-\alpha$, consumption growth is decreasing in k. The exponent on k in the consumption growth equation becomes $\frac{\alpha-1+\beta}{1-\beta} < 1$ as $\beta < 1-\alpha$. We end up with a phase diagram similar to Figure 2.1 but with the intercept in the origin. If however $\beta = 1-\alpha$ we have

$$\begin{cases} \frac{\dot{c}_t}{c_t} = \alpha(1-\xi) k_t^{\alpha-1} (\xi\alpha)^{\frac{1-\alpha}{\alpha}} k_t^{1-\alpha} - \rho = (1-\xi)\alpha^{1/\alpha}\xi^{\frac{1-\alpha}{\alpha}} - \rho \equiv \gamma_c \\ \dot{k}_t = (1-\xi\alpha) k_t (\xi\alpha)^{\frac{1-\alpha}{\alpha}} - c_t \end{cases}$$

The model exhibits AK-properties in this case. Consumption growth is equal to a positive constant provided that $(1-\xi)\alpha^{1/\alpha}\xi^{\frac{1-\alpha}{\alpha}} > \rho$. The reason is that when $\beta = 1-\alpha$ production in aggregate is linear in k (no diminishing returns).

Notice that we can recover analytical solutions for the two differential equations above:

$$\begin{cases} c_t = c_0 e^{\gamma_c t} \\ \dot{k}_t = (1-\xi\alpha)(\xi\alpha)^{\frac{1-\alpha}{\alpha}} k_t - c_0 e^{\gamma_c t} \end{cases}$$

Where c_0 has to be determined. Define the coefficient $(1-\xi\alpha)(\xi\alpha)^{\frac{1-\alpha}{\alpha}} \equiv \psi$. The equation for capital is a non-homogeneous first order linear differential equation, with constant coefficients. Hereafter, a solution method is described. Rewrite it as follows:

$$\dot{k}_t - \psi k_t = -c_0 e^{\gamma_c t}$$

Multiply both sides of the equation by the integrating factor $e^{-\psi t}$:

$$\left(\dot{k}_t - \psi k_t \right) e^{-\psi t} = -c_0 e^{(\gamma_c - \psi)t}$$

Notice that the left hand side is now equal to$\frac{d}{dt}\left[k_t e^{-\psi t}\right]$; integrate both side with respect to time:

$$\int \frac{d}{dt}\left[k_t e^{-\psi t}\right] dt = k_t e^{-\psi t} = \int -c_0 e^{(\gamma_c-\psi)t} dt$$

$$k_t e^{-\psi t} = -\frac{1}{(\gamma_c - \psi)} c_0 e^{(\gamma_c-\psi)t} + constant$$

$$k_t = -\frac{1}{(\gamma_c - \psi)} c_0 e^{\gamma_c t} + (constant)\, e^{\psi t}$$

We are left with two undetermined coefficients (c_0, *constant*): by exploiting the transversality condition and the initial condition for the capital stock k_0, also these values are found and the paths for (c_t, k_t) obtained.

$$\lim_{t\to\infty} \{k_t \mu_t e^{-\rho t}\} = \left\{ \left[\frac{1}{(\psi-\gamma_c)} c_0 e^{\gamma_c t} + (constant)\, e^{\psi t} \right] \frac{1}{c_0 e^{\gamma_c t}} e^{-\rho t} \right\} =$$

$$= \left\{ \left[\frac{1}{(\psi-\gamma_c)} + \frac{constant}{c_0} e^{(\psi-\gamma_c)t} \right] e^{-\rho t} \right\} = \left\{ \frac{1}{(\psi-\gamma_c)} e^{-\rho t} + \frac{constant}{c_0} e^{(\psi-\gamma_c-\rho)t} \right\} = 0$$

The first term in the last expression tends to zero as $t \to \infty$, while we have to analyse better the second one.Substituting back the definitions of ψ and γ_c gets:

$$\psi - \gamma_c - \rho = (1-\xi\alpha)(\xi\alpha)^{\frac{1-\alpha}{\alpha}} - (1-\xi)\alpha^{1/\alpha}\xi^{\frac{1-\alpha}{\alpha}}$$

$$(\xi\alpha)^{\frac{1-\alpha}{\alpha}}(1-\xi\alpha) > (\xi\alpha)^{\frac{1-\alpha}{\alpha}}(1-\xi)\alpha \leftrightarrow \alpha < 1$$

Hence, to satisfy the TVC, we must impose *constant*$= 0$. Evaluating the equation for capital at time 0, we can pin down c_0: $c_0 = k_0(\psi - \gamma_c)$. Hence the equations governing our economy are:

$$\begin{cases} c_t = k_0(\psi-\gamma_c) e^{\gamma_c t} \\ k_t = k_0 e^{\gamma_c t} \end{cases}$$

Notice that in this model we can get a closed form solution for the policy function for consumption:

$$c_t = (\psi - \gamma_c)\, k_t$$

2 A Growth Model with Leisure

a) The Hamiltonian of this problem is:

$$H = \log c - \theta \frac{l^{1+\eta}}{1+\eta} + \mu \left[k^{\alpha} l^{1-\alpha} - c - \delta k\right]$$

The optimality conditions are:

$$\begin{aligned}
H_c &= 0 \Rightarrow c^{-1} = \mu \quad \text{log-differentiate } \frac{\dot{\mu}}{\mu} = -\frac{\dot{c}}{c} && (8)\\
H_l &= 0 \Rightarrow \theta l^{\eta} = \mu (1-\alpha) k^{\alpha} l^{-\alpha} && (9)\\
H_k &= -\dot{\mu} + \rho\mu \Rightarrow \mu \left[\alpha k^{\alpha-1} l^{1-\alpha} - \delta - \rho\right] = -\dot{\mu} && (10)\\
H_{\mu} &= \dot{k} \Rightarrow \dot{k} = k^{\alpha} l^{1-\alpha} - c - \delta k && (11)\\
TVC &: \lim_{t\to\infty} k\mu e^{-\rho t} = 0
\end{aligned}$$

Notice that if compared to the standard Ramsey model, we have an additional condition (9) related to the new control variable of our problem: l.

This says that the marginal disutility of labour must equal the marginal contribution of labour to production, converted in utility terms through the costate variable μ.

b) Using (8) and (9) we obtain a relation between labour, consumption and capital that must be fulfilled at an optimum:

$$\theta l^{\eta} = \frac{(1-\alpha) k^{\alpha} l^{-\alpha}}{c}. \tag{12}$$

This expression, considering variables in terms of units of labour (i.e. $\widehat{x} \equiv x/l$), can be rewritten as follows:

$$\theta l^{\eta+1} = \frac{(1-\alpha)\widehat{k}^{\alpha}}{\widehat{c}} \tag{13}$$

Using (8) into (10) we obtain:

$$\frac{\dot{c}}{c} = \alpha k^{\alpha-1} l^{1-\alpha} - \delta - \rho = \alpha \widehat{k}^{\alpha-1} - \delta - \rho$$

or that

$$\frac{\dot{\widehat{c}}}{\widehat{c}} = \alpha \widehat{k}^{\alpha-1} - \delta - \rho - \frac{\dot{l}}{l}. \tag{14}$$

Consider (11): divide both sides by k, so that

$$\frac{\dot{k}}{k} = k^{\alpha-1} l^{1-\alpha} - \frac{c}{k} - \delta = \widehat{k}^{\alpha-1} - \frac{\widehat{c}}{\widehat{k}} - \delta.$$

Hence,

$$\frac{\dot{\widehat{k}}}{\widehat{k}} = \widehat{k}^{\alpha-1} - \frac{\widehat{c}}{\widehat{k}} - \delta - \frac{\dot{l}}{l}. \tag{15}$$

To study the steady-state of the model, we need to impose the following conditions: $\dot{\widehat{k}} = \dot{\widehat{c}} = \dot{l} = 0$.

From (14) we obtain the value of capital in steady state:

$$\widehat{k}^* = \left[\frac{\alpha}{\delta+\rho}\right]^{\frac{1}{1-\alpha}}$$

and from (13) the steady state value of labour:

$$l^* = \left[\left(\frac{1-\alpha}{\theta}\right)\frac{\widehat{k}^{*\alpha}}{\widehat{c}^*}\right]^{\frac{1}{1+\eta}}.$$

Notice that allowing for endogenous labour supply does not alter the steady-state capital level. This result does not hold in general, since it relies on the separability of preferences between consumption and leisure.

c) Consider (13), take logs and differentiate w.r.t. time:

$$\ln\theta + (1+\eta)\ln l = \ln(1-\alpha) + \alpha\ln\widehat{k} - \ln\widehat{c}$$

$$(1+\eta)\frac{\dot{l}}{l} = \alpha\frac{\dot{\widehat{k}}}{\widehat{k}} - \frac{\dot{\widehat{c}}}{\widehat{c}} \rightarrow \frac{\dot{l}}{l} = \frac{\alpha}{(1+\eta)}\frac{\dot{\widehat{k}}}{\widehat{k}} - \frac{1}{(1+\eta)}\frac{\dot{\widehat{c}}}{\widehat{c}}$$

Along the optimal path, the labour supply growth rate depends positively on capital growth rate (the higher k the higher the productivity of labour) and negatively on consumption growth rate (the higher c the lower the marginal utility of an additional unit of time devoted to production). The elasticity parameter η gives a measure of the labour supply reaction to these effects.

d) The dynamic system governing our economy is as follows:

$$\begin{cases} \frac{\dot{\widehat{c}}}{\widehat{c}} = \alpha\widehat{k}^{\alpha-1} - \delta - \rho - \frac{\dot{l}}{l} \\ \frac{\dot{\widehat{k}}}{\widehat{k}} = \widehat{k}^{\alpha-1} - \frac{\widehat{c}}{\widehat{k}} - \delta - \frac{\dot{l}}{l} \\ \frac{\dot{l}}{l} = \frac{\alpha}{(1+\eta)}\frac{\dot{\widehat{k}}}{\widehat{k}} - \frac{1}{(1+\eta)}\frac{\dot{\widehat{c}}}{\widehat{c}} \end{cases}$$

Substituting the first two equations into the last one gets:

$$\frac{\dot{l}}{l} = \frac{\alpha}{(1+\eta)}\left[\widehat{k}^{\alpha-1} - \frac{\widehat{c}}{\widehat{k}} - \delta - \frac{\dot{l}}{l}\right] - \frac{1}{(1+\eta)}\left[\alpha\widehat{k}^{\alpha-1} - \delta - \rho - \frac{\dot{l}}{l}\right]$$

$$\left[\frac{1+\eta+\alpha-1}{1+\eta}\right]\frac{\dot{l}}{l} = \frac{1}{(1+\eta)}\left[\left(-\frac{\widehat{c}}{\widehat{k}} - \delta\right) + \delta + \rho\right]$$

$$\frac{\dot{l}}{l} = \frac{\alpha}{(\alpha+\eta)}\left[-\frac{\widehat{c}}{\widehat{k}} - \delta + \frac{\delta+\rho}{\alpha}\right]$$

Notice that from (15) $\frac{\widehat{c}^*}{\widehat{k}^*} = \frac{\delta+\rho}{\alpha} - \delta = \frac{\delta(1-\alpha)+\rho}{\alpha}$, so that:

$$\frac{\dot{l}}{l} = \frac{\alpha}{(\alpha+\eta)}\left[\frac{\widehat{c}^*}{\widehat{k}^*} - \frac{\widehat{c}}{\widehat{k}}\right]$$

e) Assume that $\widehat{k}_0 < \widehat{k}^*$. Then from the text we know that $\frac{\widehat{c}^*}{\widehat{k}^*} < \frac{\widehat{c}}{\widehat{k}}$ or that $\frac{\dot{l}}{l} < 0$. There are two effects determining the path of labour supply along the transition. 1) A substitution effect: $\widehat{k}_0 < \widehat{k}^*$ implies that wages are higher in steady state than along the transition, which would tend to increase the labour supply over time. 2) A wealth effect: $\widehat{c}_0 < \widehat{c}^*$ implies that the marginal utility of producing an extra unit of consumption using additional labour is lower than in the initial phase. This drives down the labour supply along the transition. Since $\frac{\dot{l}}{l} < 0$, the wealth effect dominates and poor households work more than rich ones. By inspection of (15), we can see that the growth rate of capital per unit of labour is higher than in the basic model.