ECON 815
Macroeconomic Theory
Winter Term 2012/13

Assignment 2 - ANSWER KEY

1 A Solow Model

(a) Note that $Y = K^\alpha L^{1-\alpha} = Lk^\alpha$. Using this in the fundamental growth equation $sY = \delta K + \dot{K}$ we get $\dot{K} = sLk^\alpha - \delta K$. Dividing by L yields:

$$\frac{\dot{K}}{L} = sk^\alpha - \delta k \tag{1}$$

Then, time-differentiating $K = kL$ we find

$$\dot{K} = \dot{k}L + \dot{L}k \Rightarrow \dot{k} = \frac{\dot{K}}{L} - \frac{\dot{L}}{L}k \tag{2}$$

Now, log-differentiating $L = e^{(g+n)t}$ with respect to t gives

$$\frac{d}{dt}\ln(L) = \frac{\dot{L}}{L} = (g+n) \tag{3}$$

Combining (1), (2) and (3) we find the expression we are looking for:

$$\dot{k} = sk^\alpha - (\delta + g + n)\,k \tag{4}$$

The first term on the right hand side of (4) is saving per effective unit of labor and the second term is the effective depreciation of capital.

(b) From (4) we have:

$$\gamma_k = \frac{\dot{k}}{k} = sk^{\alpha-1} - (\delta + g + n)$$

Since γ_k, s, δ, g, and n are constant in a steady-state, $k^{\alpha-1}$ must also be constant. Since $\alpha < 1$ (that is $\alpha \neq 1$) this is only possible if k is constant i.e. $\gamma_k = 0$[1].

If $\alpha = 1$, $k^{\alpha-1}$ is constant for any growth rate of k. Hence assuming constant returns to capital enables us to have a positive steady-state growth rate (a balanced growth path). In this case we obtain sustained growth in the long run, since the incentives to accumulate capital (the interest rate) do not decrease with k anymore.

The steady-state growth of K can be found from the definition $K = Lk = e^{(g+n)t}k$. Log differentiate this to get:

$$\frac{d}{dt}\ln(K) = \frac{\dot{K}}{K} = \frac{d}{dt}\left[(g+n)t + \ln(k)\right] = (g+n) + \frac{\dot{k}}{k} = g + n \Rightarrow \gamma_K = g + n.$$

The capital stock grows at a rate equal to population growth plus technology growth. Intuitively, since capital per effective worker is constant, the overall capital stock must growth fast enough to compensate for the increasing number of effective workers.

(c) The steady-state value of capital, k^*, can be derived imposing $\dot{k} = 0$ in equation (4). We then get

$$s\,(k^*)^\alpha - (\delta + g + n)\,k^* = 0 \qquad \text{which implies} \qquad k^* = \left(\frac{s}{\delta + g + n}\right)^{\frac{1}{1-\alpha}}$$

[1] More formally, we have to study how the growth rate varies with k, $\frac{d}{dk}\gamma_k(k) = (\alpha - 1)\,sk^{\alpha-2} < 0$ and impose the condition for a steady state, $\frac{d}{dt}\gamma_k = (\alpha - 1)\,sk^{\alpha-2}\dot{k} = 0 \Leftrightarrow \frac{\dot{k}}{k} = 0$.

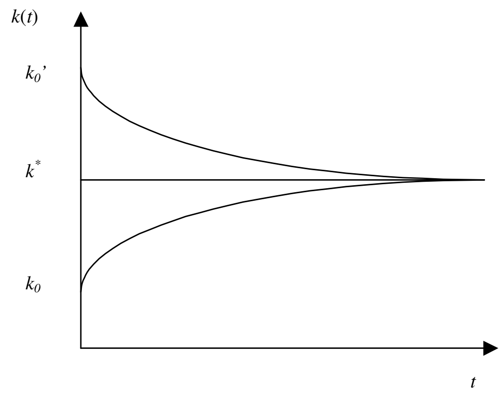


Figure 1:

(d) From now on $g = 0$. We want $\gamma_y = \frac{\dot{y}}{y}$ at a given point in time as a function of the parameters and y_t. We know that $y = k^\alpha$. Taking the log and then the time derivative we find:

$$\frac{d}{dt}\log y = \frac{\dot{y}}{y} = \gamma_y = \frac{d}{dt}\alpha \log k = \frac{\alpha \dot{k}}{k} = \alpha\gamma_k$$

We already know that $\gamma_k = sk^{\alpha-1} - (\delta + n)$. To get rid of k we simply substitute from the production function $k = y^{1/\alpha}$ to arrive at: $\gamma_k = sy^{\frac{\alpha-1}{\alpha}} - (\delta + n)$. Therefore,

$$\gamma_y = \alpha\gamma_k = \alpha\left[sy^{\frac{\alpha-1}{\alpha}} - (\delta + n)\right]$$

(e) In (a) we found the law of motion for k: $\dot{k} = sk^\alpha - (\delta + n)\,k$. An equation of this form is called a Bernoulli differential equation and can be solved by making the substitution $z = k^{1-\alpha}$. Note that $\dot{z} = (1-\alpha)k^{-\alpha}\dot{k}$, using this yields:

$$\dot{z} = (1-\alpha)s - (1-\alpha)(n+\delta)z \tag{5}$$

which is a linear first order differential equation. Note that z equal the capital-output ratio, which therefore evolves according to a linear differential equation. The general solution for (5) is:

$$z(t) = ce^{-(1-\alpha)(n+\delta)t} + \frac{s}{n+\delta}$$

where the last term is the steady-state. For $t = 0$ we obtain $c = z(0) - \frac{s}{n+\delta}$, so the exact solution is

$$z(t) = \left[z(0) - \frac{s}{n+\delta}\right]e^{-(1-\alpha)(n+\delta)t} + \frac{s}{n+\delta}$$

In terms of $k(t)$ we have:

$$k(t) = \left\{\left[k(0)^{1-\alpha} - \frac{s}{n+\delta}\right]e^{-(1-\alpha)(n+\delta)t} + \frac{s}{n+\delta}\right\}^{\frac{1}{1-\alpha}}$$

Since $(1-\alpha)(n+\delta) > 0$, it is immediate to see that as $t \to \infty$, $k(t) \to k^* = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}$ so the solution is stable. Note also that the capital-output ratio z is a weighted average of its initial value and its steady-state value, where the weights are an exponential function of time which assign an increasing (decreasing) importance to $k^*(k(0))$ to as time goes by.

(f) Note that $\frac{\dot{k}}{k^*-k}$ is a proper definition of "speed", since it is the ratio of "space" (the relative gap covered in the interval Δt, as it goes to 0) to time. To derive it in an easy way, start from the law of motion for k, $\dot{k} = sk^\alpha - (\delta + n)\, k$, and take a linear approximation around the steady state:

$$\dot{k} \simeq \left[s\alpha \left(k^*\right)^{\alpha-1} - (\delta+n) \right] (k - k^*) = \left[s\alpha \frac{(\delta+n)}{s} - (\delta+n) \right] (k - k^*) = (1-\alpha)(n+\delta)\,(k^* - k)$$

therefore,

$$\text{Speed of Convergence} = \frac{\dot{k}}{k^* - k} = (1-\alpha)(n+\delta)$$

this is the fraction of the original distance from the steady-state covered by the converging economy in an instant of time. It is easy to show that the speed of convergence for output is the same.

The speed of convergence, in general, depends on s, n, δ, α. From our linear approximation we found:

$$\frac{\dot{k}}{k^* - k} \simeq - \left[s\alpha \left(k^*\right)^{\alpha-1} - (\delta+n) \right] \qquad \text{where} \quad k^* = \left(\frac{s}{n+\delta} \right)^{\frac{1}{1-\alpha}} \tag{6}$$

the effect of each parameter is the following:

1. s affects the speed of convergence directly because a higher saving rate speeds up capital accumulation, but also indirectly through k^* because a larger saving rate increases the steady-state capital stock which in turn decreases the marginal product of capital and slows down convergence. With Cobb-Douglas technology these two effects offset exactly each other.

2. $(\delta + n)$ affects the speed of convergence because a larger $(\delta + n)$ reduces the steady-state capital stock which in turn increases the marginal product of capital $\alpha \left(k^*\right)^{\alpha-1}$ and speeds up accumulation.

3. α affects negatively the speed of convergence because it affects the marginal productivity in the steady-state. The larger is α, the larger is k^* and the lower is marginal productivity of capital.

(g) In general, an increase in k has two effects on the savings rate:

1. a substitution effect: $k \uparrow \Rightarrow f'(k) \downarrow \Rightarrow$ there is a smaller incentive to save. This intertemporal-substitution effect tends to lower the saving rate as the economy develops.

2. an income effect: $k \uparrow \Rightarrow$ the gap between current and permanent income in the economy diminishes $\Rightarrow$ consumption tends to fall in relation to income and the savings rate tends to rise.

The net effect can be ambiguous.

To see the effect on the speed of convergence, recalculate the linear approximation taking into account that $s = s(k)$:

$$\frac{\dot{k}}{k^* - k} \simeq - \left[s'(k^*) \left(k^*\right)^{\alpha} + s(k^*)\alpha \left(k^*\right)^{\alpha-1} - (\delta+n) \right] \tag{7}$$

From eq (7) we immediately see that, once k^* is given, the speed of convergence increases if $s(k)$ is decreasing in k and decreases if $s(k)$ is increasing in k. If $s(k)$ is decreasing in k then richer countries will save less than poorer countries and poorer countries will catch up easier. If instead $s(k)$ is increasing in k, richer countries will save more than poorer countries and poorer countries will have a harder time catching up.

2 A simple Ramsey model

(a) The current value Hamiltonian is $H = \frac{c_t^{1-\gamma}}{1-\gamma} + \mu_t \left(Ak_t^\theta - c_t - (\delta+n)\, k_t \right)$, where μ_t is the current value costate (or shadow price of capital). The necessary conditions for an optimum are

$$H_c = 0 \Leftrightarrow c_t^{-\gamma} = \mu_t \tag{8}$$

$$H_k = -\dot{\mu}_t + (\rho - n)\, \mu_t \Rightarrow \mu_t \left(\theta A k_t^{\theta-1} - \delta - n \right) = -\dot{\mu}_t + (\rho - n)\, \mu_t \Rightarrow \frac{\dot{\mu}_t}{\mu_t} = \rho + \delta - \theta A k_t^{\theta-1} \tag{9}$$

$$\dot{k}_t = Ak_t^{\theta} - c_t - (\delta + n)\, k_t \tag{10}$$

and the transversality condition, $\lim_{t\to\infty} e^{-(\rho-n)t}\mu_t k_t = \lim_{t\to\infty}\left[k_t \exp\left(-\int_0^t \left[f'(k_u) - n - \delta\right] du\right)\right] = 0.$

In (10) we have a differential equation $\dot{k}_t(k_t, c_t)$. We need also a differential equation $\dot{c}_t(k_t, c_t)$. Our task is to get rid of μ_t and $\dot{\mu}_t$ in (9). Log-differentiate (8) w.r.t. time, which yields $\frac{d}{dt}\left[-\gamma \ln c_t\right] = \frac{d}{dt}\left[\ln \mu_t\right] \Rightarrow -\gamma\frac{\dot{c}_t}{c_t} = \frac{\dot{\mu}_t}{\mu_t}$. Now, use this in (9) to obtain the differential equation $\dot{c}_t(k_t, c_t)$:

$$\dot{c}_t = \frac{c_t}{\gamma}\left(\theta A k_t^{\theta-1} - \rho - \delta\right) \tag{11}$$

We now have the following non-linear system

$$\begin{cases} \dot{k}_t = Ak_t^{\theta} - c_t - (\delta + n)\, k_t \\ \dot{c}_t = \frac{c_t}{\gamma}\left(\theta A k_t^{\theta-1} - \rho - \delta\right) \end{cases}$$

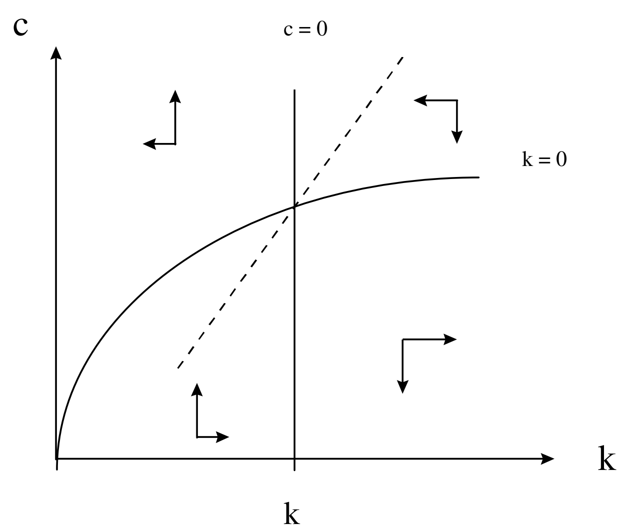
(b) In steady-state both $\dot{k} = 0$ and $\dot{c} = 0$. Imposing $\dot{k}_t = 0$ in (10) yields:

$$\dot{k}_t = 0 \Rightarrow c_t = Ak_t^{\theta} - (\delta + n)\, k_t \tag{12}$$

and the $\dot{c} = 0$ equation follows from (11),

$$\dot{c}_t = 0 \Rightarrow \theta A k_t^{\theta-1} - \rho - \delta = 0 \Rightarrow k_t = \left(\frac{\theta A}{\rho + \delta}\right)^{\frac{1}{1-\theta}} \tag{13}$$

The steady-state is given by the pair (k^*, c^*) which solves (12) and (13): $k^* = \left(\frac{\theta A}{\rho+\delta}\right)^{\frac{1}{1-\theta}}$ and $c^* = A\left(\frac{\theta A}{\rho+\delta}\right)^{\frac{\theta}{1-\theta}} - (\delta + n)\left(\frac{\theta A}{\rho+\delta}\right)^{\frac{1}{1-\theta}}$. For this to be an equilibrium also the TVC has to be satisfied, which requires that $f'(k^*) > \delta + n$. From (12) we have that $f'(k^*) = \delta + \rho$, so we need $\rho > n$.



Phase diagram. First, draw the curves given by the $\dot{k}_t = 0$ and $\dot{c}_t = 0$ equations. We now want to know what happens when we are not on these lines. Consider an increase in consumption from the $\dot{k}_t = 0$ curve. Then, eq(10) tells us that $\dot{k}_t$ becomes negative ($\frac{\partial \dot{k}}{\partial c} = -1$). So, above the $\dot{k}_t = 0$ curve capital is decreasing, which is illustrated by the left arrow in the diagram. If we instead decrease consumption compared to our initial point on $\dot{k}_t = 0$ the contrary is true, i.e., capital is increasing (illustrated by the right arrow). Now consider an initial point on the curve $\dot{c}_t = 0$. If we increase k, by (11), $\dot{c}_t$ becomes negative (since $f''(k) < 0$, $\frac{\partial \dot{c}}{\partial k} < 0$). This is illustrated by the down arrows. Analogously, if we decrease k then $\dot{c}_t$ becomes positive, which is illustrated by the up-arrows. We also see that there is a saddle path.

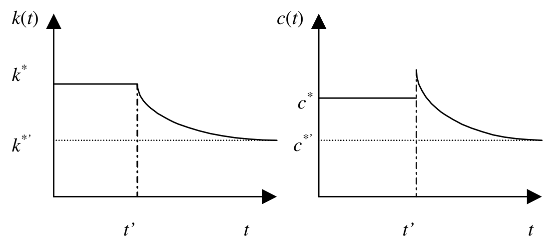
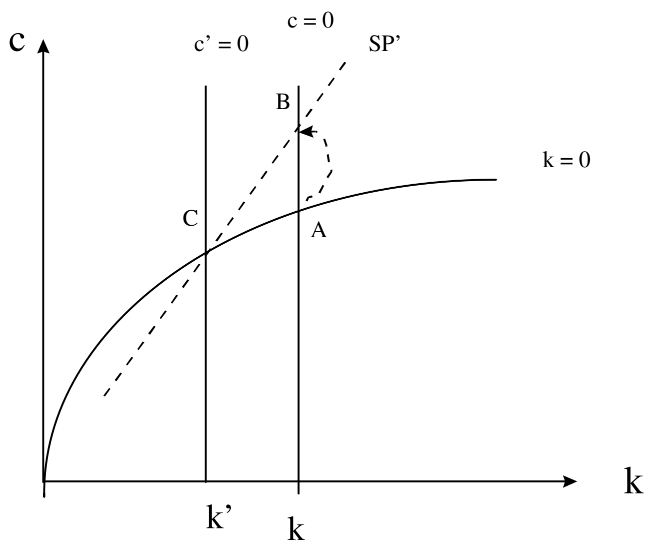


Figure 2:

(c) The discount factor ρ increases unexpectedly and permanently. First, consider the implications for the steady-state curves (12) and (13). We see that equation (12) is unaffected (there is no ρ). But eq(13) shifts to the left, so the steady-state capital stock will decrease.



How will the economy adapt? The former steady-state is now on an diverging trajectory. This is not optimal and therefore the solution is to jump to the new saddle path. We can only jump vertically and change our consumption[2]. Hence, we immediately move from point A up to the new saddle path at point B. We will now follow the saddle path towards the new steady-state, point C. The intuition is that when ρ increases we become more impatient and we prefer to consume more now even at the expense of a lower (c, k) level in the new steady-state. As we increase our consumption, investment is unable to keep the previous level of k constant. The capital stock starts decreasing and consumption will follow.

The time paths of capital and consumption are represented in the graphs above.

(d) The system in (a) without population growth and depreciation reduces to:

$$\left\{ \begin{array}{l} \dot{k}_t = Ak_t^\theta - c_t \\ \dot{c}_t = \frac{c_t}{\gamma}\left(\theta A k_t^{\theta-1} - \rho\right) \end{array} \right. \quad \text{with a steady-state} \quad \begin{array}{l} k^* = \left(\frac{\theta A}{\rho}\right)^{\frac{1}{1-\theta}} \\ c^* = A\left(\frac{\theta A}{\rho}\right)^{\frac{\theta}{1-\theta}} \end{array}$$

[2]Notice that only control variables (i.e. consumption) can 'jump'. Whereas state variables (i.e. capital stock) must obey the dynamic equation which describes their evolution over time.

A first order Taylor approximation around the steady-state gives

$$\begin{bmatrix} \dot{k}_t \\ \dot{c}_t \end{bmatrix} = \begin{bmatrix} \theta A k^{*\theta-1} & -1 \\ \frac{c^*}{\gamma}\theta(\theta-1) A k^{*\theta-2} & 0 \end{bmatrix} \begin{bmatrix} k_t - k^* \\ c_t - c^* \end{bmatrix}$$

Using the steady-state values yields:

$$\begin{bmatrix} \dot{k}_t \\ \dot{c}_t \end{bmatrix} = \begin{bmatrix} \rho & -1 \\ \frac{\rho^2}{\gamma}\frac{(\theta-1)}{\theta} & 0 \end{bmatrix} \begin{bmatrix} k_t - k^* \\ c_t - c^* \end{bmatrix} \tag{14}$$

To solve this system, we need the eigenvalues λ of the matrix of coefficients. These can be found imposing:

$$\det \begin{bmatrix} \rho - \lambda & -1 \\ \frac{\rho^2}{\gamma}\frac{(\theta-1)}{\theta} & -\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 - \rho\lambda + \frac{\rho^2}{\gamma}\frac{(\theta-1)}{\theta} = 0$$

this second order polynomial has two roots:

$$\lambda_{1,2} = \frac{\rho}{2} \pm \frac{\rho}{2}\sqrt{1 + \frac{4}{\gamma}\frac{(1-\theta)}{\theta}}$$

Note that $\frac{4}{\gamma}\frac{(1-\theta)}{\theta} > 0$ and therefore we have a negative eigenvalue and a positive one. Call λ_1 the negative eigenvalue.[3]

The solution for k of (14) has the following form:

$$k_t = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + k^*$$

where a_1 and a_2 are constant. Since $\lambda_2 > 0$, k_t will explode to infinity if $a_2 > 0$ and will go to zero if $a_2 < 0$ (recall that the lower bound for capital is zero, so that it cannot diverge to $-\infty$).

In the former case, the transversality condition would be violated (which says that over-accumulation of capital is not optimal). In the latter case, capital would be depleted in finite time and consumption would jump to zero; such a discrete change in consumption is ruled out by the Euler equation.[4]

Therefore, to meet all the optimality conditions, it must be the case that $a_2 = 0$. We still have an unknown constant, a_1. To pin it down, we use the initial condition for the capital stock: imposing $k_t = k_0$ for $t = 0$ we get $a_1 = k_0 - k^*$. Hence, the solution for k_t is $k_t = (k_0 - k^*)\, e^{\lambda_1 t} + k^* = e^{\lambda_1 t} k_0 + \left(1 - e^{\lambda_1 t}\right) k^*$, which can be seen as a weighted average of the initial condition and the steady state level, whose weights are changing over time, assigning more importance to k^* as time goes by.

How is c_t determined? Imposing $a_2 = 0$ we implicitly tied down c_t as a function of k_t. Stability requires the pair (c_t, k_t) to lie always on the stable eigenvector of the dynamic system. That is, the following relationship must hold:

$$\begin{bmatrix} \rho - \lambda_1 & -1 \\ \frac{\rho^2}{\gamma}\frac{(\theta-1)}{\theta} & -\lambda_1 \end{bmatrix} \begin{bmatrix} k_t \\ c_t \end{bmatrix} = 0 \Rightarrow c_t = (\rho - \lambda_1)\, k_t = \frac{\rho}{2}\left\{1 + \sqrt{1 + \frac{4}{\gamma}\frac{(1-\theta)}{\theta}}\right\} k_t$$

this positive relationship is the linearized saddle path of the system. Picking this particular c_t corresponds to choosing to be on the saddle path in the phase diagram analysis.

Note also that the stable eigenvalues gives the speed of convergence. Indeed, taking the time derivative of the solution for k_t and rearranging it we get:

$$\text{Speed of Convergence} = \frac{\dot{k}_t}{k^* - k_t} = -\lambda_1$$

[3] An equivalent, but less rigorous, method for 2x2 systems is to study the trace and the determinant of the matrix of coefficients, the trace being equal to the sum of the eigenvalues, the determinant to their product. If the determinant is negative, we know that one eigenvalue must be positive, while the other one must be negative. These conditions ensure saddle path stability.

[4] For a more detailed exposition see Blanchard/Fischer (1989), *Lectures on Macroeconomics*, Appendix A, Ch.2.