**ECON 815**
**Macroeconomic Theory**
Winter Term 2012/13

*Assignment 1*
*Due: In Class (DH 213) on* ***February 5th*** *2013*
**No late submissions will be accepted**
**No group submissions will be accepted**
**No electronic submissions will be accepted**

*Remarks: Write clearly and concisely. Devote some time to give the graphs, plots and tables a format easy to understand. Also the way you present your answers matters for the final grade. Even if a question is mainly analytical,* ***briefly*** *explain what you are doing, stressing the economic meaning of the various steps. Being able to convey your thoughts effectively is an asset also in real life.*

**Question 1: Differential Equations (25 Marks)**

Find the solutions of the following differential equations:

a. $\dot{y}(t) = y(t)$.

b. $\dot{y}(t) = 3t^2 y(t) + 5t^2$.

c. $\dot{y}(t) = -\frac{y(t)}{t} + \frac{[y(t)]^2}{t}$.

d. $\ddot{y}(t) = \dot{y}(t)$, knowing that $y(0) = 1$ and $\dot{y}(0) = 2$.

**Question 2: Systems of Differential Equations (40 Marks)**

a. Find the solution of the following system of differential equations, study its stability and provide a graphical representation of the phase diagram:

$$\begin{cases} \dot{y}_1(t) = y_1(t) + y_2(t) \\ \dot{y}_2(t) = -3y_2(t) \end{cases}$$

b. Determine the equilibrium points of the following non-linear system of differential equations, study its stability and provide a graphical representation of the phase diagram:

$$\begin{cases} \dot{y}_1(t) = [y_1(t)]^2 + y_2(t) \\ \dot{y}_2(t) = y_1(t) - [y_2(t)]^3 \end{cases}$$

**Question 3: Optimal Control - Exploitation of an Exhaustible Resource (35 Marks)**

Let $s(0) = s_0 > 0$ be the initial stock of a depetable resource. The utility to society of consuming the resource at the non-negative rate $c(t)$ at time $t \in [0, T]$ is $U(c(t))$, where $U(.)$ is twice continuously differentiable, increasing and strictly concave.

Social welfare is $V = \int_0^T U(c(t)) e^{-rt} dt$, where $r > 0$ is the social discount rate.

The stock of the resource evolves according to $\dot{s}(t) = -c(t)$, provided that a feasibility constraint on the control is satisfied, i.e. $0 \leq c(t) \leq \overline{c}$ for all $t \in [0, T]$. $\overline{c} > 0$ represents the maximum extraction of the resource such that the stock $s(t)$ remains non-negative.

a. Formulate the social planner's dynamic welfare maximization problem as an optimal control problem (hint: Introduce a constraint on the state endpoint $s(T)$ to take care of the feasibility constraint on $c(t)$).

b. Using the Maximum Principle, provide necessary optimality conditions that need to hold on an optimal state-control path $(s^*(t), c^*(t))$.

c. Let $\eta = -c\frac{U_{cc}(c)}{U_c(c)} > 0$ be the elasticity of the marginal utility of consumption. Re-express the optimality conditions in b to characterize the optimal growth rate of consumption $\frac{\dot{c}(t)}{c(t)}$. Discuss how $\eta$ affects the dynamic behavior of $c(t)$ and the economic meaning of the formula you have found.

d. Find the welfare maximizing control $c^*(t)$ when $U(c(t)) = \ln(c(t))$. Compute the corresponding optimal state path $s^*(t)$.

e. Re-do parts a-e when $T \to \infty$. (For each subpoint it is enough to discuss the key changes).