ECON 815 MIDTERM 2012/13

$\boxed{Q.1}$:

a) $Y(t) = AK(t)$ $\qquad y = Ak$

$$\dot{K} = sY - \delta K = sY \rightarrow \dot{k} = sy - \frac{\dot{L}}{L}k \qquad \frac{\dot{L}}{L} = n(k) = \frac{\bar{n}}{k^2}$$

$$\dot{k} = sy - \frac{\bar{n}}{k^2} \cdot k = sy - \frac{\bar{n}}{k}$$

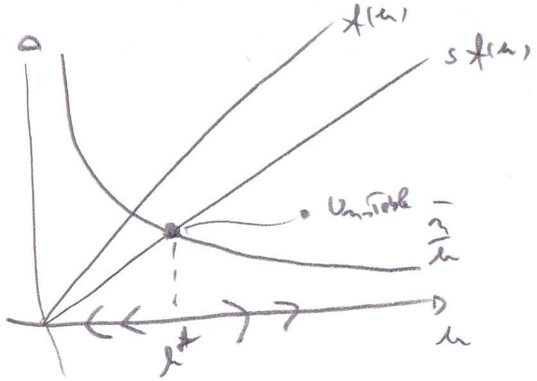
$$\dot{k} = sAk - \frac{\bar{n}}{k}$$

In SS, $\dot{k} = 0 \rightarrow sAk = \frac{\bar{n}}{k} \rightarrow k^{*2} = \left(\frac{\bar{n}}{sA}\right) \rightarrow k^* = \left(\frac{\bar{n}}{sA}\right)^{\frac{1}{2}}$

$$\frac{dk^*}{d\bar{n}} > 0$$

$$y^* = Ak^* = A\left(\frac{\bar{n}}{sA}\right)^{\frac{1}{2}}$$

b)



c) $\dot{k} = sAk - \frac{\bar{n}}{k}$ $\longrightarrow$ as if $\dot{k} = sAk - \bar{n}k^{\alpha}$ with $\alpha = -1$

$\dot{k}k^{-\alpha} = sAk^{1-\alpha} - \bar{n}$ $\longrightarrow$ let $z = k^{1-\alpha}$ then $\dot{z} = (1-\alpha)\dot{k}k^{-\alpha}$ or $\frac{\dot{z}}{1-\alpha} = sAz - \bar{n}$

$$\dot{z} = \underbrace{(1-\alpha)sA}_{\beta_0}z - \underbrace{(1-\alpha)\bar{n}}_{\beta_1} = 2sAz - 2\bar{n}$$

$z(t) = Ce^{2sAt} + \frac{\bar{n}}{sA}$, $t=0$ $z(0) = C + \frac{\bar{n}}{sA}$ $\longrightarrow$ $C = z(0) - \frac{\bar{n}}{sA}$

$$z(t) = \left[z(0) - \frac{\bar{n}}{sA}\right]e^{2sAt} + \frac{\bar{n}}{sA} = z(0)e^{2sAt} - \frac{\bar{n}}{sA}\left[e^{2sAt} - 1\right]$$

$$k^2(t) = \left[k^2(0) - \frac{\bar{n}}{sA}\right]e^{2sAt} - \frac{\bar{n}}{sA} = k^2(0)e^{2sAt} - \frac{\bar{n}}{sA}\left[e^{2sAt} - 1\right]$$

$$k(t) = \left\{k^2(0)e^{2sAt} - \frac{\bar{n}}{sA}\left[e^{2sAt} - 1\right]\right\}^{\frac{1}{2}}$$

if $k(0) = k^* = \left(\frac{\bar{n}}{sA}\right)^{\frac{1}{2}}$ $\longrightarrow$ $k(t) = k^* = \left\{\cancel{\left(\frac{\bar{n}}{sA}\right)e^{2sAt}} - \cancel{\frac{\bar{n}}{sA}e^{2sAt}} + \frac{\bar{n}}{sA}\right\}^{\frac{1}{2}}$

$k_0 > k^*$ $\quad \lim_{t\to\infty} k(t) = \underbrace{\left[k_0^2 - \frac{\bar{n}}{sA}\right]}_{>0}e^{2sAt} - \frac{\bar{n}}{sA} = +\infty$

$k_0 < k^*$ $\quad \lim_{t\to\infty} k(t) = 0$

d) Effective depreciation in our case is $\frac{\bar{n}}{k}$, if $k_0 > k^*$ growth forever. If $k_0 < k^*$ the economy shrinks for some time until it reaches $k(t) = 0$.

Q.2

① $\frac{\dot{c}}{c} = r(t) - \rho$

② $\max_{h,u} \frac{\Pi}{L} = u A h^{\alpha} - (r + u^{\delta}) h - w$

FOC: "h" $\quad \alpha A u h^{\alpha-1} = r + u^{\delta}$ [1]

"u" $\quad A h^{\alpha} = \delta u^{\delta-1} \cdot h \longrightarrow A h^{\alpha-1} = \delta u^{\delta-1} \longrightarrow u = \left[\frac{A h^{\alpha-1}}{\delta}\right]^{\frac{1}{\delta-1}}$ [2]

$$H = \begin{bmatrix} \underbrace{A\alpha(\alpha-1) u h^{\alpha-2}}_{<0} & A\alpha h^{\alpha-1} - \delta u^{\delta-1} \\ A\alpha h^{\alpha-1} - \delta u^{\delta-1} & \underbrace{-\delta(\delta-1) h}_{<0} \end{bmatrix} \longrightarrow |H| = >0 - (>0)$$

From [1]/[2] $\quad \frac{\alpha \cancel{A} u \cancel{h^{\alpha-1}}}{\cancel{A h^{\alpha}}} = \frac{r + u^{\delta}}{\delta u^{\delta-1} \cdot \cancel{h}}$

$\alpha \delta u^{\delta} = r + u^{\delta} \longrightarrow r = (-1 + \alpha\delta) u^{\delta} = (\alpha\delta - 1)\left(\frac{A h^{\alpha-1}}{\delta}\right)^{\frac{\delta}{\delta-1}}$

$$\frac{\dot{c}}{c} = (\alpha\delta - 1)\left(\frac{A h^{\alpha-1}}{\delta}\right)^{\frac{\delta}{\delta-1}} - \rho$$

$\dot{e} = w + re - c \longrightarrow \dot{h} = w + rh - c = uAh^\alpha - (\cancel{r} + u^\delta)h + \cancel{rh} - c = uAh^\alpha - u^\delta h - c$

$$= \left[\frac{h^{\alpha-1}}{\delta}\right]^{\frac{1}{\delta-1}} Ah^\alpha - \left[\frac{h^{\alpha-1}}{\delta}\right]^{\frac{\delta}{\delta-1}} h - c$$

$$= \left(\frac{1}{\delta}\right)^{\frac{1}{\delta-1}} \cdot A h^{\frac{\alpha-1}{\delta-1}+\alpha} - \left(\frac{1}{\delta}\right)^{\frac{\delta}{\delta-1}} \cdot h^{\frac{(\alpha-1)\delta}{\delta-1}+1} - c$$

$$\frac{\cancel{\alpha}-1+\alpha\delta-\cancel{\alpha}}{\delta-1} = \frac{\alpha\delta-1}{\delta-1}$$

$$\frac{\alpha\delta-\cancel{\delta}+\cancel{\delta}-1}{\delta-1} = \frac{\alpha\delta-1}{\delta-1}$$

define $d \equiv \left(\frac{1}{\delta}\right)^{\frac{1}{\delta-1}}$

$$\dot{h} = (Ad - d^\delta)h^{\frac{\alpha\delta-1}{\delta-1}} - c$$

$$\log d = \frac{1}{\delta-1}[\underbrace{\log 1}_{0} - \log\delta] < 0$$

$\frac{\alpha\delta-1}{\delta-1} > 1 \quad \delta \neq 1 \quad \alpha\delta - \cancel{1} > \delta - \cancel{1} \quad \alpha\delta > \delta \quad \alpha > 1 \quad \lightning$

$f(h)$ is concave in h

$Ad - d^\delta > 0 \rightarrow Ad > d^\delta$

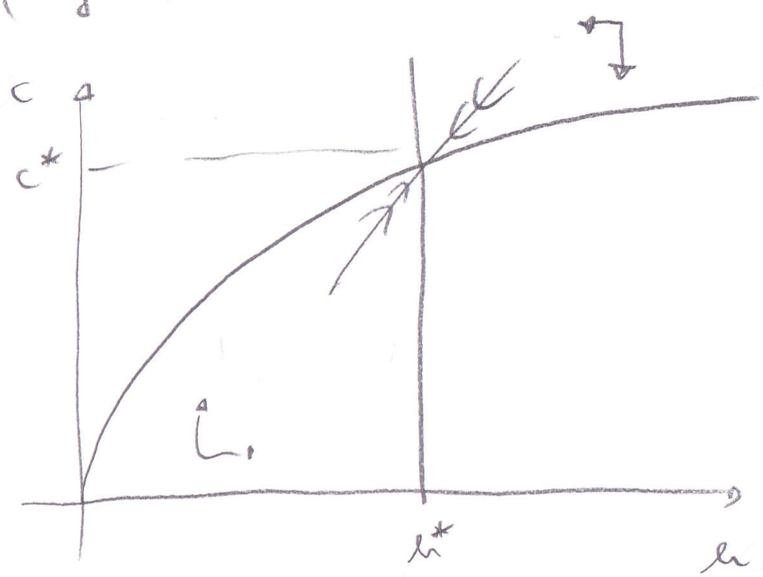
$\rightarrow A > d^{\delta-1} = \left(\frac{1}{\delta}\right)^{\frac{\delta-1}{\delta-1}} = \frac{1}{\delta}$ only if $\delta < \frac{1}{A}$

Define $(Ad - d^{\delta}) \equiv \tilde{A}$ and $\dfrac{\alpha\delta - 1}{\delta - 1} \equiv \tilde{\alpha}$ Then (c)

$$\dot{k} = \tilde{A} k^{\tilde{\alpha}} - c$$, with $\tilde{\alpha} = 1$ iff $\alpha\delta - 1 = \delta - 1 \to \alpha = 1$

$$\begin{cases} \dot{k} = \tilde{A} k^{\tilde{\alpha}} - c \\ \dfrac{\dot{c}}{c} = (\alpha\delta - 1)\left(\dfrac{A k^{\alpha-1}}{\delta}\right)^{\frac{\delta}{\delta-1}} - \rho = (\alpha\delta - 1)\, d^{\delta} \cdot A^{\frac{\delta}{\delta-1}} \cdot k^{\frac{\delta(\alpha-1)}{\delta-1}} - \rho = (\alpha\delta - 1)\, d A^{\frac{\delta}{\delta-1}} k^{\tilde{\alpha}-1} - \rho \end{cases}$$

$\to \tilde{\alpha} - 1$



$$\begin{bmatrix} d\dot{k} \\ d\dot{c} \end{bmatrix} = \begin{bmatrix} \tilde{\alpha} \tilde{A} k^{\tilde{\alpha}-1} & -1 \\ (\alpha\delta - 1) d A^{\frac{\delta}{\delta-1}} \cdot \underbrace{(\tilde{\alpha} - 1)}_{<0} k^{\tilde{\alpha}-2} & 0 \end{bmatrix} \begin{bmatrix} k - k^* \\ c - c^* \end{bmatrix}$$

$|A| = 0 - (-1) \cdot (<0) < 0 \to \lambda_1 \cdot \lambda_2 < 0$ and we have saddle path stability.