ECON 815 – MIDTERM 2011/12 Q.1

(1) Output per-capita is $y(t) = \dfrac{K^{\alpha}(t)\cdot[(1-\bar{u})\,L(t)]^{1-\alpha}}{L(t)} = (1-\bar{u})^{1-\alpha}\, k^{\alpha}(t)$

Capital per-capita evolves according to:

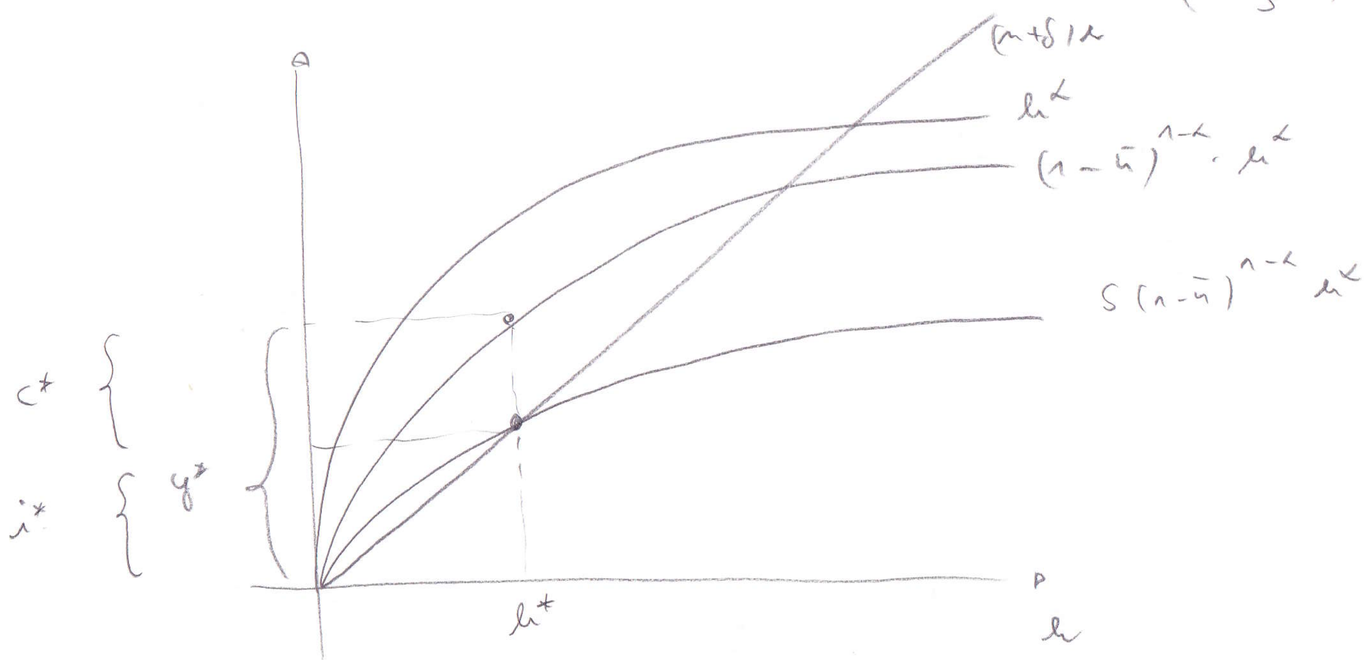
$$\dot{k}(t) = s\,y(t) - (\delta+n)\,k(t) = s\,(1-\bar{u})^{1-\alpha}\,k^{\alpha}(t) - (\delta+n)\,k(t)$$

The steady-state is when $\dot{k}(t) = 0 \rightarrow s(1-\bar{u})^{1-\alpha}(k^*)^{\alpha} = (\delta+n)\,k^*$

$$k^* = (1-\bar{u})\left(\frac{n+\delta}{s}\right)^{-\frac{1}{1-\alpha}}$$

$$y^* = (1-\bar{u})^{1-\alpha}\cdot(1-\bar{u})^{\alpha}\cdot\left(\frac{n+\delta}{s}\right)^{-\frac{\alpha}{1-\alpha}} = (1-\bar{u})\left(\frac{n+\delta}{s}\right)^{-\frac{\alpha}{1-\alpha}}$$

(2)



(3) Just like in simple Slow model, we are dealing with a Bernoulli Equation, with $z(t) = k(t)^{1-\alpha}$

$$\dot{k}(t) = s(1-\bar{u})^{1-\alpha} k(t)^{\alpha} - (\delta+n)k(t)$$

Let's consider the substitution $k(t) = z(t)^{\frac{1}{1-\alpha}}$ and $\dot{k}(t) = \frac{\dot{z}(t)}{(1-\alpha)k(t)^{-\alpha}}$

$$\frac{\dot{z}(t)}{(1-\alpha)\,k(t)^{-\alpha}} = s(1-\bar{u})^{1-\alpha} k(t)^{\alpha} - (\delta+n)k(t)$$

$$\dot{z}(t) = (1-\alpha)s(1-\bar{u})^{1-\alpha} - (1-\alpha)s(\delta+n)z(t)$$

$$z(t) = C_z e^{-(1-\alpha)s(\delta+n)} + \frac{s(1-\bar{u})^{1-\alpha}}{\delta+n} \quad \text{because} \quad z^* = \frac{s(1-\bar{u})^{1-\alpha}}{\delta+n}$$

for $t=0$ $\quad z(0) = C_z + \frac{s(1-\bar{u})^{1-\alpha}}{\delta+n}$ or $C_z = z(0) - \frac{s(1-\bar{u})^{1-\alpha}}{\delta+n}$

and

$$z(t) = \left[z(0) - \frac{s(1-\bar{u})^{1-\alpha}}{\delta+n}\right]\cdot e^{-(1-\alpha)s(\delta+n)t} + \frac{s(1-\bar{u})^{1-\alpha}}{\delta+n}$$

$$K(t) = \left\{\left[K(0)^{1-\alpha} - \frac{s(1-\bar{u})^{1-\alpha}}{\delta+n}\right]\cdot e^{-(1-\alpha)s(\delta+n)t} + \frac{s(1-\bar{u})^{1-\alpha}}{\delta+n}\right\}^{\frac{1}{1-\alpha}}$$

(4) With $\hat{k} \equiv \frac{K(t)}{L(t)\cdot[1-u(t)]}$ we get that $\frac{\dot{K}(t)}{[1-u(t)]\cdot L(t)} = s\,\frac{\hat{K}(t)[1-u(t)\cdot L(t)]^{1-\alpha}}{[1-u(t)]\cdot L(t)} - \delta\,\frac{K(t)}{[1-u(t)]\,L(t)} = s\,\hat{k}(t)^{\alpha} - \delta\hat{k}(t)$

Since $\dot{K}(t) = \dot{\hat{k}}[1-u(t)]\cdot L(t) + \hat{k}[1-\dot{u}(t)]\cdot L(t) + \hat{k}[1-u(t)]\cdot\dot{L}(t)$ we have

$$\dot{\hat{k}}(t) + \frac{[1-\dot{u}(t)]}{[1-u(t)]}\hat{k}(t) + \frac{\dot{L}(t)}{L(t)}\cdot\hat{k}(t) = s\hat{k}(t)^{\alpha} - \delta\hat{k}(t) \quad \text{or} \quad \dot{\hat{k}}(t) = s\hat{k}(t)^{\alpha} - \left(\delta - \frac{\dot{u}(t)}{1-u(t)} + \frac{\dot{L}(t)}{L(t)}\right)\hat{k}(t)$$

$-\frac{\dot{u}(t)}{1-u(t)} + \frac{\dot{L}(t)}{L(t)} = \text{Constant} \equiv K$ with $u(t) = \cos(t) + 1$, $\dot{u}(t) = -\sin(t)$ and $-\frac{\dot{u}(t)}{1-u(t)} = \frac{\sin(t)}{\cos(t)} = \operatorname{tg}(t) \rightarrow \frac{\dot{L}(t)}{L(t)} = K - \operatorname{tg}(t)$

| EGN 815 – MIDTERM 2011/2012 Q.2 |

1)
$$H(\cdot) = \log c(t) \cdot e^{-\rho t} + \lambda(t)\left[k(t)^{\alpha} - c(t) - \frac{\delta}{2} k(t)^2\right]$$

Notice that in this case we are: ① Solving for the SPP ② The definition of investment is slightly more general than usual → $I(t) = \dot{k}(t) + \delta(k(t)) \cdot k(t) = \dot{k}(t) + \frac{\delta}{2} k(t)^2$

The user cost of capital in this economy grows with the level of development, because depreciation is a convex function of the capital stock.
This leads to a lower incentive to accumulate capital compared to the case with constant depreciation.
Hence the law of motion for capital, the aggregate resource constraint is:

$$\dot{k}(t) = k^{\alpha}(t) - \frac{\delta}{2} k^2(t) - c(t)$$

The F.O.C. are:

$$\begin{cases} \dfrac{\partial H(\cdot)}{\partial c} = 0 & \dfrac{1}{c(t)} \cdot e^{-\rho t} = \lambda(t) \quad [1] \\ \dfrac{\partial H}{\partial k} = -\dot{\lambda} & \left[\alpha k(t)^{\alpha-1} - \delta k(t)\right] \cdot \lambda(t) = -\dot{\lambda}(t) \quad [2] \\ \lim\limits_{t \to \infty} \lambda(t) k(t) = 0 \end{cases}$$

From [1] and [2] we get $\dfrac{\dot{c}(t)}{c(t)} = \alpha k(t)^{\alpha-1} - \delta k(t) - \rho$

With "log" preferences $\theta = 1$, hence it does not appear in the expression for consumption growth. Moreover, the equation tells us that the social planner will optimally set a slower consumption growth with a growing capital stock, because of the added cost of depreciation, which make saving and investment more and more costly.

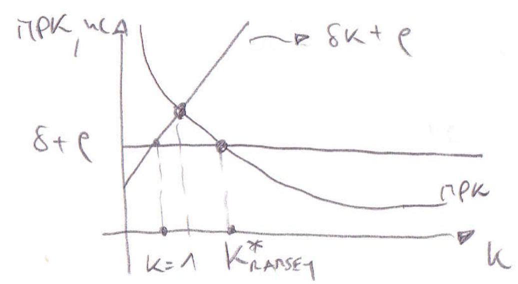
2|3) The economy is represented by the following pair of differential equations:

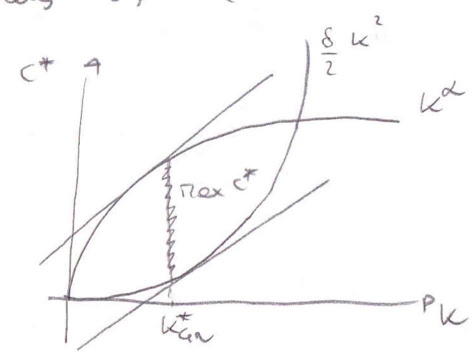
$$\begin{cases} \dfrac{\dot{c}(t)}{c(t)} = \alpha k(t)^{\alpha-1} - \delta k(t) - \rho & [3] \\ \dot{k}(t) = k(t)^{\alpha} - \dfrac{\delta}{2} k(t)^{2} - c(t) & [4] \end{cases}$$

Plus the TVC.

In order to show the phase diagram, we need to consider $\dot{c} = 0 = \dot{k}$

From [3] $\dot{c} = 0 \rightarrow \alpha k^{\alpha-1} = \delta k + \rho$, which tells us that the MPK is likely to be higher than in the benchmark case





As long as the MPK schedule intersects the $\delta k + \rho$ one for $k \geq 1$, then the capital stock in the steady state will be less than in the constant depreciation case.

If that is the case, we also know that this economy will have a $k^* < k^*_{GR} < k^*_{GR,RAMSEY}$

Notice also that the Golden rule here is different, because now it implies $k^*_{GR} = \left[\frac{\alpha}{\delta}\right]^{\frac{1}{2-\alpha}}$ or $f'(k^*_{GR}) = \delta k^*_{GR}$

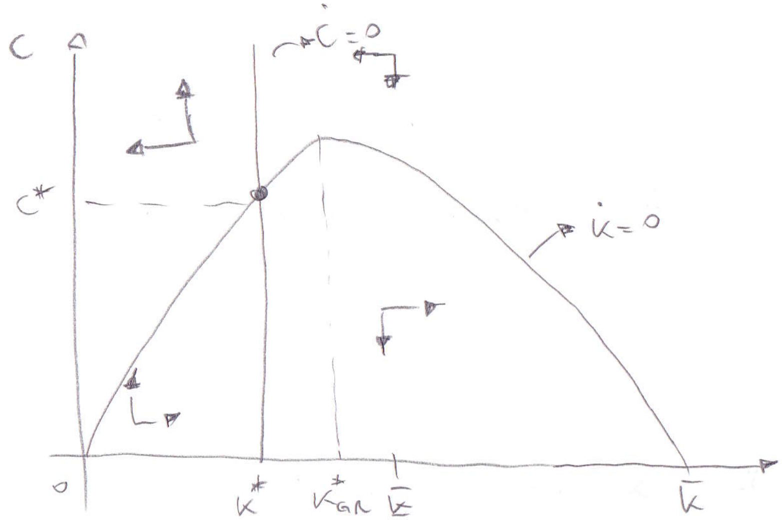
The economy will be always in a situation which is dynamically efficient.

From [4] $\dot{k} = 0 \rightarrow c = k^{\alpha} - \frac{\delta}{2} k^2$, which tells us that $c = 0$ $\begin{cases} k = 0 \\ \bar{k} \text{ such that } (\bar{k})^{\alpha} = \frac{\delta}{2}(\bar{k})^2 \rightarrow \bar{k} = \left(\frac{2}{\delta}\right)^{\frac{1}{2-\alpha}} \end{cases}$

While the maximum of the $\dot{c}=0$ locus will be such that $k = k^*_{GR}$.

Qualitatively the phase diagram is very similar to the standard model, however the $\dot{k}=0$ locus is no longer symmetric. The arrows of motion are identical to the Ramsey case.

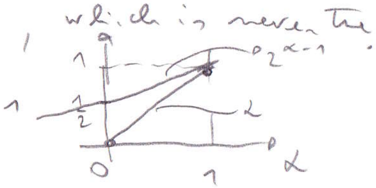
It follows that there is just one steady-state, and there might be saddle-path stability.



$\dot{k}=0$ is not symmetric because the MPK and the depreciation rate change at different rate with k.

A way to see this is that the midpoint does not coincide with k^*_{GR}. $\frac{\bar{k}}{2} = \left(\frac{1}{2^{2-\alpha}}\right)^{\frac{1}{2-\alpha}} \bar{k} = \left(\frac{2^{\alpha-1}}{\delta}\right)^{\frac{1}{2-\alpha}}$

$$k^*_{GR} > \frac{\bar{k}}{2} \quad \text{if} \quad \left[\frac{\alpha}{\delta}\right]^{\frac{1}{2-\alpha}} > \left[\frac{2^{\alpha-1}}{\delta}\right]^{\frac{1}{2-\alpha}} \longrightarrow \begin{cases} \alpha > 2^{\alpha-1}, & \text{which is never true} \\ \text{for } 0<\alpha<1 & \end{cases}$$



4) The linearized system is:

$$\begin{bmatrix} d\dot{c} \\ d\dot{k} \end{bmatrix} = \begin{bmatrix} 0 & \alpha(\alpha-1)(k^*)^{\alpha-2} - \delta \\ -1 & \underbrace{\alpha (k^*)^{\alpha-1} - \delta k^*}_{\rho} \end{bmatrix} \cdot \begin{bmatrix} c(t) - c^* \\ k(t) - k^* \end{bmatrix}$$

The term $\underbrace{\alpha(\alpha-1)(k^*)^{\alpha-2} - \delta}_{B} < 0$ and let's call it B

$$\begin{vmatrix} 0-\lambda & B \\ -1 & \rho-\lambda \end{vmatrix} = 0 \longrightarrow -\lambda(\rho-\lambda) + B = 0 \longrightarrow \lambda^2 - \rho\lambda + B = 0 \qquad \lambda_{1,2} = \frac{\rho \pm \sqrt{\rho^2 - 4B}}{2}$$

with $\lambda_1 < 0$ because $B<0$ and $\rho^2 + 4B > 0$, and $\lambda_2 > 0$ $\longrightarrow$ Saddle-path stability.