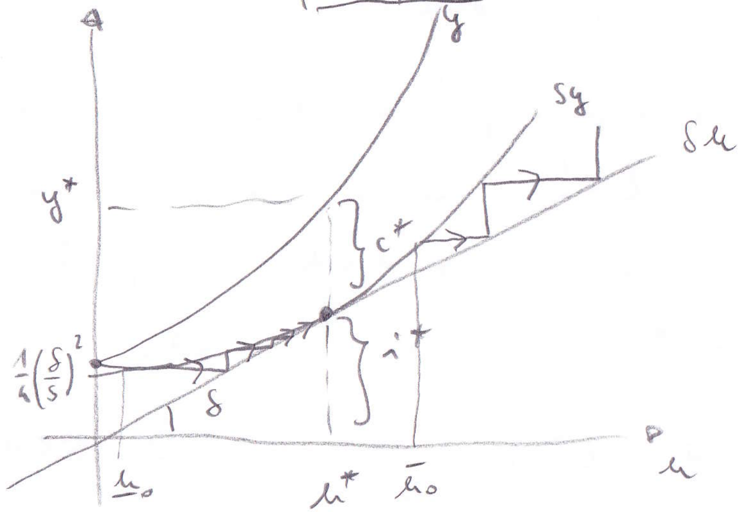
ECON815 MidTerm 2010/11

QUESTION 1

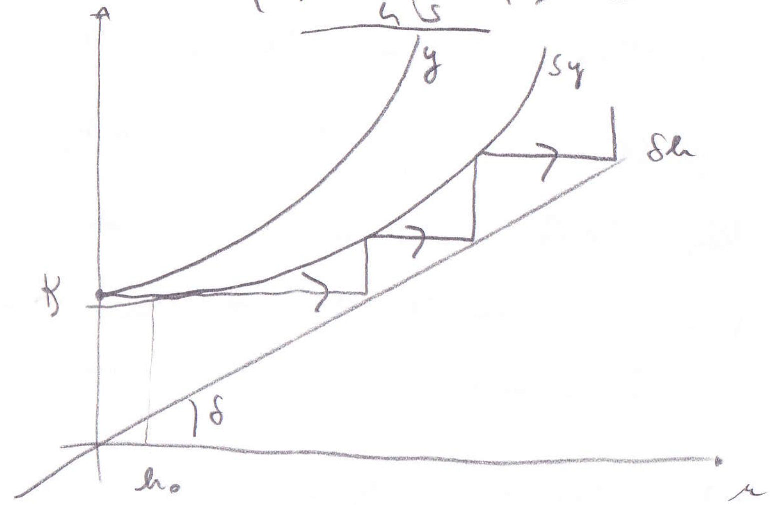
$y = k^2 + K$, $K > 0$

① $K = \frac{1}{4}\left(\frac{\delta}{s}\right)^2$ CASE 1



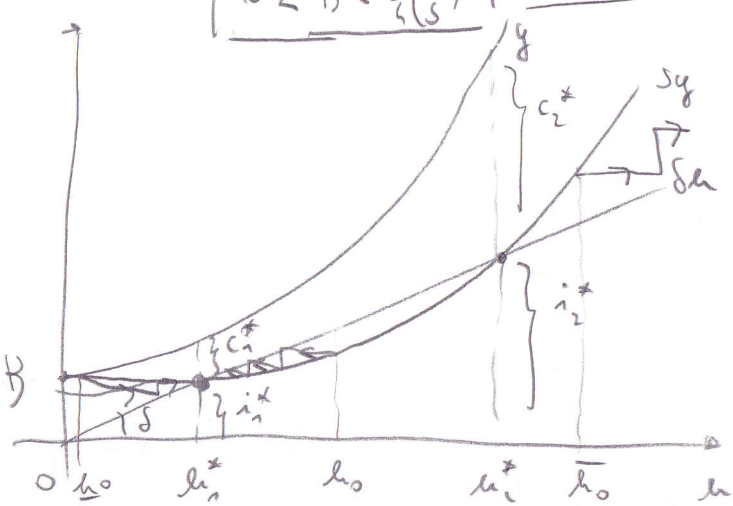
There is a steady-state, which is unique

$K > \frac{1}{4}\left(\frac{\delta}{s}\right)^2$ CASE 2



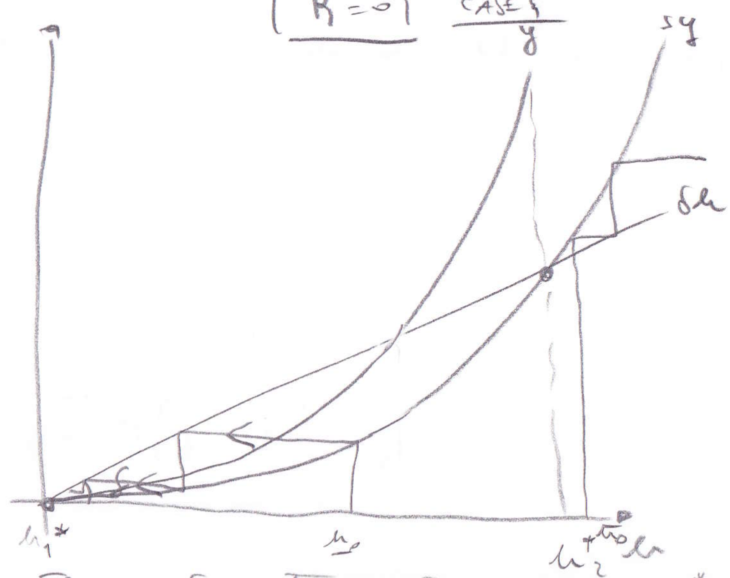
There is not steady-state

$0 < K < \frac{1}{4}\left(\frac{\delta}{s}\right)^2$ CASE 3



There are Two steady-states, both economically meaningful

$K = 0$ CASE 4



There are Two steady states, $k_1^* = 0$, $k_2^* > 0$

- In case ① there is only one steady state, which is unstable if $k_0 > k^*$, and stable if $k_0 < k^*$. Poor economies grow for a little while, then they stop growing when they reach k^*. Rich economies grow forever, with a growth rate that increases over time. This is due to the fact that the production function shows increasing returns to scale.

- In case ② there is no steady-state. Economies grow forever.

- In case ③ there are multiple steady-states. There is a poverty trap, because the low steady-state is globally stable. If you start from $k_0 < k_1^*$ or $k_0 < k_2^*$ there is going to be transitional growth (negative in the second case) and the economy will be stuck at k_1^* forever, once it gets there. Economies that start from $k_0 > k_2^*$ will grow forever instead.

- In case ④, there is only one meaningful steady state: $k_2^* > 0$. For values of $K \leq 0$ the first (global) steady-state implies either a zero or a negative k_1^*.

② Impose the s.s. condition:

$$sy^* = \delta k^*$$

$$s f(k^*) = \delta k^*$$

$$s(k^*)^2 + sK = \delta k^*$$

$$s(k^*)^2 - \delta k^* + sK = 0$$

$$k_{1,2} = \frac{\frac{\delta}{s} \pm \sqrt{\left(\frac{\delta}{s}\right)^2 - 4K}}{2} =$$

if $K = \frac{1}{4}\left(\frac{\delta}{s}\right)^2$: $\quad k^* = \frac{\delta}{2s}$

if $K = \frac{1}{8}\left(\frac{\delta}{s}\right)^2$: $\quad k_1^* = \left(1 - \sqrt{\tfrac{1}{2}}\right) \cdot \left(\frac{\delta}{2s}\right)$, $\quad k_2^* = \left(1 + \sqrt{\tfrac{1}{2}}\right)\left(\frac{\delta}{2s}\right)$

when $h^* = \frac{\delta}{2s}$ $\longrightarrow$ $y^* = \left(\frac{\delta}{2s}\right)^2 + \frac{1}{4}\left(\frac{\delta}{s}\right)^2 = 2\left(\frac{\delta}{2s}\right)^2 = \frac{1}{2}\left(\frac{\delta}{s}\right)^2$

$$c^* = (1-s)\cdot\frac{1}{2}\left(\frac{\delta}{s}\right)^2$$

$$i^* = \frac{s}{2}\left(\frac{\delta}{s}\right)^2 = \frac{\delta^2}{2s}$$

(3) See the discussion above.

(4) $K = \frac{1}{8}\left(\frac{\delta}{s}\right)^2$ corresponds to case 3.

$$y_1^* = (h_1^*)^2 + K = \left(1 - \sqrt{\tfrac{1}{2}}\right)^2\left(\frac{\delta}{2s}\right)^2 + \frac{1}{8}\cdot\left(\frac{\delta}{s}\right)^2$$

$$i_1^* = s\cdot y_1^*$$

$$c_1^* = (1-s)\, y_1^*$$

$$y_2^* = (h_2^*)^2 + K = \left(1 + \sqrt{\tfrac{1}{2}}\right)^2\left(\frac{\delta}{2s}\right)^2 + \frac{1}{8}\left(\frac{\delta}{s}\right)^2$$

$$i_2^* = s\cdot y_2^*$$

$$c_2^* = (1-s)\, y_2^*$$

QUESTION 2

There are two consumption goods, hence two control variables in this case.

$$\max_{\{c_1(t),\, c_2(t),\, a(t)\}_{t=0}^{\infty}} \int_0^{\infty} \left[\beta \log c_1(t) + (1-\beta) \log c_2(t)\right] e^{-\rho t}\, dt$$

s. To

[1] $c_1(t) \geq 0$

[2] $c_2(t) \geq 0$

$$\dot{a}(t) = r\, a(t) + w(t) - \overset{p_1(t)=1}{\downarrow} c_1(t) - \overset{p_2(t)=1}{\downarrow} c_2(t)$$

[1] & [2] are always satisfied because of the log(.) utilities.

The utility function is separable both across goods and across time. This implies that the marginal utility of consumption is not affected by past consumption decisions and by the consumption choice on the other good.

$$H(\cdot) = \left[\beta \log c_1(t) + (1-\beta) \log c_2(t)\right] e^{-\rho t} + \lambda(t)\left[r(t)a(t) + w(t) - c_1(t) - c_2(t)\right]$$

$$\begin{cases} \dfrac{\partial H}{\partial c_1} = 0 & \dfrac{\beta}{c_1(t)} \cdot e^{-\rho t} = \lambda(t) \\ \dfrac{\partial H}{\partial c_2} = 0 & \dfrac{1-\beta}{c_2(t)} \cdot e^{-\rho t} = \lambda(t) \\ \dfrac{\partial H}{\partial a} = -\dot{\lambda} & r(t)\, \lambda(t) = -\dot{\lambda}(t) \rightarrow -\dfrac{\dot{\lambda}(t)}{\lambda(t)} = r(t) \\ \text{TVC} & \lim_{t\to\infty} \lambda(t) a(t) = 0 \end{cases}$$

$$\frac{\beta}{c_1(t)} = \frac{1-\beta}{c_2(t)} \rightarrow c_1(t) = \left(\frac{\beta}{1-\beta}\right) c_2(t)$$

This condition tells us that $\gamma_{c_1} \equiv \dfrac{\dot{c}_1}{c_1} = \dfrac{\dot{c}_2}{c_2} \equiv \gamma_{c_2}$

The usual steps lead us to

$$\frac{\dot{c}_1(t)}{c_1(t)} = \frac{\dot{\lambda}(t)}{\lambda(t)} - \rho = r(t) - \rho$$

and

$$\frac{\dot{c}_2(t)}{c_2(t)} = \frac{\dot{\lambda}(t)}{\lambda(t)} - \rho = r(t) - \rho$$

so that $\gamma_{c_1} = \gamma_{c_2} = r(t) - \rho$: the usual comments go through for both consumption goods

(2) For the GE we just have to consider the first order conditions of the firm.

$$F(K, L) = K^\alpha \cdot L = \left(\frac{K}{L}\right)^\alpha L^{1+\alpha} = k^\alpha \cdot L^{1+\alpha}$$

Marginal Product of Labor $= K^\alpha > 0$ hence $L(t) = L, \forall t$.

Define the profit function as

$$\pi = \frac{1}{L} \cdot \left\{ k^\alpha \cdot L^\alpha - w - (r + \delta) k \right\}$$

$$\frac{\partial \pi}{\partial k} = 0 \qquad \alpha k^{\alpha - 1} \cdot L^\alpha = r + \delta \qquad [3]$$

From $\pi = 0$ $\qquad w(t) = k^\alpha L^\alpha - \alpha k^{\alpha-1} L^\alpha \cdot k = (1-\alpha) k^\alpha L^\alpha \qquad [4]$

Notice that $\quad MPL = K^\alpha = k^\alpha L^\alpha > (1-\alpha) k^\alpha L^\alpha = w(t)$, which confirms our result that $L(t) = L, \forall t$.

Impose the asset market equilibrium $e = k$, hence the budget constraint is going to be:

$$\dot{k}(t) = r(t) \cdot k(t) + w(t) - c_1(t) - c_2(t) \qquad [5]$$

Substituting [3] and [4] into [5] gets

$$\dot{k} = [\alpha k^{\alpha-1} L^\alpha - \delta] k + (1-\alpha) k^\alpha L^\alpha - c_1 - c_2 = k^\alpha L^\alpha - \delta k - c_1 - c_2 \qquad [6]$$

[6] is the aggregate law of motion for the economy. Notice that it depends on the two types of consumption.

We have three equations describing the evolution of the economy over time

$$\begin{cases} \dfrac{\dot{c}_1}{c_1} = \alpha k^{\alpha-1} L^\alpha - \delta - \rho \\ \dfrac{\dot{c}_2}{c_2} = \alpha k^{\alpha-1} L^\alpha - \delta - \rho \\ \dot{k} = k^\alpha L^\alpha - \delta k - c_1 - c_2 = k^\alpha L^\alpha - \delta k - c \end{cases}$$, with $c_1 + c_2 \equiv c$ being aggregate consumption.

(3) For the S.S. we impose $\dot{c}_1 = \dot{c}_2 = \dot{k} = 0$

From $\dot{c}_1 = \dot{c}_2 = 0$ we get the same solution, that is

$$(k^*)^{\alpha-1} = \frac{\rho+\delta}{\alpha L^\alpha} \rightarrow k^* = \left[\frac{\alpha L^\alpha}{\rho+\delta}\right]^{\frac{1}{1-\alpha}}$$

Consider aggregate consumption $c^* = (k^*)^\alpha \cdot L^\alpha - \delta k^* = \left[\frac{\alpha L^\alpha}{\rho+\delta}\right]^{\frac{\alpha}{1-\alpha}} \cdot L^\alpha - \delta \left[\frac{\alpha L^\alpha}{\rho+\delta}\right]^{\frac{1}{1-\alpha}}$

$$c^* = \left[\frac{\alpha L \cdot L^{\alpha-1}}{\rho+\delta}\right]^{\frac{\alpha}{1-\alpha}} L^\alpha - \delta \left[\frac{\alpha L^\alpha}{\rho+\delta}\right]^{\frac{1}{1-\alpha}} = \left(\frac{\alpha L}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}} - \delta \left(\frac{\alpha L^\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}$$

(6)

Now use $c^* = c_1^* + c_2^*$ together with $c_1^* = \left(\frac{\beta}{1-\beta}\right) c_2^* \;\to\; c^* = \left(\frac{\beta}{1-\beta}\right) c_2^* + c_2^* = \frac{c_2^*}{1-\beta}$

$$c_2^* = (1-\beta)\, c^* = (1-\beta)\, L^{\frac{\alpha}{1-\alpha}} \left[\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}} - \delta\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}\right]$$

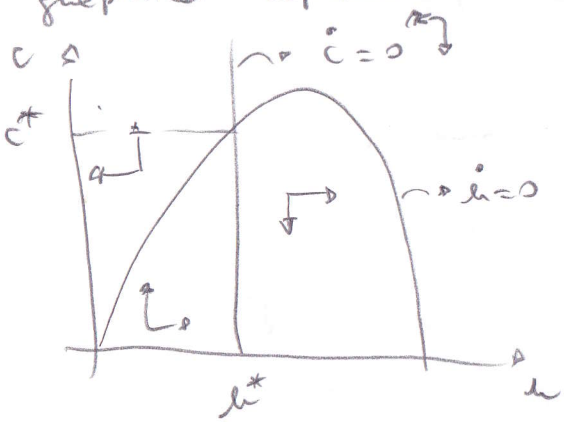
$$c_1^* = \beta\, c^* = \beta\, L^{\frac{\alpha}{1-\alpha}} \left[\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}} - \delta\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}\right]$$

(4) $c = c_1 + c_2$, hence $\dot{k} = k^\alpha L^\alpha - \delta k - c$ is the law of motion for capital

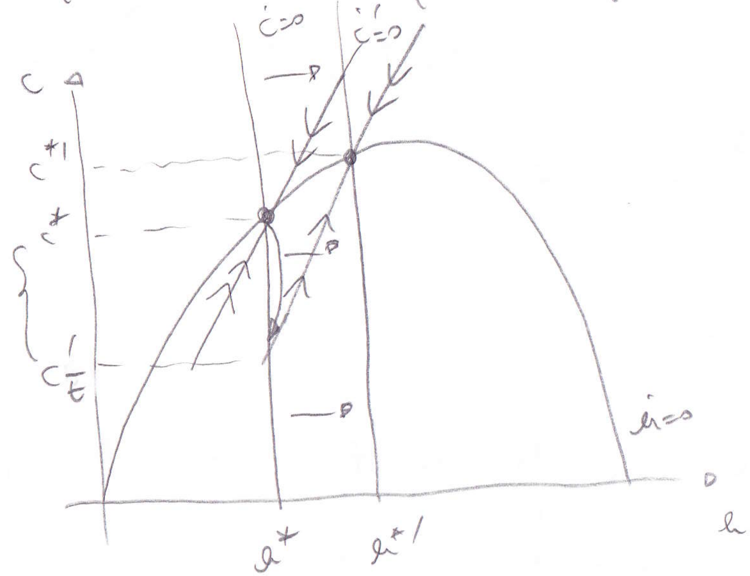
$$\dot{c} = \dot{c}_1 + \dot{c}_2 = \left[\alpha k^{\alpha-1} L^\alpha - \delta - \rho\right] \cdot c_1 + \left[\alpha k^{\alpha-1} L^\alpha - \delta - \rho\right] \cdot c_2$$

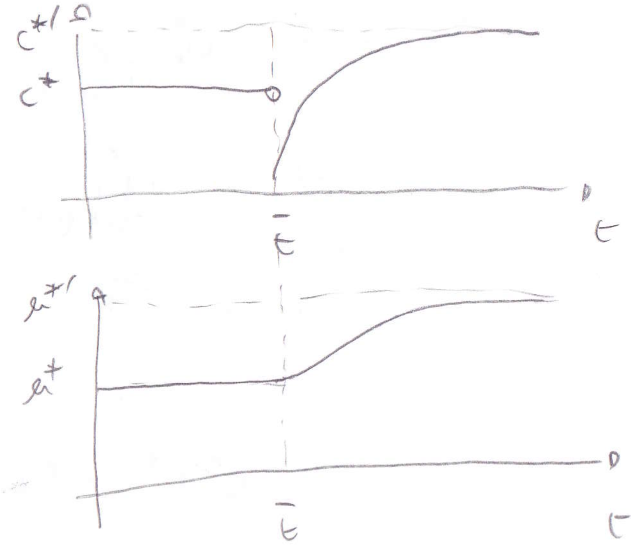
$$= \left[\alpha k^{\alpha-1} L^\alpha - \delta - \rho\right](c_1 + c_2) = \left[\alpha k^{\alpha-1} L^\alpha - \delta - \rho\right] c$$

So the graphical representation is very similar to the standard Ramsey case.



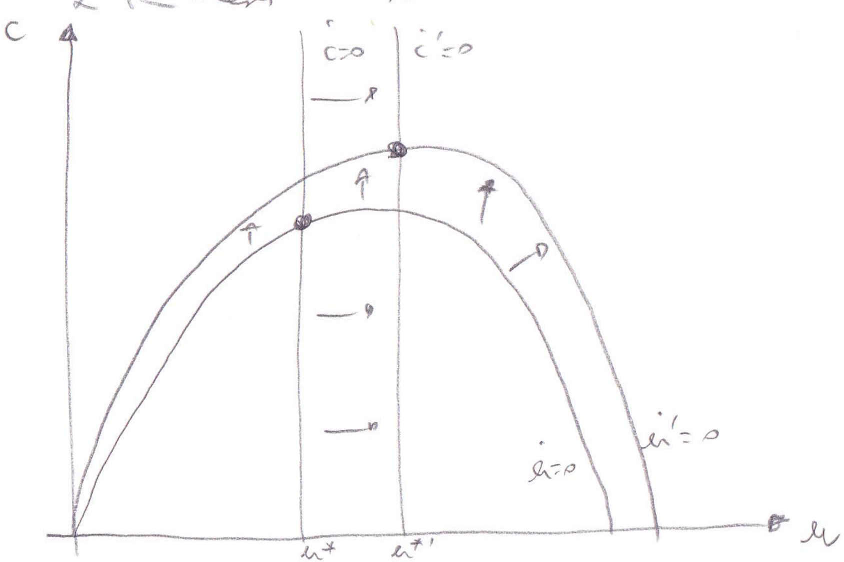
(5) ρ decreases to $\rho' \rightarrow \rho' < \rho$.





Consumption jumps as soon as the households become more patient. In this example it decreases from the initial steady-state value to the new "initial" level $c'_{\bar{t}}$ that lies on the stable branch leading to the new steady-state. Consumption rises smoothly during the transition, just like capital. The drop in consumption is needed to allow a higher investment, leading to a higher steady-state capital stock $k^{*\prime}$.

α increases to $\alpha' \rightarrow \alpha' > \alpha$



The steady-state capital $k^{*\prime}$ increases because

$$\frac{dk^*}{d\alpha} = \left(\frac{1}{1-\alpha}\right)\left[\frac{\alpha L^\alpha}{\rho+\delta}\right]^{\frac{1}{1-\alpha}-1} \cdot \left(\frac{L^\alpha + \alpha L^{\alpha-1}}{\rho+\delta}\right) > 0$$

The dynamics of consumption and capital are similar to the previous case. However, this time we don't know if consumption will jump up or down, because also production increases at $\bar{t}$. It all boils down on how steep the stable arms are.