ECON 815 MIDTERM (2010) Q. 1

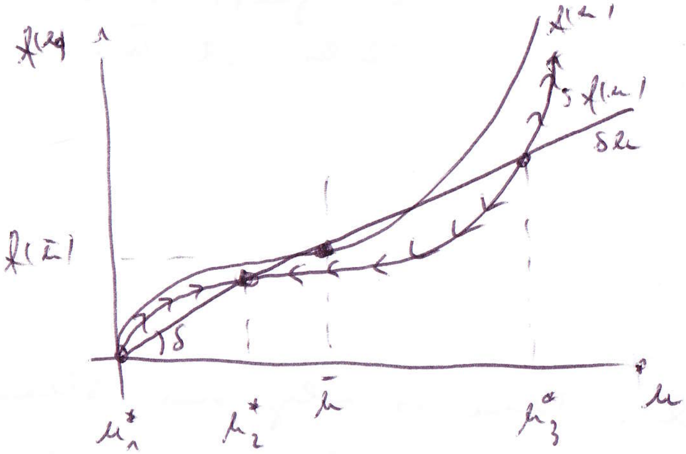
1) This is a simple Solow model, hence in order to study its law of motion and steady-states it's enough to consider the equilibrium condition

$$\underbrace{S}_{\text{savings}} = \underbrace{I}_{\text{gross investment}}, \quad \text{with} \quad S = sF(K,L) \quad \text{and} \quad I = \dot{K} + \delta K$$

In per-capita terms we get:

$$sf(k) = \dot{k} + \delta k \quad \rightarrow \quad \dot{k} = sf(k) - \delta k$$

we have a steady state k^* every time $sf(k^*) = \delta k^*$, which happens 3 times with the assumed technology.



We have a S.S. at $k_1^* = 0$, which is unstable and not economically relevant. k_2^* is another S.S., which is stable. k_3^* is the last S.S., which is unstable.

The economy possesses multiple steady-states, with a "poverty-trap" like property.

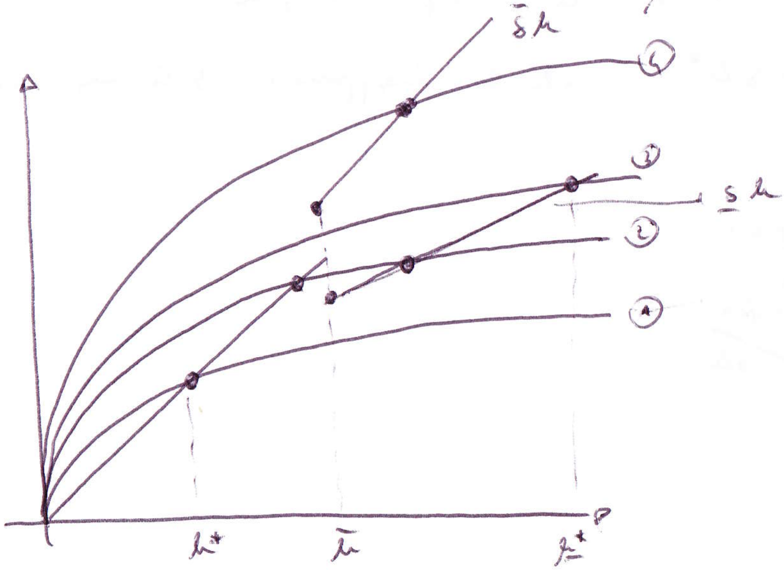
(2) $\dot{k} > 0$ every time $sf(k) > \delta k$, which happens for values of $k \le k_1^*$ and $k > k_3^*$

~~xxx~~ Yes the economy can grow forever: if the initial condition k_0 is high enough k is going to grow forever.

No, the model is not consistent with absolute convergence. It is enough for one economy to have $k_0 \ge k_3^*$ to make catching up impossible.

Poor and medium-rich economies are going to converge to k_2^*, while rich economies are going to diverge.

(3) We have to consider ~~three~~ four cases, depending on where the $sf(k)$ function lies.

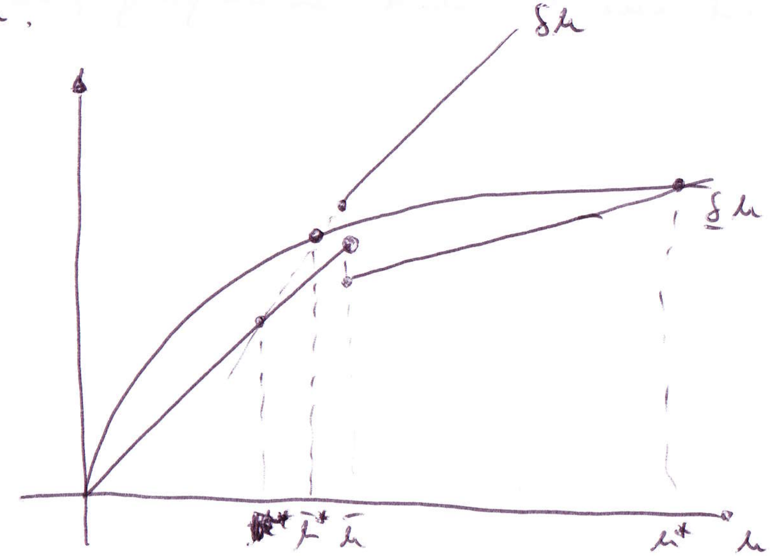


Notice that at $\bar{k}$ there is a discontinuity in the investment schedule. It jumps down or up, but there are two "holes" at $\bar{k}$.

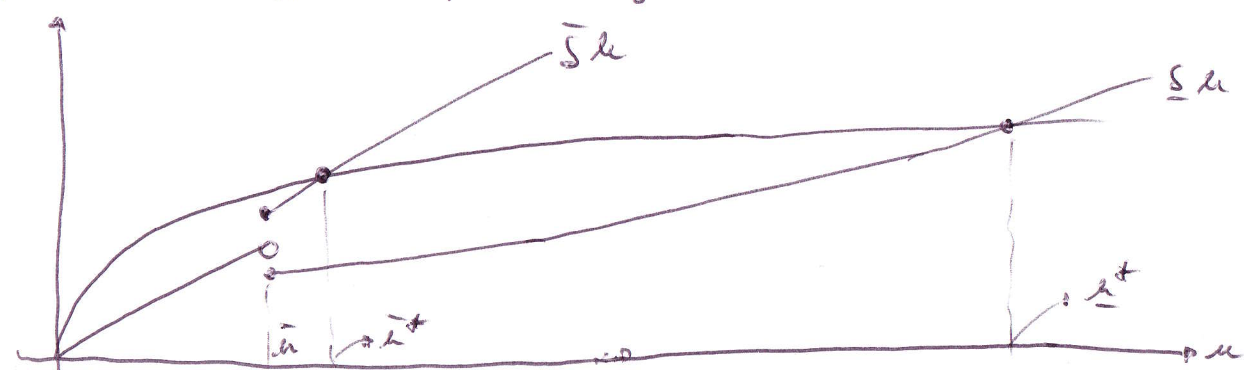
Case 1: It's just the standard model. There is only one steady state, which happens to be before $\bar{k}$. The economy will reach k^* and stop growing, so it will never experience the change in the depreciation rate.

Case 2: There are two potential steady states now, but the higher one will never happen, since the economy will be again stuck in a steady state with $k^* < \bar{k}$

Case ③ : The saving schedule "intersects" the investment line in the upper hole at $\bar{k}$. which means the following: if the economy finds itself with the low depreciation rate $\underline{\delta}$, a S.S. exists at $\underline{k}^*$. Otherwise, if the economy finds itself with $\bar{\delta}$ a SS does not exist with $k > \bar{k}$.
However, the economy will shrink back to $\bar{k}^*$ (if the process is reversible), or will be stuck at $\bar{k}$.



Case ① : There are two S.S. The economy is going to converge either to $\bar{k}^*$ or $\underline{k}^*$ depending on the next value of δ that applies after having reached $\bar{k}$



N.B.: Some students used the average $\delta = p \cdot \underline{\delta} + (1-p)\bar{\delta}$ to discuss this point. This is wrong: The economy is deterministic until $\bar{h}$ and also after $\bar{h}$ is reached in cases ③ and ⑤.

Once uncertainty is resolved, we know where the economy will actually be. Uncertainty here doesn't matter, because there are no optimizing agents (HH's or firms) hence they do not have to make contingency plans for the future evolution of the economy.

ECON 815 MIDTERM (2010) Q.2

1. $$\sigma \equiv IES \equiv - \frac{u'(c(t))}{c(t)\, u''(c(t))}$$

$$u'(c(t)) = [c(t) - \bar{c}]^{-\theta} \quad ; \quad u''(c(t)) = -\theta\,[c(t) - \bar{c}]^{-(\theta+1)}$$

$$IES = \cancel{-}\, \frac{[c(t) - \bar{c}]^{-\theta}}{c(t)\cdot \theta\,[c(t) - \bar{c}]^{-\theta-1}} = \frac{1}{\theta}\cdot\left[\frac{c(t) - \bar{c}}{c(t)}\right] = \frac{1}{\theta}\left[1 - \frac{\bar{c}}{c(t)}\right]$$

$\frac{\partial IES}{\partial \bar{c}} < 0$ Agents are less willing to substitute $c(t)$ over time. They want a smoother profile for $c(t)$.

For a given $\bar{c}$, instead, $\frac{\partial IES}{\partial c(t)} = \frac{1}{\theta}\cdot\frac{\bar{c}}{[c(t)]^2} > 0$. This means that the level of consumption matters for the smoothness of its profile over time. More precisely, for low levels of consumption people are not willing to have very "unstable" consumption paths, while they are willing to face ~~more~~ less and less smooth consumption paths, when they are enjoying a high level of consumption.

2) This is the SPP, hence the dynamic constraint is the feasibility constraint for the whole economy, that is it's the aggregate resource constraint. Moreover, the planner decides directly on the allocations, so there are no markets or prices to deal with. Firms are not optimizing anything either here.

$$\max_{\{c(t),\, I(t)\}} U(0) = \int_0^\infty \frac{[c(t) - \bar{c}]^{1-\theta} - 1}{1-\theta} \cdot L(t) \cdot e^{-\rho t} \, dt$$

s.to

- $Y(t) = C(t) + I(t) = F(K(t), L(t))$
- K_0
- $I(t) = \dot{K}(t) + \delta K(t)$
- ~~[illegible]~~

So the problem boils down to

$$\max_{\{c(t)\}} U(0) = \int_0^\infty \frac{[c(t) - \bar{c}]^{1-\theta} - 1}{1-\theta} e^{-\rho t} \, dt$$

s.to

$\dot{k}(t) = f(k(t)) - \delta k(t) - c(t)$

k_0 given

The associated Hamiltonian is: (assume $L_0 = 1$ just to simplify the notation a little).

$$J(\cdot) = \frac{[c(t) - \bar{c}]^{1-\theta} - 1}{1-\theta} e^{-\rho t} + \lambda(t) \left[f(k(t)) - \delta k(t) - c(t) \right]$$

We can take F.O.C. and consider the TVC.

$$\frac{\partial J(\cdot)}{\partial c(t)} = 0 \qquad [c(t) - \bar{c}]^{-\theta} \cdot e^{-\rho t} = \lambda(t) \qquad [1]$$

$$\frac{\partial J(\cdot)}{\partial k(t)} = -\dot{\lambda}(t) \qquad \lambda(t) \cdot [f'(k(t)) - \delta] = -\dot{\lambda}(t) \longrightarrow \lambda(t) = \lambda(0) \cdot e^{-\int_0^t [f'(k(s)) - \delta] ds} \qquad [2]$$

$$\lim_{t \to \infty} k(t) \lambda(t) = 0$$

From [1]:

$$-\theta \log [c(t) - \bar{c}] - \rho t = \log \lambda(t)$$

$$-\theta \left[\frac{1}{c(t) - \bar{c}}\right] \cdot \dot{c}(t) - \rho = \frac{\dot{\lambda}(t)}{\lambda(t)}$$

$$-\theta \cdot \frac{c(t)}{c(t) - \bar{c}} \cdot \frac{\dot{c}(t)}{c(t)} - \rho = \frac{\dot{\lambda}(t)}{\lambda(t)} \qquad [3]$$

From [2] $\lim_{t \to \infty} k(t) \cdot \lambda(0) \cdot e^{-\int_0^t [f'(k(s)) - \delta] ds}$ and $\frac{\dot{\lambda}(t)}{\lambda(t)} = -[f'(k(t)) - \delta]$ [4]

Equating [3] and [4] gets:

$$-\theta \left[\frac{c(t)}{c(t) - \bar{c}}\right] \cdot \frac{\dot{c}(t)}{c(t)} - \rho = -[f'(k(t)) - \delta]$$

or

$$\frac{\dot{c}(t)}{c(t)} = \underbrace{\left[\frac{f'(k(t)) - \delta - \rho}{\theta}\right]}_{\equiv A} \cdot \underbrace{\left[\frac{c(t) - \bar{c}}{c(t)}\right]}_{\equiv B} \qquad [5]$$

[5] is the expression for consumption growth. It consists of two parts: A and B. A is the formula we

get in the benchmark model with CEIS preferences.

Here the term B appears because of the non-homotheticity of preferences. Consumption growth not only is affected by the capital stock, but also by the level of consumption the economy is already enjoying and by the subsistence level.

Notice that if $c(t) \to \infty$, then ~~the~~ γ_c is asymptotically equal to the one in the benchmark model.

For any finite $c(t)$, consumption growth is going to be lower in this model.

To see this notice that $B = 1 - \frac{\bar{c}}{c(t)}$ can only take values between 0 and 1.

$B \to 1$ only when $c(t) \to \infty$, while $B = 0$ when $c(t) = \bar{c}$. B acts as a weight on the benchmark consumption growth, dampening it.

3) To get the steady-state, we have to impose:

$$\begin{cases} \dot{k} = 0 \\ \dot{c} = 0 \end{cases}$$

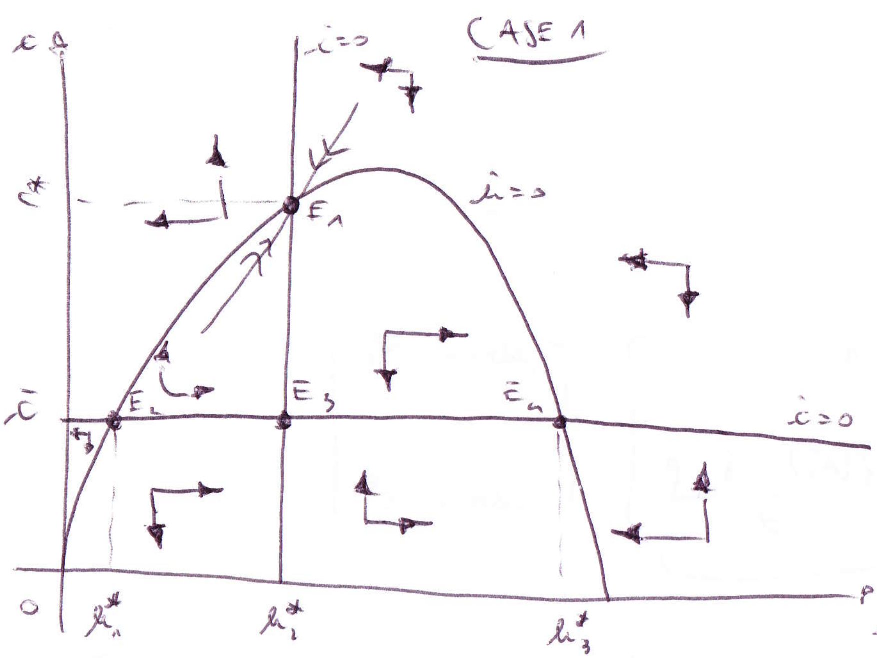
$\dot{k} = 0$ is the usual one, while $\dot{c} = 0$ now leads to:

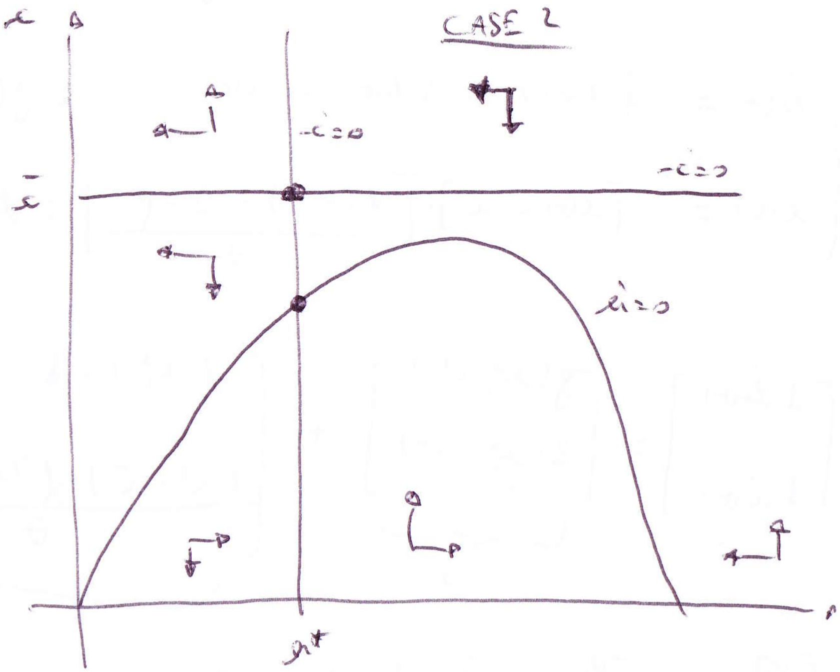
$$f'(k) - \delta - \rho = 0 \quad \&/\text{or} \quad c = \bar{c}$$

It follows that $\dot{c} > 0$ when:

$$\begin{cases} f'(k) - \delta - \rho > 0 \quad \& \quad c - \bar{c} > 0 \\ f'(k) - \delta - \rho < 0 \quad \& \quad c - \bar{c} < 0 \end{cases}$$

Graphically we have to consider two cases:





To be more precise, we cannot consider the regions in the phase diagram where $c<\bar{c}$: $\bar{c}$ is a subsistence level, so that $c(t)$ can never fall short of $\bar{c}$. In case 1 k_2^* is the usual steady-state. However, there are others that we have to consider. Potentially k_1^* and k_3^* are other S.S. with an associated level of consumption equal to $c^*=\bar{c}$. These are not equilibria, though. Every time $\theta>0$, these would imply a utility for the HH $=-\infty$. Hence a benevolent social planner would never pick them. E_2, E_3, E_4 can be ruled out, so the only admissible S.S. is E_1, with k_2^* and $c^*=f(k_2^*)-\delta k_2^*$, $[f'(k_2^*)=\delta+\rho]$

Notice that for very low initial conditions for capital $k_0\in[0,k_1^*]$ the economy is in "trouble". Only bad equilibria are feasible, or others where the Euler equation is violated. Case 2 is uninteresting, because the economy does not produce enough to satisfy the subsistence level $\bar{c}$.

4) We can linearize the system around the only economically meaningful S.S.: (k_2^*, c^*)

$$\begin{cases} \dot{k}(t) = f(k(t)) - \delta k(t) - c(t) \equiv g(k, c) \\ \dot{c}(t) = [c(t) - \bar{c}] \cdot \left[\dfrac{f'(k(t)) - \delta - \rho}{\theta} \right] \equiv h(k, c) \end{cases}$$

$$\begin{bmatrix} d\,k(t) \\ d\,c(t) \end{bmatrix} = \underbrace{\cancel{\begin{bmatrix} g(k^*, c^*) \\ h(k^*, c^*) \end{bmatrix}}}_{=0} + \underbrace{\begin{bmatrix} f'(k_2^*) - \delta & -1 \\ \dfrac{(c^* - \bar{c}) \cdot f''(k_2^*)}{\theta} & \dfrac{f'(k_2^*) - \delta - \rho}{\theta} \end{bmatrix}}_{\equiv A} \cdot \begin{bmatrix} k(t) - k_2^* \\ c(t) - c^* \end{bmatrix}$$

Following the usual steps, we can compute the eigenvalues of A, which leads us to conclude about saddle-path stability. One eigenvalue is positive, while the second is negative.