ECON 815 – Midterm 09

Q.1

1) $y_j(t) = k_j^{\alpha}(t)\,[A(t)]^{\eta}$; $0 < \alpha < 1$

$$A(t) = A_0 \frac{\sum_{j=1}^{N} k_j(t)}{N}$$

The profit for firm j are:

$$\pi_j = [y_j - (r+\delta)k_j - w]\cdot L$$

$$= [k_j^{\alpha}(t)\,[A(t)]^{\eta} - (r+\delta)k_j - w]L$$

$$\frac{\partial \pi_j}{\partial k_j} = 0 \longrightarrow \alpha[k_j(t)]^{\alpha-1}[A(t)]^{\eta}] = r+\delta$$

$$k_j(t) = \left[\frac{r+\delta}{\alpha[A(t)]^{\eta}}\right]^{\frac{1}{\alpha-1}} = \left[\frac{\alpha[A(t)]^{\eta}}{r+\delta}\right]^{\frac{1}{1-\alpha}}$$

Since α, A, r and δ are the same for every $j \to k_j = k$

$k_j = k$ means that

$$A(t) = A_0 \frac{\sum_{j=1}^{N} k}{N} = A_0 \cdot \frac{Nk}{N} = A_0 \cdot k$$

Let's impose the equilibrium condition on the goods' market:

$$S = I \qquad (\text{Savings} = \text{Investment})$$

or

$$\frac{S}{L} = \frac{I}{L}$$

$$\frac{S}{L} = \frac{s \cdot Y(K,L)}{L} = s \cdot \sum_{j=1}^{N} y_j = s \cdot \sum_{j=1}^{N} k_j^{\alpha} \cdot A^{\eta} = s \cdot N \cdot k^{\alpha} A^{\eta}$$

$$\frac{I}{L} = \frac{\dot{K} + \delta K}{L} \rightarrow \frac{\sum_{j=1}^{N} \dot{K}_j}{L} + \delta \frac{\sum_{j=1}^{N} K_j}{L} = \sum_{j=1}^{N} \frac{\dot{K}_j}{L} + \delta \sum_{j=1}^{N} k_j =$$

$$= \sum_{j=1}^{N} (\dot{k}_j + n k_j) + \delta \sum_{j=1}^{N} k_j$$

$$k_j = k, \ \forall j \rightarrow \frac{I}{L} = N \cdot [\dot{k} + nk] + \delta N k$$

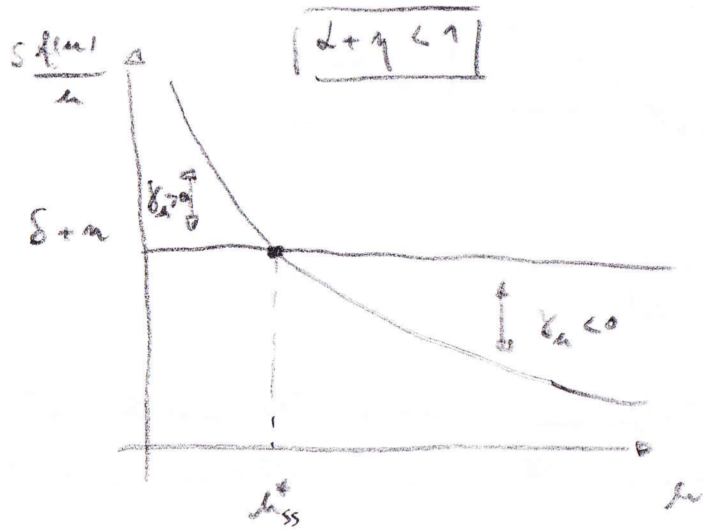
Hence $\frac{S}{L} = \frac{I}{L}$ leads to

$$s \cancel{N} \cdot k^{\alpha} A^{\eta} = \cancel{N} [\dot{k} + nk] + \delta \cancel{N} k$$

$$\dot{k} = s k^{\alpha} A^{\eta} - (\delta + n) k = s k^{\alpha+\eta} \cdot A_0^{\eta} - (\delta + n) k$$

2) $\gamma_k = \frac{\dot{k}}{k} = s k^{\alpha+\eta-1} \cdot A_0^{\eta} - (\delta + n)$

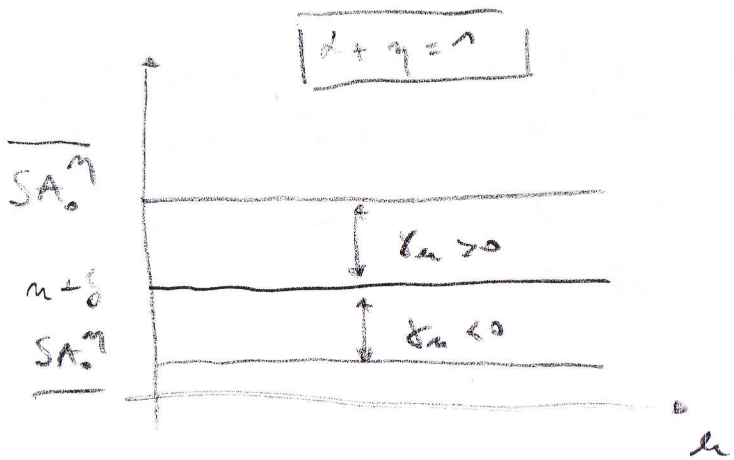
a)



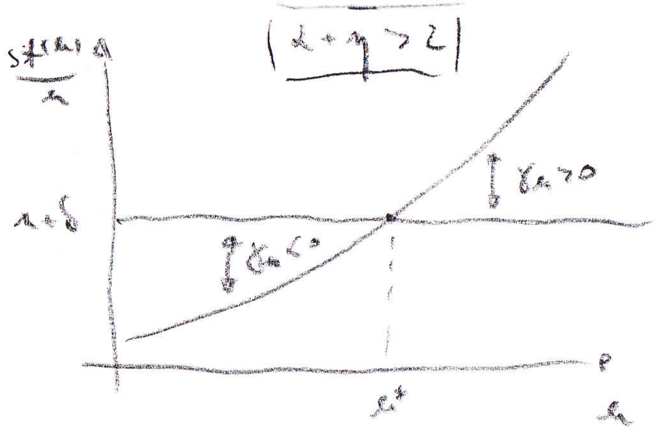
The S.S is stable. The growth rate is the distance between the term $\frac{s f(k)}{k}$ and the horizontal line given by $(n+\delta)$.

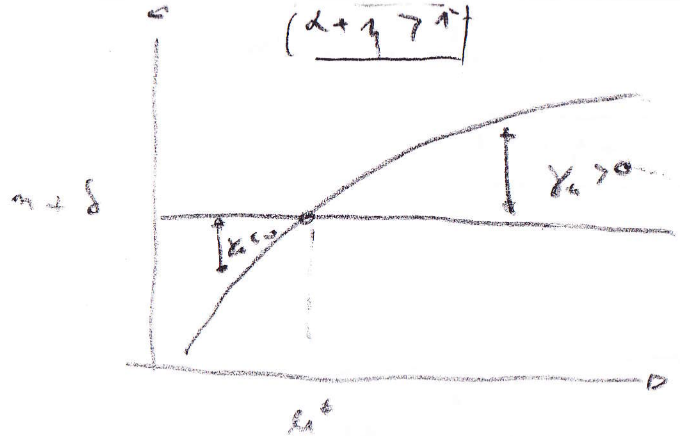
With diminishing returns $\alpha + \eta < 1$ we are back to the standard neoclassical result: the growth rate converges to zero in the long run.

b)



When $\alpha + \eta = 1$ there are constant returns to k, so we get a balanced growth path with either $\gamma_k > 0$ or $\gamma_k < 0$. The second case is uninteresting since the c)

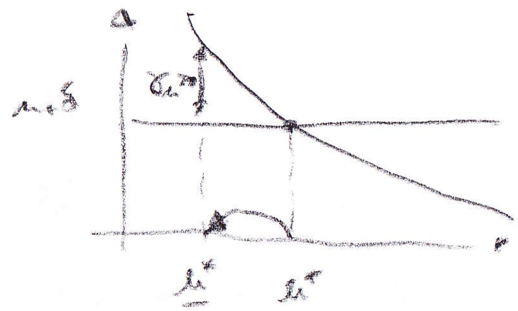




With increasing returns we get an unstable equilibrium. Hence the growth rate will either explode or converge to a zero growth / zero capital equilibrium: the final outcome depends on the initial condition k_0.

3) A higher saving rate will shift upward the curve $\frac{s f(k)}{k}$.
For the of decreasing and increasing returns the long-run growth rate will be unaffected.
However, with constant returns, a higher s implies a faster growth rate.

4) With decreasing returns we have the standard result from the neoclassical growth model. As part of the capital stock is destroyed, the growth rate accelerates until the steady state is reached again. There we have zero growth and an income level identical to the preshock values. There are no long run effects.



With increasing returns the growth rate will decrease in the periods immediately after the shock.

If the new capital stock is to the right of the unstable equilibrium, as k increases growth accelerates and tends to infinity, just like in the pre-shock situation.

If the new capital stock is to the left of the unstable equilibrium, the growth rate would start to decrease and the economy would move toward a point where the capital stock is zero.

With constant returns, $\dot{\gamma}_k$ is not affected by k. Even after the shock the economy grows at the "old" balanced growth rate.

This implies that the loss of capital and of income is permanent.

I.e. $y(t) = y(0)\, e^{\gamma t}$ before the shock while $\hat{y}(t) = \frac{\hat{y}(0)\, e^{\gamma t}}{2}$ after the shock,

since $y(t) = A_0^{\eta} \cdot k$ so that $\hat{y} = \frac{y}{2} = A_0^{\eta} \cdot \frac{k}{2}$.

$\boxed{Q.4}$

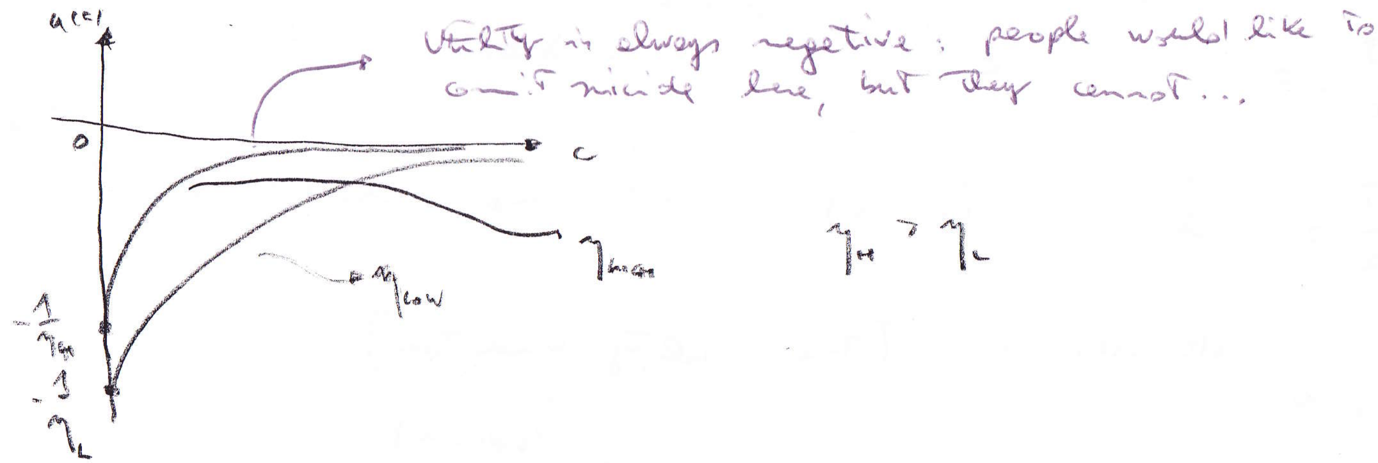
1) $u(c) = -\frac{1}{\eta} \cdot e^{-\eta c}$

$u'(c) = e^{-\eta c}$

$u''(c) = -\eta e^{-\eta c}$

• $\frac{\partial}{\partial \eta} u''(c) = e^{-\eta c} + \eta^2 e^{-\eta c} > 0$ No! $\frac{\partial}{\partial \eta} u''(c) = -e^{-\eta c} + \eta c e^{-\eta c} = e^{-\eta c}[\eta c - 1] \gtrless 0$

The higher η, the more concave the utility function and the higher the desire to smooth consumption over time.



• The IES is given by $\sigma = -\frac{u'(c)}{c \cdot u''(c)} = -\frac{e^{-\eta c}}{c \cdot (-\eta) \cdot e^{-\eta c}} = \frac{1}{c\eta}$

$\frac{\partial \sigma}{\partial c} < 0$ The higher the consumption level, the lower the IES, or agents are less willing to substitute consumption over time, that is they want a smoother profile for c.

2-3) $\max_{c(t)} U(0) = \int_0^{\infty} -\frac{1}{\eta} e^{-\eta c(t)} \cdot e^{-(\rho-n)t}\, dt$

s.To

$\dot{a}(t) = [r(t) - n]\, a(t) + w(t) - c(t)$

a_0 given

$c(t) \geq 0$

$\lim_{t\to\infty} a(t) \cdot e^{-\int_0^t [r(v)-n]\, dv} \geq 0$ [No-Ponzi Game]

$$J = -\frac{1}{\eta} e^{-\eta c(t)} \cdot e^{-(\rho-n)t} + \lambda(t) \cdot \{[r(t)-n]\, a(t) + w(t) - c(t)$$

$$\frac{\partial J}{\partial c} = 0 \qquad e^{-\eta c} \cdot e^{-(\rho-n)t} = \lambda(t) \qquad [1]$$

$$\frac{\partial J}{\partial a} = -\dot{\lambda} \qquad [r-n] \cdot \lambda = -\dot{\lambda} \rightarrow \lambda(t) = e^{-\int_0^t [r(v)-n]\, dv} \qquad [2]$$

$\lim_{t\to\infty} \lambda(t)\, a(t) = 0$ [Transversality condition]

Given [2] $\lim_{t\to\infty} \lambda(t)\, a(t) = e^{-\int_0^t [r(v)-n]} \cdot a(t) = 0$ which shows that once the

TVC is satisfied, also the No-Ponzi game is.

log-differentiating [1] gets:

$$\frac{d}{dt}\log\left[e^{-\eta c}\cdot e^{-(\rho-n)t}\right] = \frac{\dot{\lambda}}{\lambda} = -r(t) + n \quad \{\text{from } [2]\}$$

$$-\eta\dot{c} - (\rho - \cancel{n}) = -r(t) + \cancel{n}$$

$$\dot{c} = \frac{r(t)-\rho}{\eta} \quad [3] \qquad \text{or} \qquad \frac{\dot{c}}{c} = \frac{r(t)-\rho}{c\eta}$$

The only difference from the CEIS case $u(c) = \dfrac{c^{1-\theta}-1}{1-\theta}$ is that here we get just the rate of change of consumption $\dot{c}$ rather than $\dfrac{\dot{c}}{c}$.

For firms we have the standard set up, hence:

$$f'(k) = r(t) + \delta \quad [4] \quad \text{and} \quad w(t) = f(k) - f'(k)\cdot k \quad [5]$$

Substituting the equilibrium condition [4] into [3] gives us the dynamic behavior of consumption over time:

$$\dot{c}(t) = \frac{f'(k(t)) - \delta - \rho}{\eta} = \frac{\alpha\, k(t)^{\alpha-1} - \delta - \rho}{\eta}$$

Imposing the equilibrium condition in the asset market $a(t) = k(t)$ gets:

$$\dot{k}(t) = w(t) + [r(t) - n]\, a(t)\, k(t) - c(t)$$

Substituting the equilibrium prices [4] and [5] gets:

$$\dot{k}(t) = f(k) - \cancel{f'(k)} \cdot k + [f'(k) - \delta - n]k - c$$

or

$$\dot{k}(t) = k^{\alpha}(t) - (\delta + n)k(t) - c(t)$$

Hence the pair of DE describing the equilibrium behavior of the economy is:

$$\begin{cases} \dot{c}(t) = \dfrac{\alpha k^{\alpha-1}(t) - \delta - \rho}{\eta} \\ \dot{k}(t) = k^{\alpha}(t) - (\delta + n)k(t) - c(t) \end{cases}$$

we have to remember that the TVC must be satisfied, hence

$$\lim_{t \to \infty} k(t) \cdot e^{-\int_0^t [f'[k(v)] - \delta - n] dv} = 0$$

which boils down to [in a balanced growth path]

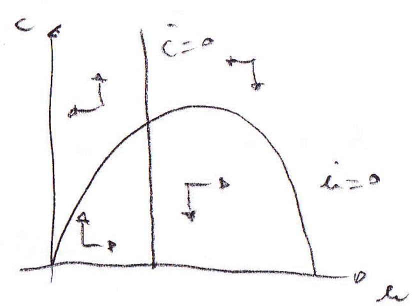
$f'(k(t)) > \delta + n$ which is satisfied every time $\rho > n$

since $\dot{c} = 0$ implies $f'(k(t)) = \delta + \rho$

b) The phase diagram is similar to the one in the Ramsey model.

The $\dot{k}(t) = 0$ locus is exactly the same, since $\dot{k}(t) = 0 \rightarrow c(t) = f(k) - (\delta + n)k$

The $\dot{c}(t) = 0$ locus is the same as well. Hence



The linearized system is the following one:

$$\begin{cases} d\dot{k} = \dfrac{\partial \dot{k}}{\partial k} \cdot (k - k^*) + \dfrac{\partial \dot{k}}{\partial c} \cdot (c - c^*) \\ d\dot{c} = \dfrac{\partial \dot{c}}{\partial k} \cdot (k - k^*) + \dfrac{\partial \dot{c}}{\partial c} \cdot (c - c^*) \end{cases}$$

or

$$\dot{k} = \left\{ \alpha [k^*]^{\alpha-1} - \delta + n \right\} \cdot (k - k^*) - (c - c^*)$$

$$\dot{c} = \frac{\alpha(\alpha-1)(k^*)^{\alpha-2}}{\eta} \cdot (k - k^*) + 0$$

In the steady-state

$\alpha (k^*)^{\alpha-1} = \delta + \rho$ or $k^* = \left[\dfrac{\alpha}{\delta+\rho}\right]^{\frac{1}{1-\alpha}}$ so that

$$\begin{bmatrix} \dot{k} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} \alpha \left(\dfrac{\alpha}{\delta+\rho}\right)^{\frac{\alpha-1}{1-\alpha}} = \alpha \dfrac{(\delta+\rho)}{\alpha} \equiv A & -1 \\ \dfrac{\alpha(\alpha-1) \cdot \left(\dfrac{\alpha}{\delta+\rho}\right)^{\frac{\alpha-2}{1-\alpha}}}{\eta} \equiv B & 0 \end{bmatrix} \cdot \begin{bmatrix} k - k^* \\ c - C^* \end{bmatrix}$$

Notice that $A > 0$ and $B < 0$. Hence the two eigenvalues $e_{1,2}$ associated to the matrix of coefficients are opposite in sign.

$$\begin{vmatrix} A-e & -1 \\ B & -e \end{vmatrix} = -(A-e)\cdot e + B = e^2 - Ae + B$$

$$e_{1,2} = \frac{(\delta+\rho) \pm \sqrt{(\delta+\rho)^2 - 4A\cdot(A-1)}}{2} \quad \begin{cases} e_1 > 0 \\ e_2 < 0 \end{cases}$$

We have saddle-path stability.

5/ In order to get the solution for $k(t)$, we need to kill the explosive behavior implied by $e_1 > 0$.

The solution can be expressed as:

$$k(t) = \chi_1 \cdot e^{e_1 t} + \chi_2 e^{e_2 \cdot t} + k^*$$

We need to impose $\chi_1 = 0$ so that we are left with

$$k(t) = \chi_2 e^{e_2 t} + k^*$$

The speed of convergence is affected negatively by η. We can see that by showing that e_2 decreases with an increase in η; the speed of convergence depends on the value of the eigenvalue, the lower e_2 the slower the speed of convergence.

Finally, the saddle-path stability implies that if we pick appropriately $c(0)$ for a given $k(0)$ $\lim_{t\to\infty} k(t) = k^*$. Since k does not change anymore after the steady-state has been reached the TVC is satisfied as long as $\rho > n$.