QUEEN'S UNIVERSITY AT KINGSTON
Department of Economics

ECONOMICS 351* - Winter Term 2008

Introductory Econometrics

Winter Term 2008 **MID-TERM EXAM: ANSWERS** M.G. Abbott

DATE: **Monday March 3, 2008.**

TIME: **80 minutes; 2:30 p.m. - 3:50 p.m.**

INSTRUCTIONS: The exam consists of **FIVE (5)** questions. Students are required to answer **ALL FIVE (5)** questions.
Answer all questions in the exam booklets provided. Be sure your ***student number*** is printed clearly on the front of all exam booklets used.
Do not write answers to questions on the front page of the first exam booklet.
Please label clearly each of your answers in the exam booklets with the appropriate number and letter.
Please write legibly.
Tables of percentage points of the t-distribution and F-distribution are given on the last two pages of the exam.

MARKING: The marks for each question are indicated in parentheses immediately above each question. **Total marks** for the exam **equal 100**.

GOOD LUCK!

All questions pertain to the simple (two-variable) linear regression model for which the population regression equation can be written in conventional notation as:

$$Y_i = \beta_0 + \beta_1 X_i + u_i \qquad \textbf{(1)}$$

where Y_i and X_i are observable variables, β_0 and β_1 are unknown (constant) regression coefficients, and u_i is an unobservable random error term. The Ordinary Least Squares (OLS) sample regression equation corresponding to regression equation (1) is

$$Y_i = \hat{\beta}_0 + \hat{\beta}_1 X_i + \hat{u}_i \qquad (i = 1, ..., N) \qquad \textbf{(2)}$$

where $\hat{\beta}_0$ is the OLS estimator of the intercept coefficient β_0, $\hat{\beta}_1$ is the OLS estimator of the slope coefficient β_1, $\hat{u}_i$ is the OLS residual for the i-th sample observation, and N is sample size (the number of observations in the sample).

QUESTIONS: **Answer ALL FIVE questions.**

(14 marks)

1. State the Ordinary Least Squares (OLS) estimation criterion. State the OLS normal equations. Derive the OLS normal equations from the OLS estimation criterion.

<u>ANSWER to Question 1</u>:

(2 marks)

- State the Ordinary Least Squares (OLS) estimation criterion. **(2 marks)**

The OLS coefficient estimators are **those formulas or expressions for $\hat{\beta}_0$ and $\hat{\beta}_1$ that <u>minimize</u> the sum of <u>squared</u> residuals RSS** for any given sample of size N.

The **OLS estimation criterion** is therefore:

$$\underset{\{\hat{\beta}_j\}}{\text{Minimize}}\ \text{RSS}\left(\hat{\beta}_0, \hat{\beta}_1\right) = \sum_{i=1}^{N} \hat{u}_i^2 = \sum_{i=1}^{N}\left(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i\right)^2$$

(4 marks)

- State the OLS normal equations. **(4 marks)**

The ***first*** **OLS normal equation** can be written in *any one* of the following forms:

$$\sum_{i=1}^{N} Y_i - N\hat{\beta}_0 - \hat{\beta}_1 \sum_{i=1}^{N} X_i = 0$$

$$-N\hat{\beta}_0 - \hat{\beta}_1 \sum_{i=1}^{N} X_i = -\sum_{i=1}^{N} Y_i \qquad \textbf{(N1)}$$

$$N\hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^{N} X_i = \sum_{i=1}^{N} Y_i$$

The ***second*** **OLS normal equation** can be written in *any one* of the following forms:

$$\sum_{i=1}^{N} X_i Y_i - \hat{\beta}_0 \sum_{i=1}^{N} X_i - \hat{\beta}_1 \sum_{i=1}^{N} X_i^2 = 0$$

$$-\hat{\beta}_0 \sum_{i=1}^{N} X_i - \hat{\beta}_1 \sum_{i=1}^{N} X_i^2 = -\sum_{i=1}^{N} X_i Y_i \qquad \textbf{(N2)}$$

$$\hat{\beta}_0 \sum_{i=1}^{N} X_i + \hat{\beta}_1 \sum_{i=1}^{N} X_i^2 = \sum_{i=1}^{N} X_i Y_i$$

Question 1 (continued)

(8 marks)

- Show how the OLS normal equations are derived from the OLS estimation criterion.

(4 marks)

<u>Step 1</u>: Partially differentiate the $RSS(\hat{\beta}_0, \hat{\beta}_1)$ function with respect to $\hat{\beta}_0$ and $\hat{\beta}_1$, using

$$\hat{u}_i = Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i \quad \Rightarrow \quad \frac{\partial \hat{u}_i}{\partial \hat{\beta}_0} = -1 \quad \text{and} \quad \frac{\partial \hat{u}_i}{\partial \hat{\beta}_1} = -X_i .$$

$$\frac{\partial RSS}{\partial \hat{\beta}_0} = \sum_{i=1}^{N} 2\hat{u}_i \left(\frac{\partial \hat{u}_i}{\partial \hat{\beta}_0} \right) = \sum_{i=1}^{N} 2\,\hat{u}_i(-1) = -2\sum_{i=1}^{N} \hat{u}_i = -2\sum_{i=1}^{N} \left(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i \right)$$ **(2 marks)** (1)

$$\frac{\partial RSS}{\partial \hat{\beta}_1} = \sum_{i=1}^{N} 2\hat{u}_i \left(\frac{\partial \hat{u}_i}{\partial \hat{\beta}_1} \right) = \sum_{i=1}^{N} 2\hat{u}_i(-X_i) = -2\sum_{i=1}^{N} X_i \hat{u}_i$$

$$= -2\sum_{i=1}^{N} X_i \left(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i \right) \quad \text{since } \hat{u}_i = Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i$$ **(2 marks)** (2)

$$= -2\sum_{i=1}^{N} \left(X_i Y_i - \hat{\beta}_0 X_i - \hat{\beta}_1 X_i^2 \right).$$

(4 marks)

Step 2: Obtain the first-order conditions (FOCs) for a minimum of the RSS function by **setting the partial derivatives (1) and (2) *equal to zero*** and then dividing each equation by −2 and re-arranging:

$$\frac{\partial RSS}{\partial \hat{\beta}_0} = 0 \quad \Rightarrow \; -2\sum_{i=1}^{N}\hat{u}_i = 0 \quad \Rightarrow \; -2\sum_{i=1}^{N}\left(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i\right) = 0$$

$$\Rightarrow \; \sum_{i=1}^{N}\left(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i\right) = 0$$

$$\Rightarrow \; \sum_{i=1}^{N} Y_i - N\hat{\beta}_0 - \hat{\beta}_1 \sum_{i=1}^{N} X_i = 0$$

$$\Rightarrow \; \sum_{i=1}^{N} Y_i = N\hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^{N} X_i \qquad \textbf{(N1)} \qquad \textbf{(2 marks)}$$

$$\frac{\partial RSS}{\partial \hat{\beta}_1} = 0 \quad \Rightarrow \; -2\sum_{i=1}^{N} X_i\hat{u}_i = 0 \quad \Rightarrow \; -2\sum_{i=1}^{N} X_i\left(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i\right) = 0$$

$$\Rightarrow \; \sum_{i=1}^{N} X_i\left(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i\right) = 0$$

$$\Rightarrow \; \sum_{i=1}^{N}\left(X_i Y_i - \hat{\beta}_0 X_i - \hat{\beta}_1 X_i^2\right) = 0$$

$$\Rightarrow \; \sum_{i=1}^{N} X_i Y_i - \hat{\beta}_0 \sum_{i=1}^{N} X_i - \hat{\beta}_1 \sum_{i=1}^{N} X_i^2 = 0$$

$$\Rightarrow \; \sum_{i=1}^{N} X_i Y_i = \hat{\beta}_0 \sum_{i=1}^{N} X_i + \hat{\beta}_1 \sum_{i=1}^{N} X_i^2 \qquad \textbf{(N2)} \qquad \textbf{(2 marks)}$$

(12 marks)

2. Answer parts (a), (b) and (c) below.

(6 marks)

(a) Write the expression (or formula) for $\text{Var}(\hat{\beta}_1)$, the variance of $\hat{\beta}_1$. Define all terms that enter the formula for $\text{Var}(\hat{\beta}_1)$.

ANSWER:

$$\text{Var}(\hat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^{N} x_i^2} = \frac{\sigma^2}{\sum_{i=1}^{N} (X_i - \bar{X})^2} \qquad \text{where} \quad x_i \equiv X_i - \bar{X}, \quad i = 1, \ldots, N$$

and $\sigma^2 = \text{Var}(u_i | X_i) = E(u_i^2 | X_i)$ is the constant error variance.

(3 marks)

(b) Which of the following factors makes $\text{Var}(\hat{\beta}_1)$ *smaller*?

ANSWER: Correct answers are highlighted in bold.

(1) a smaller sample

(2) less sample variation of the X_i values around their sample mean $\bar{X}$

(3) a larger value of the error variance

(4) a smaller value of the error variance

(5) a larger sample

(6) more sample variation of the X_i values around their sample mean $\bar{X}$

(3 marks)

(c) How do you compute an unbiased estimator of $\text{Var}(\hat{\beta}_1)$?

ANSWER: An unbiased estimator of $\text{Var}(\hat{\beta}_1)$ is

$$\hat{\text{Var}}(\hat{\beta}_1) = \frac{\hat{\sigma}^2}{\sum_{i=1}^{N} x_i^2} = \frac{\hat{\sigma}^2}{\sum_{i=1}^{N} (X_i - \bar{X})^2}$$

where $\hat{\sigma}^2 = \frac{RSS}{N-2} = \frac{\sum_{i=1}^{N} \hat{u}_i^2}{N-2}$ is an unbiased estimator of the error variance σ^2.

(10 marks)

3. Explain what is meant by each of the following statements about the estimator $\hat{\theta}$ of the population parameter θ.

(a) $\hat{\theta}$ is an unbiased estimator of θ.
(b) $\hat{\theta}$ is an efficient estimator of θ.

What is the difference between the minimum variance and efficiency properties of the estimator $\hat{\theta}$?

<u>ANSWER to Question 3</u>:

(5 marks)

- (a) $\hat{\theta}$ is an unbiased estimator of θ.

  $\hat{\theta}$ is an unbiased estimator of θ if the mean of the sampling (probability) distribution of $\hat{\theta}$ is equal to θ, i.e., to the true population value of θ:

  $\hat{\theta}$ is an ***unbiased*** **estimator** of θ if $E(\hat{\theta}) = \theta$.

(5 marks)

- (b) $\hat{\theta}$ is an efficient estimator of θ.

  The estimator $\hat{\theta}$ is an efficient estimator if it is ***unbiased*** and has ***smaller variance*** than *any other* ***unbiased*** estimator of the parameter θ.

  If $\tilde{\theta}$ is *any other* ***unbiased*** *estimator* of θ, then $\hat{\theta}$ is an ***efficient*** **estimator** of θ if

  $$\text{Var}(\hat{\theta}) \le \text{Var}(\tilde{\theta}) \qquad \text{where } E(\hat{\theta}) = \theta \text{ and } E(\tilde{\theta}) = \theta.$$

(32 marks)

4. A researcher is using data for a sample of 274 male employees to investigate the relationship between employees' hourly wage rates Y_i (measured in *dollars per hour*) and their years of formal education X_i (measured in *years*). The population regression equation takes the form of equation (1): $Y_i = \beta_0 + \beta_1 X_i + u_i$. Preliminary analysis of the sample data produces the following sample information:

$$N = 274 \qquad \sum_{i=1}^{N} Y_i = 1945.26 \qquad \sum_{i=1}^{N} X_i = 3504.00 \qquad \sum_{i=1}^{N} Y_i^2 = 18536.73$$

$$\sum_{i=1}^{N} X_i^2 = 47272.00 \qquad \sum_{i=1}^{N} X_i Y_i = 26204.65 \qquad \sum_{i=1}^{N} x_i y_i = 1328.04$$

$$\sum_{i=1}^{N} y_i^2 = 4726.38 \qquad \sum_{i=1}^{N} x_i^2 = 2461.72 \qquad \sum_{i=1}^{N} \hat{u}_i^2 = 4009.93$$

where $x_i \equiv X_i - \overline{X}$ and $y_i \equiv Y_i - \overline{Y}$ for i = 1, ..., N. Use the above sample information to answer all the following questions. **Show explicitly all formulas and calculations**.

<u>ANSWERS to Question 4</u>:

(12 marks)

(a) Use the above information to compute OLS estimates of the intercept coefficient β_0 and the slope coefficient β_1.

- $\hat{\beta}_1 = \dfrac{\sum_i x_i y_i}{\sum_i x_i^2} = \dfrac{1328.04}{2461.72} = \mathbf{0.53948} = \underline{\mathbf{0.539}}$ **(6 marks)**

- $\hat{\beta}_0 = \overline{Y} - \hat{\beta}_1 \overline{X}$

$$\overline{Y} = \frac{\sum_{i=1}^{N} Y_i}{N} = \frac{1945.26}{274} = 7.09949 \quad \text{and} \quad \overline{X} = \frac{\sum_{i=1}^{N} X_i}{N} = \frac{3504.00}{274} = 12.7883$$

Therefore

$$\hat{\beta}_0 = \overline{Y} - \hat{\beta}_1 \overline{X} = 7.09949 - (0.53948)(12.7883) = 7.09949 - 6.89903 = \underline{\mathbf{0.20046}}$$

(6 marks)

ANSWERS to Question 4 (continued):

(4 marks)

(b) Interpret the slope coefficient estimate you calculated in part (a) – i.e., explain in words what the numeric value you calculated for $\hat{\beta}_1$ means.

Note: $\hat{\beta}_1$ = **0.53948**. Y_i is measured in ***dollars per hour***, and X_i is measured in ***years***.

The estimate **0.53948** of β_1 means that a **1-year *increase* (decrease) in years of education** X_i is associated on average with **an *increase* (decrease) in hourly wage rate equal to *0.53948 dollars per hour***, or ***53.95 cents per hour***.

(4 marks)

(c) Calculate an estimate of σ^2, the error variance.

$$RSS = \sum_{i=1}^{N} \hat{u}_i^2 = 4009.93; \qquad N - 2 = 274 - 2 = 272$$

$$\hat{\sigma}^2 = \frac{RSS}{N-2} = \frac{\sum_{i=1}^{N} \hat{u}_i^2}{N-2} = \frac{4009.93}{274-2} = \frac{4009.93}{272} = \underline{\mathbf{14.7424}}$$ **(4 marks)**

(6 marks)

(d) Compute the value of R^2, the coefficient of determination for the estimated OLS sample regression equation. Briefly explain what the calculated value of R^2 means.

(4 marks)

$$R^2 = \frac{ESS}{TSS} = \frac{\sum_{i=1}^{N} y_i^2 - \sum_{i=1}^{N} \hat{u}_i^2}{\sum_{i=1}^{N} y_i^2} = \frac{4726.38 - 4009.93}{4726.38} = \frac{716.45}{4726.38} = \underline{\mathbf{0.1516}}$$

OR

$$R^2 = 1 - \frac{RSS}{TSS} = 1 - \frac{\sum_{i=1}^{N} \hat{u}_i^2}{\sum_{i=1}^{N} y_i^2} = 1 - \frac{4009.93}{4726.38} = 1 - 0.8484 = \underline{\mathbf{0.1516}}$$

(2 marks)

Interpretation of R^2 = 0.1516: The value of 0.1516 indicates that **15.16 percent of the total sample (or observed) variation in Y_i** (hourly wage rates) **is *attributable to,* or *explained by,* the sample regression function or the regressor X_i** (years of formal education).

(6 marks)

(e) Calculate the sample value of the F-statistic for testing the null hypothesis H_0: $\beta_1 = 0$ against the alternative hypothesis H_1: $\beta_1 \neq 0$. (Note: You are not required to obtain or state the inference of this test.)

- F-statistic for $\hat{\beta}_1$ is $F(\hat{\beta}_1) = \dfrac{(\hat{\beta}_1 - \beta_1)^2}{\hat{Var}(\hat{\beta}_1)}$ (1) **(2 marks)**

- From part (a), $\hat{\beta}_1 = 0.53948$; $\hat{Var}(\hat{\beta}_1) = \dfrac{\hat{\sigma}^2}{\sum_{i=1}^{N} x_i^2} = \dfrac{14.7424}{2461.72} = 0.0059887$.

- Calculate the ***sample value* of the F-statistic** (1) under H_0: set $\beta_1 = 0$, $\hat{\beta}_1 = 0.53948$ and $\hat{Var}(\hat{\beta}_1) = 0.0059887$ in (1).

$$F_0(\hat{\beta}_1) = \frac{(\hat{\beta}_1 - \beta_1)^2}{\hat{Var}(\hat{\beta}_1)} = \frac{(0.53948 - 0)^2}{0.0059887} = \frac{0.291039}{0.0059887} = \underline{\mathbf{48.60}}$$

(4 marks)

Alternative Answer to 4(e): use the ANOVA F-statistic

- ANOVA F-statistic is: $\text{ANOVA} - F_0 = \dfrac{ESS/1}{RSS/N-2} = \dfrac{\sum_i y_i^2 - \sum_i \hat{u}_i^2}{\hat{\sigma}^2}$ **(2 marks)**

- $\sum_{i=1}^{N} y_i^2 = 4726.38$; $\sum_{i=1}^{N} \hat{u}_i^2 = 4009.93$; from part (c), $\hat{\sigma}^2 = \sum_{i=1}^{N} \hat{u}_i^2 \Big/ (N-2) = \underline{\mathbf{14.7424}}$

- Calculate the ***sample value* of the ANOVA F-statistic**.

$$\text{ANOVA} - F_0 = \frac{\sum_i y_i^2 - \sum_i \hat{u}_i^2}{\hat{\sigma}^2} = \frac{4726.38 - 4009.93}{14.7424} = \frac{716.45}{14.7424} = \underline{\mathbf{48.60}}$$

(4 marks)

(32 marks)

5. You have been commissioned to investigate the relationship between the annual salaries of Chief Executive Officers (CEOs) of firms and the annual profits of their firms. The dependent variable is ***salary*** $salary_i$, the annual salary of the CEO of the i-th firm, measured in *thousands of dollars per year*. The explanatory variable is ***$profits_i$***, the annual profits of the i-th firm, measured in *millions of dollars per year*. The model you propose to estimate is given by the population regression equation

$$\text{salary}_i = \beta_0 + \beta_1 \text{profits}_i + u_i .$$

Your research assistant has used observations on ***$salary_i$*** and ***$profits_i$*** for a sample of 177 corporations to estimate the following OLS sample regression equation, where the figures in parentheses below the coefficient estimates are the *estimated standard errors* of the coefficient estimates:

$$\underset{(45.798)}{\text{salary}_i = 746.92} + \underset{(0.10094)}{0.57230}\,\text{profits}_i + \hat{u}_i \qquad (i = 1, ..., N) \qquad N = 177 \qquad \textbf{(3)}$$

$\leftarrow$ (standard errors)

<u>ANSWERS to Question 5</u>:

(8 marks)

(a) Perform a test of the null hypothesis H_0: $\beta_1 = 0$ against the alternative hypothesis H_1: $\beta_1 \neq 0$ at the 1% significance level (i.e., for significance level $\alpha = 0.01$). Show how you calculated the test statistic. State the decision rule you use, and the inference you would draw from the test. Briefly state the conclusion you would draw from the test.

H_0: $\beta_1 = 0$
H_1: $\beta_1 \neq 0$ a ***two-sided*** **alternative hypothesis** $\Rightarrow$ a ***two-tailed*** **test**

- Test statistic is either $t(\hat{\beta}_1) = \dfrac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} \sim t[N-2]$ *or* $F(\hat{\beta}_1) = \dfrac{(\hat{\beta}_1 - \beta_1)^2}{\hat{Var}(\hat{\beta}_1)} \sim F[1, N-2]$.

- $\hat{\beta}_1 = \mathbf{0.57230}$; $\hat{se}(\hat{\beta}_1) = \mathbf{0.10094}$; $\hat{Var}(\hat{\beta}_1) = \left(\hat{se}(\hat{\beta}_1)\right)^2 = \mathbf{0.0101889}$

- Calculate the ***sample value*** **of** ***either*** **the t-statistic** ***or*** **the F-statistic** under H_0: set $\beta_1 = 0$, $\hat{\beta}_1 = \mathbf{0.57230}$, $\hat{se}(\hat{\beta}_1) = \mathbf{0.10094}$**,** and $\hat{Var}(\hat{\beta}_1) = \mathbf{0.0101889}$.

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} = \frac{0.57230 - 0.0}{0.10094} = \frac{0.57230}{0.10094} = 5.6697 = \underline{\mathbf{5.67}}$$

or **(3 marks)**

ANSWER to Question 5(a) -- continued:

$$F_0(\hat{\beta}_1) = \frac{(\hat{\beta}_1 - \beta_1)^2}{\hat{Var}(\hat{\beta}_1)} = \frac{(0.57230 - 0.0)^2}{0.0101889} = \frac{0.327527}{0.0101889} = 32.14550 = \underline{\mathbf{32.15}}$$

- **Null distribution** of $t_0(\hat{\beta}_1)$ is **t[N – 2] = t[177 – 2] = t[175]**
- **Null distribution** of $F_0(\hat{\beta}_1)$ is **F[1, N – 2] = F[1, 177 – 2] = F[1, 175]**

Decision Rule: At significance level α, **(2 marks)**

- ***reject* H₀** if $F_0(\hat{\beta}_1) > F_\alpha[1,175]$ *or* $\left|t_0(\hat{\beta}_1)\right| > t_{\alpha/2}[175]$,
  i.e., if either (1) $t_0(\hat{\beta}_1) > t_{\alpha/2}[175]$ or (2) $t_0(\hat{\beta}_1) < -t_{\alpha/2}[175]$;

- ***retain* H₀** if $F_0(\hat{\beta}_1) \le F_\alpha[1,175]$ *or* $\left|t_0(\hat{\beta}_1)\right| \le t_{\alpha/2}[175]$,
  i.e., if $-t_{\alpha/2}[175] \le t_0(\hat{\beta}_1) \le t_{\alpha/2}[175]$.

Critical values of t[175]-distribution or F[1, 175]-distribution: in t-table, use **df = 120** or **df = ∞** or any number between.

- ***two-tailed* 1 percent critical value** = $t_{\alpha/2}[175] = t_{0.005}[175] = 2.604 = \underline{\mathbf{2.60}}$ **(1 mark)**
  $= t_{0.005}[120] = 2.617 = \underline{\mathbf{2.62}}$
  $= t_{0.005}[\infty] = 2.576 = \underline{\mathbf{2.58}}$

Critical values of F[1, 175]-distribution: in F-table, use **denominator df = 120** or **df = 200** or any number between.

- **1 percent critical value** = $F_\alpha[1,175] = F_{0.01}[1, 175] = 6.782 = \underline{\mathbf{6.78}}$
  $= F_{0.01}[1, 120] = \underline{\mathbf{6.85}}$
  $= F_{0.01}[1, 200] = \underline{\mathbf{6.76}}$

Inference: **(1 mark)**

- At **1 percent significance level**, i.e., for α = 0.01,

$\left|t_0(\hat{\beta}_1)\right| = \mathbf{5.67} > \mathbf{2.60} = t_{0.005}[175] \Rightarrow$ ***reject* H₀** vs. H₁ **at 1 percent level**.

$F_0(\hat{\beta}_1) = \mathbf{32.15} > \mathbf{6.78} = F_{0.01}[1, 175] \Rightarrow$ ***reject* H₀** vs. H₁ **at 1 percent level**.

<u>ANSWER to Question 5(a) -- continued</u>:

- **<u>Inference</u>: At the 1% significance level**, the null hypothesis $\beta_1 = 0$ is ***rejected*** in favour of the alternative hypothesis $\beta_1 \neq 0$.

<u>Conclusion implied by test outcome</u>: **(1 mark)**

Rejection of the null hypothesis $\beta_1 = 0$ against the alternative hypothesis $\beta_1 \neq 0$ means that **the sample evidence favours the existence of a relationship between** CEOs' ***salaries*** and their firms' ***profits***.

<u>Question 5(a) – Alternative Answer</u> -- uses ***confidence interval*** **approach**

- The **two-sided (1 – α)-level,** or **100(1 – α) percent, confidence interval for β_1** is:

$$\hat{\beta}_1 - t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_1) \;\leq\; \beta_1 \leq \hat{\beta}_1 + t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_1)$$

$$\hat{\beta}_{1L} \;\leq\; \beta_1 \leq \hat{\beta}_{1U}$$

- Required results and intermediate calculations:

$N - K = 177 - 2 = 175$; $\hat{\beta}_1 = \mathbf{0.57230}$; $\hat{se}(\hat{\beta}_1) = \mathbf{0.10094}$

$1 - \alpha = 0.99 \Rightarrow \alpha = 0.01 \Rightarrow \boldsymbol{\alpha/2 = 0.005}$: $t_{\alpha/2}[N-2] = \mathbf{t_{0.005}[175]} = \underline{\mathbf{2.604}}$ **(1 mark)**

$t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_1) = t_{0.005}[175]\,\hat{se}(\hat{\beta}_1) = 2.604(0.10094) = \mathbf{0.262848}$

- **Lower 99% confidence limit for β_1** is: **(2 marks)**

$$\hat{\beta}_{1L} = \hat{\beta}_1 - t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_1) = \hat{\beta}_1 - t_{0.005}[1720]\,\hat{se}(\hat{\beta}_1)$$

$= 0.57230 - \mathbf{2.604}(0.10094) = 0.57230 - 0.262848 = 0.309452 = \underline{\mathbf{0.3095}}$

$= 0.57230 - \mathbf{2.62}(0.10094) = 0.57230 - 0.264463 = 0.307837 = \underline{\mathbf{0.3078}}$

$= 0.57230 - \mathbf{2.58}(0.10094) = 0.57230 - 0.260425 = 0.311875 = \underline{\mathbf{0.3119}}$

- **Upper 99% confidence limit for β_1** is: **(2 marks)**

$$\hat{\beta}_{1U} = \hat{\beta}_1 + t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_1) = \hat{\beta}_1 + t_{0.005}[1720]\,\hat{se}(\hat{\beta}_1)$$

$= 0.57230 + \mathbf{2.604}(0.10094) = 0.57230 + 0.262848 = 0.835148 = \underline{\mathbf{0.8351}}$

$= 0.57230 + \mathbf{2.62}(0.10094) = 0.57230 + 0.264463 = 0.836763 = \underline{\mathbf{0.8368}}$

$= 0.57230 + \mathbf{2.58}(0.10094) = 0.57230 + 0.260425 = 0.832725 = \underline{\mathbf{0.8327}}$

Question 5(a) – Alternative Answer (continued)

- **Two-sided 99% confidence interval for β_1** is therefore:

  $\mathbf{0.3095 \le \beta_1 \le 0.8351}$ *or* $\mathbf{0.3078 \le \beta_1 \le 0.8368}$ *or* $\mathbf{0.3119 \le \beta_1 \le 0.8327}$

- **Decision Rule:** At significance level α, **(1 mark)**

  - ***reject*** $\mathbf{H_0}$ if the ***hypothesized*** **value b_1 of β_1** specified by H_0 **lies *outside*** the two-sided $(1-\alpha)$-level confidence interval for β_1, i.e., if either
    (1) $b_1 < \hat{\beta}_1 - t_{\alpha/2}[175]\hat{se}(\hat{\beta}_1)$ or (2) $b_1 > \hat{\beta}_1 + t_{\alpha/2}[175]\hat{se}(\hat{\beta}_1)$.

  - ***retain*** $\mathbf{H_0}$ if the ***hypothesized*** **value b_1 of β_1** specified by H_0 **lies *inside*** the two-sided $(1-\alpha)$-level confidence interval for β_1, i.e., if
    $\hat{\beta}_1 - t_{\alpha/2}[175]\hat{se}(\hat{\beta}_1) \le b_1 \le \hat{\beta}_1 + t_{\alpha/2}[175]\hat{se}(\hat{\beta}_1)$.

Inference: **(1 mark)**

♦ At **1 percent significance level**, i.e., for $\alpha = 0.01$,

$$b_1 = \mathbf{0} < \mathbf{0.3094} = \hat{\beta}_{1L} = \hat{\beta}_1 - t_{0.005}[175]\hat{se}(\hat{\beta}_1)$$

$\Rightarrow$ ***reject*** $\mathbf{H_0}$ vs. H_1 **at 1 percent level**.

♦ **Inference: At the 1% significance level**, the null hypothesis $\beta_1 = 0$ is ***rejected*** in favour of the alternative hypothesis $\beta_1 \neq 0$.

Conclusion implied by test outcome: **(1 mark)**

Rejection of the null hypothesis $\beta_1 = 0$ against the alternative hypothesis $\beta_1 \neq 0$ means that **the sample evidence favours the existence of a relationship between** CEOs' ***salaries*** and their firms' ***profits***.

(8 marks)

(b) Perform a test of the proposition that CEOs' annual salaries are positively related to their firms' profits, i.e., that an increase in firms' profits *increases* the annual salaries of their CEOs. Use the 1 percent significance level (i.e., $\alpha = 0.01$). State the null hypothesis H_0 and the alternative hypothesis H_1. Show how you calculated the test statistic. State the decision rule you use, and the inference you would draw from the test.

ANSWER to Question 5(b):

Null and Alternative Hypotheses:

H_0: $\beta_1 = 0$
H_1: $\beta_1 > 0$ $\Rightarrow$ a ***right-tailed*** **t-test** **(1 mark)**

- Test statistic is $t(\hat{\beta}_1) = \dfrac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} \sim t[N-2]$; $\hat{\beta}_1 = \mathbf{0.57230}$ and $\hat{se}(\hat{\beta}_1) = \mathbf{0.10094}$

- Calculate the ***sample value*** **of the t-statistic** under H_0:
  set $\beta_1 = 0$, $\hat{\beta}_1 = \mathbf{0.57230}$, $\hat{se}(\hat{\beta}_1) = \mathbf{0.10094}$ in (1).

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} = \frac{0.57230 - 0.0}{0.10094} = \frac{0.57230}{0.10094} = 5.6697 = \underline{\mathbf{5.67}}$$ **(2 marks)**

- **Null distribution** of $t_0(\hat{\beta}_1)$ is **t[N – 2] = t[177– 2] = t[175]** **(1 mark)**

Decision Rule: At significance level α, **(1 mark)**

- ***reject*** $\mathbf{H_0}$ if $t_0(\hat{\beta}_1) > t_\alpha[175]$,
- ***retain*** $\mathbf{H_0}$ if $t_0(\hat{\beta}_1) \le t_\alpha[175]$.

Critical value of t[175]-distribution: from t-table, use **df = 120** or **df = ∞** or any number in between.

- ***right-tail*** **1 percent critical value** $= t_{0.01}[175] = \underline{\mathbf{2.348}} = \underline{\mathbf{2.35}}$ **(1 mark)**
  $= t_{0.01}[120] = \underline{\mathbf{2.358}} = \underline{\mathbf{2.36}}$
  $= t_{0.01}[\infty] = \underline{\mathbf{2.326}} = \underline{\mathbf{2.33}}$

ANSWER to Question 5(b) -- continued:

Inference: **(2 marks)**

- At **1 percent significance level**, i.e., for $\alpha = 0.01$,

  $t_0(\hat{\beta}_1) = \mathbf{5.67} > \mathbf{2.348} = t_{0.01}[175] \quad \Rightarrow \quad$ ***reject*** $\mathbf{H_0}$ vs. H_1 **at 1 percent level**.

- **Inference: At the 1% significance level**, the null hypothesis $\boldsymbol{\beta_1 = 0}$ **is *rejected*** in favour of the alternative hypothesis $\boldsymbol{\beta_1} > \mathbf{0}$.

(8 marks)

(c) Compute the two-sided 95% confidence interval for the slope coefficient β_1.

ANSWER to Question 5(c):

- The **two-sided (1 – α)-level,** or **100(1 – α) percent, confidence interval for β_1** is computed as

$$\hat{\beta}_1 - t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_1) \le \beta_1 \le \hat{\beta}_1 + t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_1)$$ **(2 marks)**

where

- $\hat{\beta}_{1L} = \hat{\beta}_1 - t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_1)$ = the ***lower*** **100(1 – α)% confidence limit for β_1**
- $\hat{\beta}_{1U} = \hat{\beta}_1 + t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_1)$ = the ***upper*** **100(1 – α)% confidence limit for β_1**
- $t_{\alpha/2}[N-2]$ = the **α/2 critical value of the t-distribution with N–2 degrees of freedom**.

- Required results and intermediate calculations:

$N - K = 177 - 2 = 175$; $\hat{\beta}_1 = \mathbf{0.57230}$; $\hat{se}(\hat{\beta}_1) = \mathbf{0.10094}$

$1 - \alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow \boldsymbol{\alpha/2 = 0.025}$: $t_{\alpha/2}[N-2] = \mathbf{t_{0.025}[175] = 1.974}$

$= \mathbf{t_{0.025}[120] = 1.980}$

$= \mathbf{t_{0.025}[\infty] = 1.960}$

$t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_1) = t_{0.025}[175]\,\hat{se}(\hat{\beta}_1) = 1.974(0.10094) = \mathbf{0.199256}$

- **Lower 95% confidence limit for β_1** is: **(3 marks)**

$\hat{\beta}_{1L} = \hat{\beta}_1 - t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_1) = \hat{\beta}_1 - t_{0.025}[175]\,\hat{se}(\hat{\beta}_1)$

$= 0.57230 - \mathbf{1.974}(0.10094) = 0.57230 - 0.199256 = 0.373044 = \underline{\mathbf{0.3730}}$

$= 0.57230 - \mathbf{1.980}(0.10094) = 0.57230 - 0.199861 = 0.372439 = \underline{\mathbf{0.3724}}$

$= 0.57230 - \mathbf{1.960}(0.10094) = 0.57230 - 0.197842 = 0.374458 = \underline{\mathbf{0.3745}}$

ANSWER to Question 5(c) -- continued:

- **Upper 95% confidence limit for β_1** is: **(3 marks)**

$$\hat{\beta}_{1U} = \hat{\beta}_1 + t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_1) = \hat{\beta}_1 + t_{0.025}[175]\,\hat{se}(\hat{\beta}_1)$$

$$= 0.57230 + \mathbf{1.974}(0.10094) = 0.57230 + 0.199256 = 0.771556 = \underline{\mathbf{0.7716}}$$

$$= 0.57230 + \mathbf{1.980}(0.10094) = 0.57230 + 0.199861 = 0.772161 = \underline{\mathbf{0.7722}}$$

$$= 0.57230 + \mathbf{1.960}(0.10094) = 0.57230 + 0.197842 = 0.770142 = \underline{\mathbf{0.7701}}$$

- **Two-sided 95% confidence interval for β_1** is therefore:

$0.373 \le \beta_1 \le 0.772$ *or* **$0.372 \le \beta_1 \le 0.772$** *or* **$0.375 \le \beta_1 \le 0.770$**

- **Result:** The **two-sided 95% confidence interval for β_1** is: **[0.373, 0.772]**

(8 marks)

(d) Perform a test of the proposition that an increase in profits of 1 million dollars per year is associated on average with an increase in CEO salary of 500 dollars per year. Use the 5 percent significance level (i.e., $\alpha = 0.05$). State the null hypothesis H_0 and the alternative hypothesis H_1. Show how you calculated the test statistic. State the decision rule you use, and the inference you would draw from the test.

ANSWER to Question 5(d):

H_0: $\beta_1 = 0.500$

H_1: $\beta_1 \neq 0.500$ $\Rightarrow$ a ***two-tailed*** **test** **(1 mark)**

- Test statistic is either $t(\hat{\beta}_1) = \dfrac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} \sim t[N-2]$ *or* $F(\hat{\beta}_1) = \dfrac{(\hat{\beta}_1 - \beta_1)^2}{\hat{Var}(\hat{\beta}_1)} \sim F[1, N-2]$.

- $\hat{\beta}_1 = \mathbf{0.57230}$; $\hat{se}(\hat{\beta}_1) = \mathbf{0.10094}$; $\hat{Var}(\hat{\beta}_1) = \left(\hat{se}(\hat{\beta}_1)\right)^2 = \mathbf{0.0101889}$

- Calculate the ***sample value*** **of *either* the t-statistic *or* the F-statistic** under H_0: set $\beta_1 = 0$, $\hat{\beta}_1 = \mathbf{0.57230}$, $\hat{se}(\hat{\beta}_1) = \mathbf{0.10094}$, and $\hat{Var}(\hat{\beta}_1) = \mathbf{0.0101889}$.

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} = \frac{0.57230 - 0.500}{0.10094} = \frac{0.07230}{0.10094} = 0.716267 = \underline{\mathbf{0.716}}$$

or **(3 marks)**

$$F_0(\hat{\beta}_1) = \frac{(\hat{\beta}_1 - \beta_1)^2}{\hat{Var}(\hat{\beta}_1)} = \frac{(0.57230 - 0.500)^2}{0.0101889} = \frac{0.00522729}{0.0101889} = 0.513039 = \underline{\mathbf{0.513}}$$

- **Null distribution** of $t_0(\hat{\beta}_1)$ is **t[N – 2] = t[177 – 2] = t[175]**
- **Null distribution** of $F_0(\hat{\beta}_1)$ is **F[1, N – 2] = F[1, 177 – 2] = F[1, 175]**

Decision Rule: At significance level α, **(2 marks)**

- ***reject*** **H_0** if $F_0(\hat{\beta}_1) > F_\alpha[1, 175]$ *or* $\left|t_0(\hat{\beta}_1)\right| > t_{\alpha/2}[175]$,

  i.e., if either (1) $t_0(\hat{\beta}_1) > t_{\alpha/2}[175]$ or (2) $t_0(\hat{\beta}_1) < -t_{\alpha/2}[175]$;

- ***retain*** **H_0** if $F_0(\hat{\beta}_1) \le F_\alpha[1, 175]$ *or* $\left|t_0(\hat{\beta}_1)\right| \le t_{\alpha/2}[175]$,

  i.e., if $-t_{\alpha/2}[175] \le t_0(\hat{\beta}_1) \le t_{\alpha/2}[175]$.

<u>ANSWER to Question 5(d) -- continued</u>:

Critical values of t[175]-distribution or F[1, 175]-distribution: in t-table, use **df = 120** or **df = ∞** or any number between.

- ***two-tailed*** **<u>5 percent</u> critical value** = $t_{\alpha/2}[175] = t_{0.025}[175] = 1.974 =$ **<u>1.97</u>** **(1 mark)**
  $= t_{0.025}[120] = 1.980 =$ **<u>1.98</u>**
  $= t_{0.025}[\infty] = 1.960 =$ **<u>1.96</u>**

Critical values of F[1, 175]-distribution: in F-table, use **denominator df = 120** or **df = 200** or any number between.

- **<u>5 percent</u> critical value** = $F_{\alpha}[1, 175] = F_{0.05}[1, 175] = 3.895 =$ **<u>3.90</u>**
  $= F_{0.05}[1, 120] =$ **<u>3.92</u>**
  $= F_{0.05}[1, 200] =$ **<u>3.89</u>**

<u>Inference</u>: **(1 mark)**

- At **5 percent significance level**, i.e., for $\alpha = 0.05$,

  $\left|t_0(\hat{\beta}_1)\right| = \mathbf{0.716} < \mathbf{1.97} = t_{0.025}[175] \quad \Rightarrow$ ***retain*** **H_0** vs. H_1 **at 5 percent level**.

  $F_0(\hat{\beta}_1) = \mathbf{0.513} < \mathbf{3.90} = F_{0.05}[1, 175] \quad \Rightarrow$ ***retain*** **H_0** vs. H_1 **at 5 percent level**.

- **<u>Inference</u>: At the 5% significance level**, the null hypothesis $\beta_1 = 0.500$ is ***retained (not rejected)*** against the alternative hypothesis $\beta_1 \neq 0500$.

<u>Question 5(d) – ALTERNATIVE ANSWER</u> -- uses ***confidence interval*** **approach**

H_0: $\beta_1 = 0.500$
H_1: $\beta_1 \neq 0.500$ $\Rightarrow$ a ***two-tailed*** **test** **(1 mark)**

- The **two-sided (1 – α)-level,** or **100(1 – α) percent, confidence interval for β_1** is:

$$\hat{\beta}_1 - t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_1) \le \beta_1 \le \hat{\beta}_1 + t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_1)$$
$$\hat{\beta}_{1L} \le \beta_1 \le \hat{\beta}_{1U}$$

- Required results and intermediate calculations:

$N - K = 177 - 2 = 175$; $\hat{\beta}_1 = \mathbf{0.57230}$; $\hat{se}(\hat{\beta}_1) = \mathbf{0.10094}$

$1 - \alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow \boldsymbol{\alpha/2 = 0.025}$: $t_{\alpha/2}[N-2] = \mathbf{t_{0.025}[175] = 1.974}$
$= \mathbf{t_{0.025}[120] = 1.980}$
$= \mathbf{t_{0.025}[\infty] = 1.960}$

$$t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_1) = t_{0.025}[175]\,\hat{se}(\hat{\beta}_1) = 1.974(0.10094) = \mathbf{0.199256}$$

- **Lower 95% confidence limit for β_1** is: **(2 marks)**

$$\hat{\beta}_{1L} = \hat{\beta}_1 - t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_1) = \hat{\beta}_1 - t_{0.025}[175]\,\hat{se}(\hat{\beta}_1)$$

$= 0.57230 - \mathbf{1.974}(0.10094) = 0.57230 - 0.199256 = 0.373044 = \underline{\mathbf{0.3730}}$

$= 0.57230 - \mathbf{1.980}(0.10094) = 0.57230 - 0.199861 = 0.372439 = \underline{\mathbf{0.3724}}$

$= 0.57230 - \mathbf{1.960}(0.10094) = 0.57230 - 0.197842 = 0.374458 = \underline{\mathbf{0.3745}}$

- **Upper 95% confidence limit for β_1** is: **(2 marks)**

$$\hat{\beta}_{1U} = \hat{\beta}_1 + t_{\alpha/2}[N-2]\,\hat{se}(\hat{\beta}_1) = \hat{\beta}_1 + t_{0.025}[175]\,\hat{se}(\hat{\beta}_1)$$

$= 0.57230 + \mathbf{1.974}(0.10094) = 0.57230 + 0.199256 = 0.771556 = \underline{\mathbf{0.7716}}$

$= 0.57230 + \mathbf{1.980}(0.10094) = 0.57230 + 0.199861 = 0.772161 = \underline{\mathbf{0.7722}}$

$= 0.57230 + \mathbf{1.960}(0.10094) = 0.57230 + 0.197842 = 0.770142 = \underline{\mathbf{0.7701}}$

- **Two-sided 95% confidence interval for β_1** is therefore:

$\mathbf{0.373 \le \beta_1 \le 0.772}$ *or* $\mathbf{0.372 \le \beta_1 \le 0.772}$ *or* $\mathbf{0.375 \le \beta_1 \le 0.770}$

- **<u>Result</u>:** The **two-sided 95% confidence interval for β_1** is: **[0.373, 0.772]**

ALTERNATIVE ANSWER to Question 5(d) -- continued:

- **Decision Rule:** At significance level α, **(2 marks)**

  - ***reject* H_0 if the *hypothesized* value b_1 of β_1** specified by H_0 **lies *outside*** the two-sided $(1-\alpha)$-level confidence interval for β_1, i.e., if either
    (1) $b_1 < \hat{\beta}_1 - t_{\alpha/2}[175]\hat{se}(\hat{\beta}_1)$ or (2) $b_1 > \hat{\beta}_1 + t_{\alpha/2}[175]\hat{se}(\hat{\beta}_1)$.

  - ***retain* H_0 if the *hypothesized* value b_1 of β_1** specified by H_0 **lies *inside*** the two-sided $(1-\alpha)$-level confidence interval for β_1, i.e., if
    $\hat{\beta}_1 - t_{\alpha/2}[175]\hat{se}(\hat{\beta}_1) \le b_1 \le \hat{\beta}_1 + t_{\alpha/2}[175]\hat{se}(\hat{\beta}_1)$.

Inference: **(1 mark)**

- At **5 percent significance level**, i.e., for $\alpha = 0.05$, the hypothesized value of $\boldsymbol{\beta_1}$, b_1 = **0.500, lies *inside* the two-sided 95% confidence interval for β_1:**

  $0.3730 \le 0.500 \le 0.7716 \Rightarrow$ ***retain* H_0** vs. H_1 **at 5 percent level**.

- **Inference: At the 5% significance level**, the null hypothesis $\beta_1 = 0.500$ is ***retained (not rejected)*** against the alternative hypothesis $\beta_1 \neq 0500$.

Percentage Points of the t-Distribution

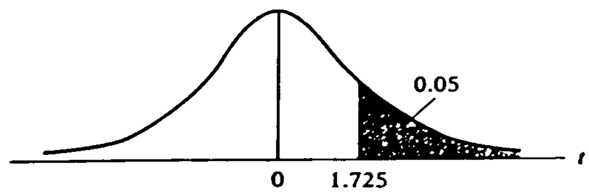
TABLE D.2
Percentage points of the t distribution

Example

$\Pr(t > 2.086) = 0.025$

$\Pr(t > 1.725) = 0.05$ for df = 20

$\Pr(|t| > 1.725) = 0.10$



df \ Pr	0.25 0.50	0.10 0.20	0.05 0.10	0.025 0.05	0.01 0.02	0.005 0.010	0.001 0.002
1	1.000	3.078	6.314	12.706	31.821	63.657	318.31
2	0.816	1.886	2.920	4.303	6.965	9.925	22.327
3	0.765	1.638	2.353	3.182	4.541	5.841	10.214
4	0.741	1.533	2.132	2.776	3.747	4.604	7.173
5	0.727	1.476	2.015	2.571	3.365	4.032	5.893
6	0.718	1.440	1.943	2.447	3.143	3.707	5.208
7	0.711	1.415	1.895	2.365	2.998	3.499	4.785
8	0.706	1.397	1.860	2.306	2.896	3.355	4.501
9	0.703	1.383	1.833	2.262	2.821	3.250	4.297
10	0.700	1.372	1.812	2.228	2.764	3.169	4.144
11	0.697	1.363	1.796	2.201	2.718	3.106	4.025
12	0.695	1.356	1.782	2.179	2.681	3.055	3.930
13	0.694	1.350	1.771	2.160	2.650	3.012	3.852
14	0.692	1.345	1.761	2.145	2.624	2.977	3.787
15	0.691	1.341	1.753	2.131	2.602	2.947	3.733
16	0.690	1.337	1.746	2.120	2.583	2.921	3.686
17	0.689	1.333	1.740	2.110	2.567	2.898	3.646
18	0.688	1.330	1.734	2.101	2.552	2.878	3.610
19	0.688	1.328	1.729	2.093	2.539	2.861	3.579
20	0.687	1.325	1.725	2.086	2.528	2.845	3.552
21	0.686	1.323	1.721	2.080	2.518	2.831	3.527
22	0.686	1.321	1.717	2.074	2.508	2.819	3.505
23	0.685	1.319	1.714	2.069	2.500	2.807	3.485
24	0.685	1.318	1.711	2.064	2.492	2.797	3.467
25	0.684	1.316	1.708	2.060	2.485	2.787	3.450
26	0.684	1.315	1.706	2.056	2.479	2.779	3.435
27	0.684	1.314	1.703	2.052	2.473	2.771	3.421
28	0.683	1.313	1.701	2.048	2.467	2.763	3.408
29	0.683	1.311	1.699	2.045	2.462	2.756	3.396
30	0.683	1.310	1.697	2.042	2.457	2.750	3.385
40	0.681	1.303	1.684	2.021	2.423	2.704	3.307
60	0.679	1.296	1.671	2.000	2.390	2.660	3.232
120	0.677	1.289	1.658	1.980	2.358	2.617	3.160
∞	0.674	1.282	1.645	1.960	2.326	2.576	3.090

Note: The smaller probability shown at the head of each column is the area in one tail; the larger probability is the area in both tails.

Source: From E. S. Pearson and H. O. Hartley, eds., *Biometrika Tables for Statisticians*, vol. 1, 3d ed., table 12, Cambridge University Press, New York, 1966. Reproduced by permission of the editors and trustees of *Biometrika*.

Source: Damodar N. Gujarati, *Basic Econometrics*, Third Edition. New York: McGraw-Hill, 1995, p. 809.

Selected Upper Percentage Points of the F-Distribution

TABLE D.3
Upper percentage points of the *F* distribution (*continued*)

df for denominator N_2		df for numerator N_1											
	Pr	1	2	3	4	5	6	7	8	9	10	11	12
	.25	1.40	1.48	1.47	1.45	1.44	1.42	1.41	1.40	1.39	1.39	1.38	1.37
22	.10	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93	1.90	1.88	1.86
	.05	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.26	2.23
	.01	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.18	3.12
	.25	1.39	1.47	1.46	1.44	1.43	1.41	1.40	1.39	1.38	1.38	1.37	1.36
24	.10	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91	1.88	1.85	1.83
	.05	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.21	2.18
	.01	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.09	3.03
	.25	1.38	1.46	1.45	1.44	1.42	1.41	1.39	1.38	1.37	1.37	1.36	1.35
26	.10	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88	1.86	1.84	1.81
	.05	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18	2.15
	.01	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	3.02	2.96
	.25	1.38	1.46	1.45	1.43	1.41	1.40	1.39	1.38	1.37	1.36	1.35	1.34
28	.10	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87	1.84	1.81	1.79
	.05	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.15	2.12
	.01	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.96	2.90
	.25	1.38	1.45	1.44	1.42	1.41	1.39	1.38	1.37	1.36	1.35	1.35	1.34
30	.10	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85	1.82	1.79	1.77
	.05	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.13	2.09
	.01	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.91	2.84
	.25	1.36	1.44	1.42	1.40	1.39	1.37	1.36	1.35	1.34	1.33	1.32	1.31
40	.10	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79	1.76	1.73	1.71
	.05	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.04	2.00
	.01	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.73	2.66
	.25	1.35	1.42	1.41	1.38	1.37	1.35	1.33	1.32	1.31	1.30	1.29	1.29
60	.10	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.68	1.66
	.05	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.95	1.92
	.01	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.56	2.50
	.25	1.34	1.40	1.39	1.37	1.35	1.33	1.31	1.30	1.29	1.28	1.27	1.26
120	.10	2.75	2.35	2.13	1.99	1.90	1.82	1.77	1.72	1.68	1.65	1.62	1.60
	.05	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.87	1.83
	.01	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.40	2.34
	.25	1.33	1.39	1.38	1.36	1.34	1.32	1.31	1.29	1.28	1.27	1.26	1.25
200	.10	2.73	2.33	2.11	1.97	1.88	1.80	1.75	1.70	1.66	1.63	1.60	1.57
	.05	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93	1.88	1.84	1.80
	.01	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.60	2.50	2.41	2.34	2.27
	.25	1.32	1.39	1.37	1.35	1.33	1.31	1.29	1.28	1.27	1.25	1.24	1.24
∞	.10	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63	1.60	1.57	1.55
	.05	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.79	1.75
	.01	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.25	2.18

Source: Damodar N. Gujarati, *Basic Econometrics*, Third Edition. New York: McGraw-Hill, 1995, p. 814.