ECON 351* -- NOTE 20

Tests of Single Linear Coefficient Restrictions: t-tests and F-tests

1. Basic Rules

1. **Tests of a *single* linear coefficient restriction** can be performed using either a **two-tailed t-test** or an **F-test**.

2. **Tests of *two or more* linear coefficient restrictions** can only be performed using an **F-test**.

2. Testing Single Linear Coefficient Restrictions

Consider the following LOG-LOG (double-log) regression equation:

$$\ln Y_i = \beta_0 + \beta_1 \ln X_{1i} + \beta_2 \ln X_{2i} + u_i \qquad \textbf{(1)}$$

- The **slope coefficients β_1 and β_2 are *elasticity coefficients***; they are therefore comparable in magnitude.

- **Common hypothesis tests:** each involves only *one* linear coefficient restriction

1. $\beta_1 = \beta_2$ or $\beta_1 - \beta_2 = 0$.

- the elasticity of Y wrt X_1 equals the elasticity of Y wrt X_2.
- the marginal effect on lnY of $\ln X_1$ equals the marginal effect on lnY of $\ln X_2$.

2. $\beta_1 = -\beta_2$ or $\beta_1 + \beta_2 = 0$.

- the elasticity of Y wrt X_1 is equal in magnitude but opposite in sign to the elasticity of Y wrt X_2.
- the marginal effect on lnY of $\ln X_1$ is equal in magnitude but opposite in sign to the marginal effect on lnY of $\ln X_2$.

3. $\beta_1 + \beta_2 = 1$. (the constant returns-to-scale hypothesis)

- the elasticities of Y wrt X_1 and X_2 sum to one; implies that if X_1 and X_2 both change by some proportion λ, then Y changes by the same proportion.
- the marginal effects on lnY of $\ln X_1$ and $\ln X_2$ sum to one.

All three of these hypotheses have a common form: each states that a **linear combination of the regression coefficients β_1 and β_2** equals some constant.

♦ A ***linear function*, or *linear combination*, of the regression coefficients β_1 and β_2** takes the general form

$c_1\beta_1 + c_2\beta_2$ where c_1 and c_2 are specified (known) constants.

Some simple examples:

1. $\beta_1 = \beta_2$ or $\beta_1 - \beta_2 = 0$.

For this case, $c_1 = 1$ and $c_2 = -1$.

2. $\beta_1 = -\beta_2$ or $\beta_1 + \beta_2 = 0$.

For this case, $c_1 = 1$ and $c_2 = 1$.

3. $\beta_1 + \beta_2 = 1$.

For this case, $c_1 = 1$ and $c_2 = 1$.

4. $\beta_1 + 2\beta_2 = 1$.

For this case, $c_1 = 1$ and $c_2 = 2$.

3. General Framework for t-tests and F-tests of Linear Coefficient Restrictions

- We want to generalize the **t-statistics** and **F-statistics** for ***individual* coefficient estimates** $\hat{\boldsymbol{\beta}}_j$.

  Recall that the **t-statistic for** $\hat{\boldsymbol{\beta}}_j$ is:

$$t\left(\hat{\beta}_j\right) = \frac{\hat{\beta}_j - \beta_j}{\hat{se}(\hat{\beta}_j)} \sim t[N-K] = t[N-K_1]$$

  Recall that the **F-statistic for** $\hat{\boldsymbol{\beta}}_j$ is:

$$F\left(\hat{\beta}_j\right) = \frac{(\hat{\beta}_j - \beta_j)^2}{V\hat{a}r(\hat{\beta}_1)} \sim F[1, N-K] = F[1, N-K_1].$$

- We now need the t-statistic and the F-statistic for **linear *combinations*, or linear *functions*, of regression coefficient estimates** such as:

  $c_1\hat{\beta}_1 + c_2\hat{\beta}_2$ where c_1 and c_2 are specified (known) constants

- A ***linear function*****, or *****linear combination*****, of the regression coefficients β_1 and β_2** takes the general form

  $c_1\beta_1 + c_2\beta_2$ where c_1 and c_2 are specified (known) constants.

- A ***linear restriction*** **on the regression coefficients β_1 and β_2** takes the general form

  $c_1\beta_1 + c_2\beta_2 = c_0$ where c_0 is also a specified (known) constant.

- The ***null*** **and** ***alternative*** **hypotheses** take the general form

  H_0: $c_1\beta_1 + c_2\beta_2 = c_0$
  H_1: $c_1\beta_1 + c_2\beta_2 \neq c_0$

- The t- and F-statistics for testing H_0 against H_1 are based on **OLS estimates of the *unrestricted* model** corresponding to the alternative hypothesis H_1.

$$\ln Y_i = \hat{\beta}_0 + \hat{\beta}_1 \ln X_{1i} + \hat{\beta}_2 \ln X_{2i} + \hat{u}_i$$

- The ***t-statistic*** **for testing H_0 against H_1** takes the general form

$$t\left(c_1\hat{\beta}_1 + c_2\hat{\beta}_2\right) = \frac{(c_1\hat{\beta}_1 + c_2\hat{\beta}_2) - (c_1\beta_1 + c_2\beta_2)}{\hat{se}(c_1\hat{\beta}_1 + c_2\hat{\beta}_2)} \sim t[N-K] = t[N-K_1]$$

  where $\hat{se}(c_1\hat{\beta}_1 + c_2\hat{\beta}_2) = \sqrt{\hat{Var}(c_1\hat{\beta}_1 + c_2\hat{\beta}_2)}$.

- The ***F-statistic*** **for testing H_0 against H_1** takes the general form

$$F\left(c_1\hat{\beta}_1 + c_2\hat{\beta}_2\right) = \frac{\left[(c_1\hat{\beta}_1 + c_2\hat{\beta}_2) - (c_1\beta_1 + c_2\beta_2)\right]^2}{\hat{Var}(c_1\hat{\beta}_1 + c_2\hat{\beta}_2)} \sim F[1, N-K] = F[1, N-K_1].$$

♦ **General formula** for computing the **estimated variance** of a **linear combination of coefficient estimates**.

The ***estimated*** **variance of the linear combination of coefficient estimates** $\mathbf{c_1\hat{\beta}_1 + c_2\hat{\beta}_2}$ is given by the formula:

$$\hat{Var}(c_1\hat{\beta}_1 + c_2\hat{\beta}_2) = c_1^2\hat{Var}(\hat{\beta}_1) + c_2^2\hat{Var}(\hat{\beta}_2) + 2c_1c_2\hat{Cov}(\hat{\beta}_1, \hat{\beta}_2)$$

where

$\hat{Var}(\hat{\beta}_1) \equiv$ the estimated variance of $\hat{\beta}_1$;

$\hat{Var}(\hat{\beta}_2) \equiv$ the estimated variance of $\hat{\beta}_2$;

$\hat{Cov}(\hat{\beta}_1, \hat{\beta}_2) \equiv$ the estimated covariance of $\hat{\beta}_1$ and $\hat{\beta}_2$.

Note: To compute $\hat{Var}(c_1\hat{\beta}_1 + c_2\hat{\beta}_2)$, you need to obtain the values of $\hat{Var}(\hat{\beta}_1)$, $\hat{Var}(\hat{\beta}_2)$ and $\hat{Cov}(\hat{\beta}_1, \hat{\beta}_2)$. These are obtained from the **estimated variance-covariance matrix** for the OLS coefficient estimates $\hat{\beta}_j$.

Examples

Evaluate the general formula

$$\hat{Var}(c_1\hat{\beta}_1 + c_2\hat{\beta}_2) = c_1^2\hat{Var}(\hat{\beta}_1) + c_2^2\hat{Var}(\hat{\beta}_2) + 2c_1c_2\hat{Cov}(\hat{\beta}_1, \hat{\beta}_2)$$

for some specific linear combinations of the two coefficient estimates $\hat{\beta}_1$ and $\hat{\beta}_2$.

1. For the linear combination $\hat{\beta}_1 - \hat{\beta}_2$,

$c_1 = 1$ and $c_2 = -1 \quad \Rightarrow \quad c_1^2 = 1,\ c_2^2 = 1,\ 2c_1c_2 = 2(1)(-1) = -2.$

$$\therefore \hat{Var}(\hat{\beta}_1 - \hat{\beta}_2) = \hat{Var}(\hat{\beta}_1) + \hat{Var}(\hat{\beta}_2) \underset{\uparrow}{-} 2\hat{Cov}(\hat{\beta}_1, \hat{\beta}_2)$$

2. For the linear combination $\hat{\beta}_1 + \hat{\beta}_2$,

$c_1 = 1$ and $c_2 = 1 \quad \Rightarrow \quad c_1^2 = 1,\ c_2^2 = 1,\ 2c_1c_2 = 2(1)(1) = 2.$

$$\therefore \hat{Var}(\hat{\beta}_1 + \hat{\beta}_2) = \hat{Var}(\hat{\beta}_1) + \hat{Var}(\hat{\beta}_2) \underset{\uparrow}{+} 2\hat{Cov}(\hat{\beta}_1, \hat{\beta}_2)$$

3. For the linear combination $\hat{\beta}_1 + 2\hat{\beta}_2$,

$c_1 = 1$ and $c_2 = 2 \quad \Rightarrow \quad c_1^2 = 1,\ c_2^2 = 2^2 = 4,\ 2c_1c_2 = 2(1)(2) = 4.$

$$\therefore \hat{Var}(\hat{\beta}_1 + 2\hat{\beta}_2) = \hat{Var}(\hat{\beta}_1) + 4\hat{Var}(\hat{\beta}_2) \underset{\uparrow}{+} 4\hat{Cov}(\hat{\beta}_1, \hat{\beta}_2)$$

4. For the linear combination $\hat{\beta}_1 - 2\hat{\beta}_2$,

$c_1 = 1$ and $c_2 = -2 \quad \Rightarrow \quad c_1^2 = 1,\ c_2^2 = (-2)^2 = 4,\ 2c_1c_2 = 2(1)(-2) = -4.$

$$\therefore \hat{Var}(\hat{\beta}_1 - 2\hat{\beta}_2) = \hat{Var}(\hat{\beta}_1) + 4\hat{Var}(\hat{\beta}_2) \underset{\uparrow}{-} 4\hat{Cov}(\hat{\beta}_1, \hat{\beta}_2)$$

4. Test of a Single Linear Coefficient Restriction: General Example

For testing a single linear combination of two (or more) regression coefficients such as $c_1\beta_1 + c_2\beta_2$, **use either a *t-test* or an *F-test***.

- The **t-statistic** for the linear combination of coefficient estimates $c_1\hat{\beta}_1 + c_2\hat{\beta}_2$ is:

$$t\left(c_1\hat{\beta}_1 + c_2\hat{\beta}_2\right) = \frac{(c_1\hat{\beta}_1 + c_2\hat{\beta}_2) - (c_1\beta_1 + c_2\beta_2)}{\hat{se}(c_1\hat{\beta}_1 + c_2\hat{\beta}_2)} \sim t[N-K] = t[N-K_1]$$

- The **F-statistic** for the linear combination of coefficient estimates $c_1\hat{\beta}_1 + c_2\hat{\beta}_2$ is:

$$F\left(c_1\hat{\beta}_1 + c_2\hat{\beta}_2\right) = \frac{\left[(c_1\hat{\beta}_1 + c_2\hat{\beta}_2) - (c_1\beta_1 + c_2\beta_2)\right]^2}{\hat{Var}(c_1\hat{\beta}_1 + c_2\hat{\beta}_2)} \sim F[1, N-K] = F[1, N-K_1].$$

❑ **<u>A Two-Tailed t-test of a Single Linear Coefficient Restriction</u>**

Null and alternative hypotheses

H_0: $\beta_1 = \beta_2 \quad \Rightarrow \quad \beta_1 - \beta_2 = 0$

H_1: $\beta_1 \neq \beta_2 \quad \Rightarrow \quad \beta_1 - \beta_2 \neq 0$

1. **Compute OLS estimates of the *unrestricted* model** corresponding to the alternative hypothesis H_1. The ***unrestricted* OLS SRE** is:

$$\ln Y_i = \hat{\beta}_0 + \hat{\beta}_1 \ln X_{1i} + \hat{\beta}_2 \ln X_{2i} + \hat{u}_i \qquad (i = 1, \ldots, N) \qquad \textbf{(1*)}$$

Retrieve the values of: $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{Var}(\hat{\beta}_1)$, $\hat{Var}(\hat{\beta}_2)$ and $\hat{Cov}(\hat{\beta}_1, \hat{\beta}_2)$.

2. **Compute *sample value* of the *t-statistic* under the null hypothesis H_0.**

The **t-statistic** for the linear coefficient combination $\hat{\beta}_1 - \hat{\beta}_2$ is

$$t\left(\hat{\beta}_1 - \hat{\beta}_2\right) = \frac{(\hat{\beta}_1 - \hat{\beta}_2)\ -\ (\beta_1 - \beta_2)}{\hat{se}(\hat{\beta}_1 - \hat{\beta}_2)} \sim t[N-K] = t[N-K_1] = t[N-3]$$

- Compute $\hat{Var}(\hat{\beta}_1 - \hat{\beta}_2) = \hat{Var}(\hat{\beta}_1) + \hat{Var}(\hat{\beta}_2) - 2\hat{Cov}(\hat{\beta}_1, \hat{\beta}_2)$.
- Compute $\hat{se}(\hat{\beta}_1 - \hat{\beta}_2) = \sqrt{\hat{Var}(\hat{\beta}_1 - \hat{\beta}_2)}$.
- Set $\beta_1 - \beta_2 = 0$, as specified by the null hypothesis H_0.

The ***sample value*** **of the t-statistic under H_0** is therefore

$$t_0\left(\hat{\beta}_1 - \hat{\beta}_2\right) = \frac{(\hat{\beta}_1 - \hat{\beta}_2)\ -\ (\beta_1 - \beta_2)}{\hat{se}(\hat{\beta}_1 - \hat{\beta}_2)} = \frac{\hat{\beta}_1 - \hat{\beta}_2}{\hat{se}(\hat{\beta}_1 - \hat{\beta}_2)}.$$

3. The ***null distribution*** **of $t_0\left(\hat{\beta}_1 - \hat{\beta}_2\right)$** is the $t[N-K] = t[N-K_1] = t[N-3]$ distribution:

$$t_0\left(\hat{\beta}_1 - \hat{\beta}_2\right) \sim t[N-K] = t[N-K_1] = t[N-3] \text{ under } H_0.$$

4. **Apply the usual *decision rule* for a *<u>two-tailed</u> t-test*.**

At significance level α (the 100α percent significance level),

- ***Reject*** **H_0** if $|t_0| > t_{\alpha/2}[N-K] = t_{\alpha/2}[N-3]$ *or* ***two-tail p-value*** **for $t_0 < \alpha$**;
- ***Retain*** **H_0** if $|t_0| \le t_{\alpha/2}[N-K] = t_{\alpha/2}[N-3]$ *or* ***two-tail p-value*** **for $t_0 \ge \alpha$**.

❑ A Two-Tailed F-test of a Single Linear Coefficient Restriction

Null and alternative hypotheses

H_0: $\beta_1 = \beta_2 \quad \Rightarrow \quad \beta_1 - \beta_2 = 0$
H_1: $\beta_1 \neq \beta_2 \quad \Rightarrow \quad \beta_1 - \beta_2 \neq 0$

1. **Compute OLS estimates of the *unrestricted* model** corresponding to the alternative hypothesis H_1. The ***unrestricted* OLS SRE** is:

$$\ln Y_i = \hat{\beta}_0 + \hat{\beta}_1 \ln X_{1i} + \hat{\beta}_2 \ln X_{2i} + \hat{u}_i \qquad (i = 1, \ldots, N) \qquad \textbf{(1*)}$$

Retrieve the values of: $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{Var}(\hat{\beta}_1)$, $\hat{Var}(\hat{\beta}_2)$ and $\hat{Cov}(\hat{\beta}_1, \hat{\beta}_2)$.

2. **Compute *sample value* of the *F-statistic* under the null hypothesis H_0.**

The **F-statistic** for the linear coefficient combination $\hat{\beta}_1 - \hat{\beta}_2$ is

$$F\left(\hat{\beta}_1 - \hat{\beta}_2\right) = \frac{\left[(\hat{\beta}_1 - \hat{\beta}_2) - (\beta_1 - \beta_2)\right]^2}{\hat{Var}(\hat{\beta}_1 - \hat{\beta}_2)} \sim F[1, N-K] = F[1, N-K_1] = F[1, N-3].$$

- Compute $\hat{Var}(\hat{\beta}_1 - \hat{\beta}_2) = \hat{Var}(\hat{\beta}_1) + \hat{Var}(\hat{\beta}_2) - 2\hat{Cov}(\hat{\beta}_1, \hat{\beta}_2)$.
- Set $\beta_1 - \beta_2 = 0$, as specified by the null hypothesis H_0.

The ***sample value* of the F-statistic under H_0** is therefore

$$F_0\left(\hat{\beta}_1 - \hat{\beta}_2\right) = \frac{\left[(\hat{\beta}_1 - \hat{\beta}_2) - (\beta_1 - \beta_2)\right]^2}{\hat{Var}(\hat{\beta}_1 - \hat{\beta}_2)} = \frac{\left(\hat{\beta}_1 - \hat{\beta}_2\right)^2}{\hat{Var}(\hat{\beta}_1 - \hat{\beta}_2)}.$$

3. The ***null distribution* of** $\mathbf{F_0\left(\hat{\beta}_1 - \hat{\beta}_2\right)}$ is the $F[1, N-K] = F[1, N-K_1] = F[1, N-3]$ distribution:

$$F_0\left(\hat{\beta}_1 - \hat{\beta}_2\right) \sim F[1, N-K] = F[1, N-K_1] = F[1, N-3] \quad \text{under } H_0.$$

Note: $F_0\left(\hat{\beta}_1 - \hat{\beta}_2\right) = \left[t_0\left(\hat{\beta}_1 - \hat{\beta}_2\right)\right]^2$ ***or*** $t_0\left(\hat{\beta}_1 - \hat{\beta}_2\right) = \sqrt{F_0\left(\hat{\beta}_1 - \hat{\beta}_2\right)}$.

***4.* Apply the usual *decision rule* for an *F-test*.**

At significance level α (the 100α percent significance level),

- ***Reject* $\mathbf{H_0}$** if $F_0 > F_\alpha[1, N-K] = F_\alpha[1, N-3]$ *or* ***p-value* for $\mathbf{F_0} < \alpha$**;
- ***Retain* $\mathbf{H_0}$** if $F_0 \le F_\alpha[1, N-K] = F_\alpha[1, N-3]$ *or* ***p-value* for $\mathbf{F_0} \ge \alpha$**.

❑ An Equivalent General F-test of a Single Linear Coefficient Restriction

Null and alternative hypotheses

$$H_0: \ \beta_1 = \beta_2 \quad \Rightarrow \quad \beta_1 - \beta_2 = 0$$
$$H_1: \ \beta_1 \neq \beta_2 \quad \Rightarrow \quad \beta_1 - \beta_2 \neq 0$$

1. **Compute OLS estimates of the *unrestricted* model** corresponding to the alternative hypothesis H_1.

The ***unrestricted*** **model** is given by the PRE

$$\ln Y_i = \beta_0 + \beta_1 \ln X_{1i} + \beta_2 \ln X_{2i} + u_i \qquad \textbf{(1)}$$

The ***unrestricted*** **OLS SRE** obtained by OLS estimation of equation (1) is

$$\ln Y_i = \hat{\beta}_0 + \hat{\beta}_1 \ln X_{1i} + \hat{\beta}_2 \ln X_{2i} + \hat{u}_i \qquad (i = 1, \ldots, N) \qquad \textbf{(1*)}$$

Retrieve the values of: $RSS_1 = RSS_U = \sum_{i=1}^{N} \hat{u}_i^2$ and $df_1 = N - K = N - K_1 = N - 3$.

2. **Formulate the *restricted* model** corresponding to the null hypothesis H_0.

Substitute the restriction $\beta_1 = \beta_2$ into the unrestricted regression equation (1):

$$\begin{aligned} \ln Y_i &= \beta_0 + \beta_1 \ln X_{1i} + \beta_2 \ln X_{2i} + u_i \\ &= \beta_0 + \beta_2 \ln X_{1i} + \beta_2 \ln X_{2i} + u_i \\ &= \beta_0 + \beta_2 (\ln X_{1i} + \ln X_{2i}) + u_i. \end{aligned}$$

<u>Result</u>: The ***restricted*** **model** is given by the PRE

$$\ln Y_i = \beta_0 + \beta_2 (\ln X_{1i} + \ln X_{2i}) + u_i \qquad \textbf{(2)}$$

3. **Estimate the *restricted* model** by OLS.

The ***restricted*** **OLS SRE** obtained by OLS estimation of equation (2) is

$$\ln Y_i = \tilde{\beta}_0 + \tilde{\beta}_2(\ln X_{1i} + \ln X_{2i}) + \tilde{u}_i \qquad (i = 1, \ldots, N) \qquad \textbf{(2*)}$$

Note that the ***restricted*** **OLS estimate of β_1** is simply $\tilde{\beta}_1 = \tilde{\beta}_2$.

Retrieve the values of $RSS_0 = RSS_R = \sum_{i=1}^{N} \tilde{u}_i^2$ and $df_0 = N - K_0 = N - 2$.

4. **Compute the *sample value* of the F-statistic** under the null hypothesis H_0.

The **required F-statistic** is:

$$F = \frac{(RSS_0 - RSS_1)/(df_0 - df_1)}{RSS_1/df_1} \sim F[df_0 - df_1, df_1] \text{ under } H_0.$$

For this particular test:

$df_0 = N - K_0 = N - 2$;
$df_1 = N - K = N - 3$.
$df_0 - df_1 = N - K_0 - (N - K) = K - K_0 = 3 - 2 = 1$.

The ***sample value*** **of the F-statistic** is therefore

$$F_0 = \frac{(RSS_0 - RSS_1)/(df_0 - df_1)}{RSS_1/df_1} = \frac{(RSS_0 - RSS_1)/1}{RSS_1/(N-3)} = \frac{(RSS_0 - RSS_1)}{RSS_1/(N-3)}.$$

5. The ***null distribution*** **of F_0** is the F[1, N–3] distribution.

6. **Apply the usual *decision rule* for an F-test**. At significance level α,

- ***Reject*** **H_0** if $F_0 > F_\alpha[1, N-K] = F_\alpha[1, N-3]$ *or* ***p-value*** **for $F_0 < \alpha$**;
- ***Retain*** **H_0** if $F_0 \le F_\alpha[1, N-K] = F_\alpha[1, N-3]$ *or* ***p-value*** **for $F_0 \ge \alpha$**.

❑ Equivalence of the t-test and F-tests of a Single Linear Coefficient Restriction

The **t-test and F-test of a *single* linear coefficient restriction are *completely equivalent***.

This equivalence follows from two facts:

1. The ***sample values* of the two test statistics** under the null hypothesis H_0 are related as follows:

$$(t_0)^2 = F_0 \quad or \qquad t_0 = \sqrt{F_0} \,.$$

The square of the sample value of the t-statistic equals the sample value of the F-statistic; *or* the sample value of the t-statistic equals the square root of the sample value of the F-statistic.

2. At significance level α, the *critical values* of the null distributions t[N−K] and F[1, N−K] are related as follows:

$$\left(t_{\alpha/2}[N-K]\right)^2 = F_\alpha[1, N-K] \qquad or \qquad t_{\alpha/2}[N-K] = \sqrt{F_\alpha[1, N-K]} \,.$$

The square of the two-tailed α/2 critical value of the t[N−K] distribution equals the α-level critical value of the F[1, N−K] distribution; *or* the two-tailed α/2 critical value of the t[N−K] distribution equals the square root of the α-level critical value of the F[1, N−K] distribution.

3. The ***p-values* for the calculated *sample values* of the test statistics t_0 and F_0** are related as follows:

two-tailed **p-value for t_0 = p-value for F_0**

where

two-tailed **p-value for t_0** = $\mathbf{Pr(|t| > |t_0|)}$

p-value for F_0 = $\mathbf{Pr(F > F_0)}$.