ECON 351* -- NOTE 19

F-Tests of Exclusion Restrictions on Regression Coefficients: Numerical Examples 2

Consider two alternative regression models of North American car prices.

- The **sample data** consist of **74 observations** on the following variables:

  $price_i$ = the price of the i-th car (in US dollars);
  wgt_i = the weight of the i-th car (in pounds);
  mpg_i = the fuel efficiency of the i-th car (in miles per gallon);
  N = 74 = the number of observations in the estimation sample.

  Data Source: Stata-format data set **auto.dta** supplied with *Stata Release 8*.

♦ **Model 1**: is given by the PRE

$$price_i = \beta_0 + \beta_1 wgt_i + \beta_2 mpg_i + u_i \qquad \textbf{(1)}$$

This model contains two explanatory variables, wgt_i and mpg_i.

Model 1 assumes that the ***marginal effect*** **of each explanatory variable** is ***a constant***; it is ***linear*** in the explanatory variables wgt_i and mpg_i.

marginal effect of *wgt*$_i$ = $\partial price_i / \partial wgt_i = \beta_1$ = a ***constant***;
marginal effect of *mpg*$_i$ = $\partial price_i / \partial mpg_i = \beta_2$ = a ***constant***.

```
. regress price wgt mpg

  Source |       SS       df       MS                  Number of obs =      74
---------+------------------------------               F(  2,    71) =   14.74
   Model |   186321280     2  93160639.9               Prob > F      =  0.0000
Residual |   448744116    71  6320339.67               R-squared     =  0.2934
---------+------------------------------               Adj R-squared =  0.2735
   Total |   635065396    73  8699525.97               Root MSE      =  2514.0

------------------------------------------------------------------------------
   price |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
     wgt |   1.746559   .6413538      2.723   0.008       .4677361    3.025382
     mpg |  -49.51222   86.15604     -0.575   0.567      -221.3025     122.278
   _cons |   1946.069    3597.05      0.541   0.590      -5226.244    9118.382
------------------------------------------------------------------------------
```

♦ **<u>Model 2</u>**: is given by the PRE

$$\text{price}_i = \beta_0 + \beta_1\text{wgt}_i + \beta_2\text{mpg}_i + \beta_3\text{wgt}_i^2 + \beta_4\text{mpg}_i^2 + \beta_5\text{wgt}_i\text{mpg}_i + u_i \qquad \textbf{(2)}$$

This model also contains the two explanatory variables wgt_i and mpg_i.

But Model 2 specifies that the ***marginal effect*** **of each explanatory variable** is **a *linear* function of both *wgt*** —

marginal effect of wgt_i =

$$\partial\,\text{price}_i / \partial\,\text{wgt}_i = \beta_1 + 2\beta_3\text{wgt}_i + \beta_5\text{mpg}_i$$ = a ***linear*** **function of** wgt_i **and** mpg_i;

marginal effect of mpg_i =

$$\partial\,\text{price}_i / \partial\,\text{mpg}_i = \beta_2 + 2\beta_4\text{mpg}_i + \beta_5\text{wgt}_i$$ = a ***linear*** **function of** wgt_i **and** mpg_i.

- The ***unrestricted*** **OLS SRE** is written in general as

$$\text{price}_i = \hat{\beta}_0 + \hat{\beta}_1\text{wgt}_i + \hat{\beta}_2\text{mpg}_i + \hat{\beta}_3\text{wgt}_i^2 + \hat{\beta}_4\text{mpg}_i^2 + \hat{\beta}_5\text{wgt}_i\text{mpg}_i + \hat{u}_i \qquad \textbf{(2*)}$$

where the $\hat{\beta}_j$ (j = 0, …, 5) are the ***unrestricted*** **OLS coefficient estimates** and the $\hat{u}_i$ (i = 1, …, N) are the ***unrestricted*** **OLS residuals**.

```
. regress price wgt mpg wgtsq mpgsq wgtmpg

  Source |       SS       df       MS                  Number of obs =      74
---------+------------------------------               F(  5,    68) =   12.84
   Model |   308384833     5  61676966.6               Prob > F      =  0.0000
Residual |   326680563    68  4804125.93               R-squared     =  0.4856
---------+------------------------------               Adj R-squared =  0.4478
   Total |   635065396    73  8699525.97               Root MSE      =  2191.8

------------------------------------------------------------------------------
   price |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
     wgt |  -31.88985   9.148215     -3.486   0.001      -50.14483   -13.63487
     mpg |  -3549.495   1126.464     -3.151   0.002      -5797.318   -1301.672
   wgtsq |   .0034574   .0008629      4.007   0.000       .0017355    .0051792
   mpgsq |   38.74472   12.62339      3.069   0.003       13.55514     63.9343
  wgtmpg |   .5421927   .1971854      2.750   0.008       .1487154    .9356701
   _cons |   92690.55   25520.53      3.632   0.001       41765.12      143616
------------------------------------------------------------------------------
```

- Recall from *Note 13* the **two alternative forms of the <u>ANOVA F-statistic</u>**:

$$\text{ANOVA} - F = \frac{ESS_1/(K-1)}{RSS_1/(N-K)} = \frac{ESS_1}{RSS_1}\frac{(N-K)}{(K-1)} \qquad \textbf{(A1)}$$

$$\text{ANOVA} - F = \frac{R_U^2/(K-1)}{(1-R_U^2)/(N-K)} = \frac{R_U^2}{(1-R_U^2)}\frac{(N-K)}{(K-1)} \qquad \textbf{(A2)}$$

- Recall from *Note 17* the **two alternative forms of the <u>general F-statistic</u>**:

$$F = \frac{(RSS_0 - RSS_1)/(df_0 - df_1)}{RSS_1/df_1} = \frac{(RSS_0 - RSS_1)}{RSS_1}\frac{df_1}{(df_0 - df_1)}. \qquad \textbf{(F1)}$$

$$F = \frac{(R_U^2 - R_R^2)/(df_0 - df_1)}{(1-R_U^2)/df_1} = \frac{(R_U^2 - R_R^2)}{(1-R_U^2)}\frac{df_1}{(df_0 - df_1)}. \qquad \textbf{(F2)}$$

$df_0 = N - K_0$ = degrees of freedom for the restricted RSS, RSS_0;
$df_1 = N - K_1 = N - K$ = degrees of freedom for the unrestricted RSS, RSS_1;
$df_0 - df_1 = N - K_0 - (N - K_1) = N - K_0 - N + K_1 = K_1 - K_0 = K - K_0$;
K_0 = the number of free regression coefficients in the restricted model;
$K = K_1$ = the number of free regression coefficients in the unrestricted model.

- ***Null distribution*** of F-statistic:

$F \sim F[df_0 - df_1, df_1] = F[K - K_0, N - K]$ under H_0.

1. The ANOVA F-Test of Zero Restrictions on *All* Slope Coefficients in Model 1

♦ **Model 1** is given by the population regression equation (PRE):

$$\text{price}_i = \beta_0 + \beta_1 \text{wgt}_i + \beta_2 \text{mpg}_i + u_i \qquad \textbf{(1)}$$

♦ The ***null*** **hypothesis** is:

H_0: $\beta_j = 0 \quad \forall \; j = 1, 2;$ $\quad \beta_1 = 0$ *and* $\beta_2 = 0$.

♦ The ***alternative*** **hypothesis** is:

H_1: $\beta_j \neq 0 \quad j = 1, 2;$ $\quad \beta_1 \neq 0$ *and/or* $\beta_2 \neq 0$.

Test statistic: For this test, we can **use *either* the ANOVA F-statistic *or* the general F-statistic**.

<u>ANOVA F-test</u>: uses *only* the *unrestricted* OLS SRE.

- The ***<u>unrestricted</u>* <u>model</u>** corresponding to the alternative hypothesis H_1 is simply PRE (1):

$$\text{price}_i = \beta_0 + \beta_1 \text{wgt}_i + \beta_2 \text{mpg}_i + u_i \qquad \textbf{(1)}$$

Number of free (unrestricted) regression coefficients in (1) is $\mathbf{K = K_1 = 3}$.

- OLS estimation of the unrestricted model (1) yields the ***unrestricted* OLS SRE**:

$$\text{price}_i = \hat{\beta}_0 + \hat{\beta}_1 \text{wgt}_i + \hat{\beta}_2 \text{mpg}_i + \hat{u}_i \qquad \textbf{(1*)}$$

(1) The **OLS decomposition equation for the *unrestricted* OLS SRE** is

$$\underset{(N-1)}{TSS} = \underset{(K-1)}{ESS_1} + \underset{(N-K)}{RSS_1}.$$

$TSS = 635065396.0$ *with* $df = N - 1 = 74 - 1 = 73$
$ESS_1 = 186321280.0$ *with* $df = K - 1 = 3 - 1 = 2$
$\mathbf{RSS_1 = 448744116.0}$ ***with*** $\mathbf{df_1 = N - K = 74 - 3 = 71}$

(2) The $\mathbf{R^2}$ **for the *unrestricted* OLS SRE** is

$$R_U^2 = \frac{ESS_1}{TSS} = 1 - \frac{RSS_1}{TSS} = \mathbf{0.2934}.$$

- Compute the ***sample value* of either of the ANOVA F-statistics**.

$$F_0 = \frac{ESS_1/(K-1)}{RSS_1/(N-K)} = \frac{186321280/(3-1)}{448744116/(74-3)} = \frac{186321280/2}{448744116/71} = \mathbf{14.74}$$

$$F_0 = \frac{R_U^2/(K-1)}{(1-R_U^2)/(N-K)} = \frac{0.2934/(3-1)}{(1-0.2934)/(74-3)} = \frac{0.2934/2}{0.7066/71} = \mathbf{14.74}$$

- ***Null distribution*** of the F-statistic: $\mathbf{F \sim F[2,71]}$ under H_0.

The ***critical values* of the F[2,71]-distribution** are:

for $\alpha = 0.05$: $F_\alpha[2,71] = F_{0.05}[2,71] = 3.13.$
for $\alpha = 0.01$: $F_\alpha[2,71] = F_{0.01}[2,71] = 4.92.$

- **Inference at α = 0.05 (5% significance level)**:

Since $F_0 = 14.74 > 3.13 = F_{0.05}[2,71]$, ***reject*** $\mathbf{H_0}$ at the ***5%*** significance level.

- **Inference at α = 0.01 (1% significance level)**:

Since $F_0 = 14.74 > 4.92 = F_{0.01}[2,71]$, ***reject*** $\mathbf{H_0}$ at the ***1%*** significance level.

<u>General F-test</u>: uses ***both* the unrestricted *and* restricted OLS SREs**.

$$F = \frac{(RSS_0 - RSS_1)/(df_0 - df_1)}{RSS_1/df_1} = \frac{(RSS_0 - RSS_1)}{RSS_1}\,\frac{df_1}{(df_0 - df_1)} \qquad \textbf{(F1)}$$

$$F = \frac{(R_U^2 - R_R^2)/(df_0 - df_1)}{(1 - R_U^2)/df_1} = \frac{(R_U^2 - R_R^2)}{(1 - R_U^2)}\,\frac{df_1}{(df_0 - df_1)} \qquad \textbf{(F2)}$$

- The ***<u>unrestricted</u>* <u>model</u>** corresponding to the ***alternative* hypothesis H_1** is simply PRE (1):

$$price_i = \beta_0 + \beta_1 wgt_i + \beta_2 mpg_i + u_i \qquad \textbf{(1)}$$

Number of free (unrestricted) regression coefficients in (1) is $\mathbf{K = K_1 = 3}$.

- OLS estimation of the unrestricted model yields the ***unrestricted* OLS SRE**:

$$price_i = \hat{\beta}_0 + \hat{\beta}_1 wgt_i + \hat{\beta}_2 mpg_i + \hat{u}_i \qquad \textbf{(1*)}$$

(1) The **OLS decomposition equation for the *unrestricted* OLS SRE** is

$$\underset{(N-1)}{TSS} = \underset{(K-1)}{ESS_1} + \underset{(N-K)}{RSS_1}.$$

$TSS = 635065396.0$ *with* $df = N - 1 = 74 - 1 = 73$
$ESS_1 = 186321280.0$ *with* $df = K - 1 = 3 - 1 = 2$
$\mathbf{RSS_1 = 448744116.0}$ ***with*** $\mathbf{df_1 = N - K = 74 - 3 = 71}$

(2) The $\mathbf{R^2}$ **for the *unrestricted* OLS SRE** is

$$R_U^2 = \frac{ESS_1}{TSS} = 1 - \frac{RSS_1}{TSS} = \mathbf{0.2934}.$$

- The ***restricted* model** corresponding to the ***null* hypothesis H_0** is obtained by setting $\beta_1 = 0$ and $\beta_2 = 0$ in the unrestricted model (1):

$$\text{price}_i = \beta_0 + u_i \qquad \textbf{(0)}$$

Number of free (unrestricted) regression coefficients is $\mathbf{K_0 = 1}$.

- OLS estimation of the restricted model yields the ***restricted* OLS SRE**:

$$\text{price}_i = \tilde{\beta}_0 + \tilde{u}_i \qquad \textbf{(0*)}$$

(1) The **OLS decomposition equation for the *restricted* OLS SRE** is

$$\underset{(N-1)}{TSS} = \underset{(K_0-1)}{ESS_0} + \underset{(N-K_0)}{RSS_0}\,.$$

TSS = 635065396.0 *with* df = N −1 = 74 − 1 = 73
ESS_0 = 0.0 *with* df = K_0 −1 = 1 − 1 = 0
RSS_0 = 635065396.0 *with* df_0 = N − K_0 = 74 − 1 = 73

(2) The **R^2 for the *restricted* OLS SRE** is

$$R_R^2 = \frac{ESS_0}{TSS} = 1 - \frac{RSS_0}{TSS} = \mathbf{0.0}.$$

- Compute the ***sample value* of either of the general F-statistics**.

  - $F = \dfrac{(RSS_0 - RSS_1)/(df_0 - df_1)}{RSS_1/df_1} \sim F[df_0 - df_1, df_1] = F[K - K_0, N - K]$

    RSS_0 = 635065396.0 and $df_0 = N - K_0 = 74 - 1 = 73$
    RSS_1 = 448744116.0 and $df_1 = N - K = 74 - 3 = 71$
    $RSS_0 - RSS_1$ = 635065396.0 – 448744116.0 = 186321280
    $df_0 - df_1 = 73 - 71 = 2$ or $K - K_0 = 3 - 1 = 2$.

    $$F_0 = \frac{(635065396 - 448744116)/(73-71)}{448744116/(74-3)} = \frac{186321280/2}{448744116/71} = \mathbf{14.74}$$

  - $F = \dfrac{(R_U^2 - R_R^2)/(df_0 - df_1)}{(1 - R_U^2)/df_1} \sim F[df_0 - df_1, df_1] = F[K - K_0, N - K]$

    $$R_U^2 = \frac{ESS_1}{TSS} = 1 - \frac{RSS_1}{TSS} = \mathbf{0.2934}; \quad R_R^2 = \frac{ESS_0}{TSS} = 1 - \frac{RSS_0}{TSS} = \mathbf{0.0}.$$

    $df_0 = N - K_0 = 74 - 1 = 73$
    $df_1 = N - K = 74 - 3 = 71$
    $df_0 - df_1 = K - K_0 = 73 - 71 = 2$ or $K - K_0 = 3 - 1 = 2$.

    $$F_0 = \frac{(0.2934 - 0.0)/(3-1)}{(1-0.2934)/(74-3)} = \frac{0.2934/2}{0.7066/71} = \mathbf{14.74}$$

- ***Null distribution*** of the F-statistic: **F ~ F[2,71]** under H_0.

  The ***critical values* of the F[2,71]-distribution** are:

  for $\alpha = 0.05$: $F_\alpha[2,71] = F_{0.05}[2,71] = 3.13$.
  for $\alpha = 0.01$: $F_\alpha[2,71] = F_{0.01}[2,71] = 4.92$.

- **Inference at α = 0.05 (5% significance level)**:

  Since $F_0 = 14.74 > 3.13 = F_{0.05}[2,71]$, ***reject*** $\mathbf{H_0}$ at the ***5%*** significance level.

- **Inference at α = 0.01 (1% significance level)**:

  Since $F_0 = 14.74 > 4.92 = F_{0.01}[2,71]$, ***reject*** $\mathbf{H_0}$ at the ***1%*** significance level.

❑ ***Conclusion -- ANOVA F-test***

Reject the ***restricted* model corresponding to $\mathbf{H_0}$ (Model 0)**

$$price_i = \beta_0 + u_i \qquad \textbf{(0)}$$

in favour of the ***unrestricted* model corresponding to $\mathbf{H_1}$ (Model 1)**

$$price_i = \beta_0 + \beta_1 wgt_i + \beta_2 mpg_i + u_i \qquad \textbf{(1)}$$

2. Tests of Exclusion Restrictions on *Subsets* of Slope Coefficients

<u>TEST 1</u>: Test Model 1 against Model 2

$$\text{price}_i = \beta_0 + \beta_1 \text{wgt}_i + \beta_2 \text{mpg}_i + u_i \qquad \textbf{(1)}$$

$$\text{price}_i = \beta_0 + \beta_1 \text{wgt}_i + \beta_2 \text{mpg}_i + \beta_3 \text{wgt}_i^2 + \beta_4 \text{mpg}_i^2 + \beta_5 \text{wgt}_i \text{mpg}_i + u_i \qquad \textbf{(2)}$$

- What coefficient restrictions does PRE (1) impose on PRE (2)?

  Answer: $\boldsymbol{\beta_3 = \beta_4 = \beta_5 = 0}$

♦ The ***null* hypothesis** is:

H_0: $\beta_j = 0 \quad \forall \; j = 3, 4, 5$; $\beta_3 = 0$ *and* $\beta_4 = 0$ *and* $\beta_5 = 0$.

The ***restricted* model implied by $\mathbf{H_0}$** is obtained by setting $\beta_3 = 0$ and $\beta_4 = 0$ and $\beta_5 = 0$ in the unrestricted model (2):

$$\text{price}_i = \beta_0 + \beta_1 \text{wgt}_i + \beta_2 \text{mpg}_i + u_i \qquad \textbf{(1)}$$

♦ The ***alternative* hypothesis** is:

H_1: $\beta_j \neq 0 \quad j = 3, 4, 5$; $\beta_3 \neq 0$ *and/or* $\beta_4 \neq 0$ *and/or* $\beta_5 \neq 0$.

The ***unrestricted* model implied by $\mathbf{H_1}$** is:

$$\text{price}_i = \beta_0 + \beta_1 \text{wgt}_i + \beta_2 \text{mpg}_i + \beta_3 \text{wgt}_i^2 + \beta_4 \text{mpg}_i^2 + \beta_5 \text{wgt}_i \text{mpg}_i + u_i \qquad \textbf{(2)}$$

Test statistic: For this test, we **can use *only* the general F-test**.

- The ***unrestricted* model** corresponding to the ***alternative* hypothesis H_1** is simply PRE (2):

$$\text{price}_i = \beta_0 + \beta_1\text{wgt}_i + \beta_2\text{mpg}_i + \beta_3\text{wgt}_i^2 + \beta_4\text{mpg}_i^2 + \beta_5\text{wgt}_i\text{mpg}_i + u_i \qquad \textbf{(2)}$$

Number of free (unrestricted) regression coefficients in (2) is $\mathbf{K = K_1 = 6}$.

- OLS estimation of the unrestricted model yields the ***unrestricted* OLS SRE**:

$$\text{price}_i = \hat\beta_0 + \hat\beta_1\text{wgt}_i + \hat\beta_2\text{mpg}_i + \hat\beta_3\text{wgt}_i^2 + \hat\beta_4\text{mpg}_i^2 + \hat\beta_5\text{wgt}_i\text{mpg}_i + \hat u_i \qquad \textbf{(2*)}$$

(1) The **OLS decomposition equation for the *unrestricted* OLS SRE** is

$$\underset{(N-1)}{TSS} = \underset{(K-1)}{ESS_1} + \underset{(N-K)}{RSS_1}.$$

TSS = 635065396.0 *with* df = N −1 = 74 − 1 = 73
ESS_1 = 308384833.0 *with* df = K −1 = 6 − 1 = 5
RSS_1 = 326680563.0 ***with*** **df_1 = N − K = 74 − 6 = 68**

(2) The **R^2 for the *unrestricted* OLS SRE** is

$$R_U^2 = \frac{ESS_1}{TSS} = 1 - \frac{RSS_1}{TSS} = \mathbf{0.4856}.$$

- The ***restricted* model** corresponding to the ***null* hypothesis H_0** is obtained by setting $\beta_3 = 0$ and $\beta_4 = 0$ and $\beta_5 = 0$ in the unrestricted model (2):

$$\text{price}_i = \beta_0 + \beta_1 \text{wgt}_i + \beta_2 \text{mpg}_i + u_i \qquad \textbf{(1)}$$

Number of free (unrestricted) regression coefficients is $\mathbf{K_0 = 3}$.

- OLS estimation of the restricted model yields the ***restricted* OLS SRE**:

$$\text{price}_i = \tilde{\beta}_0 + \tilde{\beta}_1 \text{wgt}_i + \tilde{\beta}_2 \text{mpg}_i + \tilde{u}_i \qquad \textbf{(1*)}$$

(1) The **OLS decomposition equation for the *restricted* OLS SRE** is

$$\underset{(N-1)}{TSS} = \underset{(K_0-1)}{ESS_0} + \underset{(N-K_0)}{RSS_0}.$$

TSS = 635065396.0 *with* df = N –1 = 74 – 1 = 73
ESS_0 = 186321280.0 *with* df = K_0 –1 = 3 – 1 = 2
RSS_0 = 448744116.0 ***with*** **$df_0 = N - K_0$ = 74 – 3 = 71**

(2) The **R^2 for the *restricted* OLS SRE** is

$$R_R^2 = \frac{ESS_0}{TSS} = 1 - \frac{RSS_0}{TSS} = \mathbf{0.2934}.$$

- Compute the ***sample value*** **of either of the general F-statistics**.

  - $$F = \frac{(RSS_0 - RSS_1)/(df_0 - df_1)}{RSS_1/df_1} \sim F[df_0 - df_1, df_1] = F[K - K_0, N - K]$$

    RSS_0 = 448744116.0 *with* $df_0 = N - K_0 = 74 - 3 = 71$
    RSS_1 = 326680563.0 *with* $df_1 = N - K = 74 - 6 = 68$
    $RSS_0 - RSS_1$ = 448744116.0 – 326680563.0 = 186321280
    $df_0 - df_1 = 71 - 68 = 3$ *or* $K - K_0 = 6 - 3 = 3$.

    $$F_0 = \frac{(448744116 - 326680563)/(71-68)}{326680563/68} = \frac{122063553/3}{326680563/68} = \mathbf{8.469}$$

  - $$F = \frac{(R_U^2 - R_R^2)/(df_0 - df_1)}{(1 - R_U^2)/df_1} \sim F[df_0 - df_1, df_1] = F[K - K_0, N - K]$$

    $$R_U^2 = \frac{ESS_1}{TSS} = 1 - \frac{RSS_1}{TSS} = \mathbf{0.4856}; \quad R_R^2 = \frac{ESS_0}{TSS} = 1 - \frac{RSS_0}{TSS} = \mathbf{0.\ 2934}.$$

    $df_0 = N - K_0 = 74 - 3 = 71$
    $df_1 = N - K = 74 - 6 = 68$
    $df_0 - df_1 = 71 - 68 = 3$ or $K - K_0 = 6 - 3 = 3$.

    $$F_0 = \frac{(0.4856 - 0.2934)/(71-68)}{(1 - 0.4856)/68} = \frac{0.1922/3}{0.5144/68} = \mathbf{8.469}$$

- ***Null distribution*** of the F-statistic: **F ~ F[3,68]** under H_0.

  The ***critical values*** **of the F[3,68]-distribution** are:

  for $\alpha = 0.05$: $F_\alpha[3,68] = F_{0.05}[3,68] = 2.74$.
  for $\alpha = 0.01$: $F_\alpha[3,68] = F_{0.01}[3,68] = 4.08$.

- **Inference at α = 0.05 (5% significance level)**:

  Since $F_0 = 8.469 > 2.74 = F_{0.05}[3,68]$, ***reject*** $\mathbf{H_0}$ at the ***5%*** significance level.

- **Inference at α = 0.01 (1% significance level)**:

  Since $F_0 = 8.469 > 4.08 = F_{0.01}[3,68]$, ***reject*** $\mathbf{H_0}$ at the ***1%*** significance level.

❑ ***Conclusion -- TEST 1***

Reject the ***restricted*** **model (Model 1)**

$$\text{price}_i = \beta_0 + \beta_1\text{wgt}_i + \beta_2\text{mpg}_i + u_i \qquad \textbf{(1)}$$

in favour of the ***unrestricted*** **model (Model 2)**

$$\text{price}_i = \beta_0 + \beta_1\text{wgt}_i + \beta_2\text{mpg}_i + \beta_3\text{wgt}_i^2 + \beta_4\text{mpg}_i^2 + \beta_5\text{wgt}_i\text{mpg}_i + u_i \qquad \textbf{(2)}$$

<u>TEST 2</u>: Test that *marginal effect of wgt_i* equals *zero* in Model 2

Proposition: The variable wgt_i has no effect on car price; car price is unrelated to car weight when the effect of fuel efficiency (mpg_i) on car price is controlled for.

$$price_i = \beta_0 + \beta_1 wgt_i + \beta_2 mpg_i + \beta_3 wgt_i^2 + \beta_4 mpg_i^2 + \beta_5 wgt_i mpg_i + u_i \qquad \textbf{(2)}$$

- **Marginal effect of *wgt_i* in Model 2** =

$$\partial price_i / \partial wgt_i = \beta_1 + 2\beta_3 wgt_i + \beta_5 mpg_i$$

A sufficient condition for $\partial price_i / \partial wgt_i = 0$ is: $\boldsymbol{\beta_1 = \beta_3 = \beta_5 = 0}$.

- The ***null*** **hypothesis** is:

H_0: $\beta_j = 0 \quad \forall \; j = 1, 3, 5$; $\beta_1 = 0$ *and* $\beta_3 = 0$ *and* $\beta_5 = 0$.

The ***restricted*** **model** implied by H_0 is:

$$price_i = \beta_0 + \beta_2 mpg_i + \beta_4 mpg_i^2 + u_i \qquad \textbf{(3)}$$

- The ***alternative*** **hypothesis** is:

H_1: $\beta_j \neq 0 \quad j = 1, 3, 5$; $\beta_1 \neq 0$ *and/or* $\beta_3 \neq 0$ *and/or* $\beta_5 \neq 0$.

The ***unrestricted*** **model** implied by H_1 is:

$$price_i = \beta_0 + \beta_1 wgt_i + \beta_2 mpg_i + \beta_3 wgt_i^2 + \beta_4 mpg_i^2 + \beta_5 wgt_i mpg_i + u_i \qquad \textbf{(2)}$$

Test statistic: For this test, **we can use *only* the general F-test**.

- The ***unrestricted* model** corresponding to the ***alternative* hypothesis $\mathbf{H_1}$** is simply PRE (2):

$$\text{price}_i = \beta_0 + \beta_1\text{wgt}_i + \beta_2\text{mpg}_i + \beta_3\text{wgt}_i^2 + \beta_4\text{mpg}_i^2 + \beta_5\text{wgt}_i\text{mpg}_i + u_i \qquad \textbf{(2)}$$

Number of free (unrestricted) regression coefficients in (2) is $\mathbf{K = K_1 = 6}$.

- OLS estimation of the unrestricted model yields the ***unrestricted* OLS SRE**:

$$\text{price}_i = \hat\beta_0 + \hat\beta_1\text{wgt}_i + \hat\beta_2\text{mpg}_i + \hat\beta_3\text{wgt}_i^2 + \hat\beta_4\text{mpg}_i^2 + \hat\beta_5\text{wgt}_i\text{mpg}_i + \hat u_i \qquad \textbf{(2*)}$$

(1) The **OLS decomposition equation for the *unrestricted* OLS SRE** is

$$\underset{(N-1)}{\text{TSS}} = \underset{(K-1)}{\text{ESS}_1} + \underset{(N-K)}{\text{RSS}_1}\,.$$

TSS = 635065396.0 *with* df = N −1 = 74 – 1 = 73
ESS_1 = 308384833.0 *with* df = K −1 = 6 – 1 = 5
$\mathbf{RSS_1}$ **= 326680563.0** ***with*** $\mathbf{df_1 = N - K = 74 - 6 = 68}$

(2) The $\mathbf{R^2}$ **for the *unrestricted* OLS SRE** is

$$R_U^2 = \frac{\text{ESS}_1}{\text{TSS}} = 1 - \frac{\text{RSS}_1}{\text{TSS}} = \mathbf{0.4856}.$$

- The ***restricted* model** corresponding to the ***null* hypothesis $\mathbf{H_0}$** is:

$$\text{price}_i = \beta_0 + \beta_2 \text{mpg}_i + \beta_4 \text{mpg}_i^2 + u_i \qquad \textbf{(3)}$$

Number of free (unrestricted) regression coefficients in (3) is $\mathbf{K_0 = 3}$.

- OLS estimation of the restricted model (3) yields the ***restricted* OLS SRE**:

$$\text{price}_i = \tilde{\beta}_0 + \tilde{\beta}_2 \text{mpg}_i + \tilde{\beta}_4 \text{mpg}_i^2 + \tilde{u}_i \qquad \textbf{(3*)}$$

(1) The **OLS decomposition equation for the *restricted* OLS SRE** is

$$\underset{(N-1)}{\text{TSS}} = \underset{(K_0-1)}{\text{ESS}_0} + \underset{(N-K_0)}{\text{RSS}_0}\,.$$

TSS = 635065396.0 *with* df = N −1 = 74 – 1 = 73
ESS_0 = 215835615.0 *with* df = K_0 −1 = 3 – 1 = 2
$\mathbf{RSS_0}$ = 419229781.0 ***with*** **$\mathbf{df_0 = N - K_0}$ = 74 – 3 = 71**

(2) The **$\mathbf{R^2}$ for the *restricted* OLS SRE** is

$$R_R^2 = \frac{\text{ESS}_0}{\text{TSS}} = 1 - \frac{\text{RSS}_0}{\text{TSS}} = \mathbf{0.3399}.$$

- Compute the ***sample value* of either of the general F-statistics**.

  - $F = \dfrac{(RSS_0 - RSS_1)/(df_0 - df_1)}{RSS_1/df_1} \sim F[df_0 - df_1, df_1] = F[K - K_0, N - K]$

RSS_0 = 419229781.0 *with* $df_0 = N - K_0 = 74 - 3 = 71$
RSS_1 = 326680563.0 *with* $df_1 = N - K = 74 - 6 = 68$
$RSS_0 - RSS_1$ = 419229781.0 – 326680563.0 = 92549218
$df_0 - df_1 = 71 - 68 = 3$ *or* $K - K_0 = 6 - 3 = 3$.

$$F_0 = \frac{(419229781 - 326680563)/(71-68)}{326680563/68} = \frac{92549218/3}{326680563/68} = \mathbf{6.422}$$

  - $F = \dfrac{(R_U^2 - R_R^2)/(df_0 - df_1)}{(1 - R_U^2)/df_1} \sim F[df_0 - df_1, df_1] = F[K - K_0, N - K]$

$R_U^2 = \dfrac{ESS_1}{TSS} = 1 - \dfrac{RSS_1}{TSS} = \mathbf{0.4856}$; $R_R^2 = \dfrac{ESS_0}{TSS} = 1 - \dfrac{RSS_0}{TSS} = \mathbf{0.\ 3399}$.

$df_0 = N - K_0 = 74 - 3 = 71$
$df_1 = N - K = 74 - 6 = 68$
$df_0 - df_1 = 71 - 68 = 3$ *or* $K - K_0 = 6 - 3 = 3$.

$$F_0 = \frac{(0.4856 - 0.3399)/(71-68)}{(1-0.4856)/68} = \frac{0.1457/3}{0.5144/68} = \mathbf{6.422}$$

- ***Null distribution*** of the F-statistic: $\mathbf{F \sim F[3,68]}$ under H_0.

  The ***critical values* of the F[3,68]-distribution** are:

  for $\alpha = 0.05$: $F_\alpha[3,68] = F_{0.05}[3,68] = 2.74$.
  for $\alpha = 0.01$: $F_\alpha[3,68] = F_{0.01}[3,68] = 4.08$.

- **Inference at α = 0.05 (5% significance level)**:

  Since $F_0 = 6.422 > 2.74 = F_{0.05}[3,68]$, ***reject*** $\mathbf{H_0}$ at the ***5%*** significance level.

- **Inference at α = 0.01 (1% significance level)**:

  Since $F_0 = 6.422 > 4.08 = F_{0.01}[3,68]$, ***reject*** $\mathbf{H_0}$ at the ***1%*** significance level.

❑ ***Conclusion -- TEST 2***

Reject the ***restricted*** **model (Model 3)**

$$\text{price}_i = \beta_0 + \beta_2\text{mpg}_i + \beta_4\text{mpg}_i^2 + u_i \qquad \textbf{(3)}$$

in favour of the ***unrestricted*** **model (Model 2)**

$$\text{price}_i = \beta_0 + \beta_1\text{wgt}_i + \beta_2\text{mpg}_i + \beta_3\text{wgt}_i^2 + \beta_4\text{mpg}_i^2 + \beta_5\text{wgt}_i\text{mpg}_i + u_i \qquad \textbf{(2)}$$

<u>TEST 3</u>: Test that *marginal effect of* wgt_i is *constant* in Model 2

Proposition: The variable wgt_i has a constant marginal effect on car price; the marginal effect wgt_i on car price does not vary with wgt_i and mpg_i.

$$price_i = \beta_0 + \beta_1 wgt_i + \beta_2 mpg_i + \beta_3 wgt_i^2 + \beta_4 mpg_i^2 + \beta_5 wgt_i mpg_i + u_i \qquad \textbf{(2)}$$

- **Marginal effect of** wgt_i **in Model 2** =

$$\partial price_i / \partial wgt_i = \beta_1 + 2\beta_3 wgt_i + \beta_5 mpg_i .$$

A sufficient condition for $\partial price_i / \partial wgt_i = \beta_1$ = a constant is: $\boldsymbol{\beta_3 = \beta_5 = 0}$.

- The ***null* hypothesis** is:

H_0: $\beta_j = 0 \quad \forall \; j = 3, 5;$ $\qquad \beta_3 = 0$ *and* $\beta_5 = 0$.

The ***restricted* model implied by H_0** is:

$$price_i = \beta_0 + \beta_1 wgt_i + \beta_2 mpg_i + \beta_4 mpg_i^2 + u_i \qquad \textbf{(4)}$$

- The ***alternative* hypothesis** is:

H_1: $\beta_j \neq 0 \qquad j = 3, 5;$ $\qquad \beta_3 \neq 0$ *and/or* $\beta_5 \neq 0$.

The ***unrestricted* model implied by H_1** is Model 2:

$$price_i = \beta_0 + \beta_1 wgt_i + \beta_2 mpg_i + \beta_3 wgt_i^2 + \beta_4 mpg_i^2 + \beta_5 wgt_i mpg_i + u_i \qquad \textbf{(2)}$$

Test statistic: For this test, **we can use *only* the general F-statistic**.

- The ***unrestricted* model** corresponding to the ***alternative* hypothesis $\mathbf{H_1}$** is simply PRE (2):

$$\text{price}_i = \beta_0 + \beta_1 \text{wgt}_i + \beta_2 \text{mpg}_i + \beta_3 \text{wgt}_i^2 + \beta_4 \text{mpg}_i^2 + \beta_5 \text{wgt}_i \text{mpg}_i + u_i \qquad \textbf{(2)}$$

Number of free (unrestricted) regression coefficients in (2) is $\mathbf{K = K_1 = 6}$.

- OLS estimation of the unrestricted model yields the ***unrestricted* OLS SRE**:

$$\text{price}_i = \hat\beta_0 + \hat\beta_1 \text{wgt}_i + \hat\beta_2 \text{mpg}_i + \hat\beta_3 \text{wgt}_i^2 + \hat\beta_4 \text{mpg}_i^2 + \hat\beta_5 \text{wgt}_i \text{mpg}_i + \hat u_i \qquad \textbf{(2*)}$$

(1) The **OLS decomposition equation for the *unrestricted* OLS SRE** is

$$\underset{(N-1)}{\text{TSS}} = \underset{(K-1)}{\text{ESS}_1} + \underset{(N-K)}{\text{RSS}_1}.$$

TSS = 635065396.0 *with* df = N −1 = 74 − 1 = 73
ESS_1 = 308384833.0 *with* df = K −1 = 6 − 1 = 5
$\mathbf{RSS_1}$ = 326680563.0 ***with*** **$\mathbf{df_1}$ = N − K = 74 − 6 = 68**

(2) The **$\mathbf{R^2}$ for the *unrestricted* OLS SRE** is

$$R_U^2 = \frac{\text{ESS}_1}{\text{TSS}} = 1 - \frac{\text{RSS}_1}{\text{TSS}} = \mathbf{0.4856}.$$

- The ***restricted* model** corresponding to the ***null* hypothesis H_0** is:

$$\text{price}_i = \beta_0 + \beta_1 \text{wgt}_i + \beta_2 \text{mpg}_i + \beta_4 \text{mpg}_i^2 + u_i \qquad \textbf{(4)}$$

Number of free (unrestricted) regression coefficients in (4) is $\mathbf{K_0 = 4}$.

- OLS estimation of the restricted model (4) yields the ***restricted* OLS SRE**:

$$\text{price}_i = \tilde{\beta}_0 + \tilde{\beta}_1 \text{wgt}_i + \tilde{\beta}_2 \text{mpg}_i + \tilde{\beta}_4 \text{mpg}_i^2 + \tilde{u}_i \qquad \textbf{(4*)}$$

(1) The **OLS decomposition equation for the *restricted* OLS SRE** is

$$\underset{(N-1)}{\text{TSS}} = \underset{(K_0-1)}{\text{ESS}_0} + \underset{(N-K_0)}{\text{RSS}_0}.$$

TSS = 635065396.0 *with* df = N −1 = 74 − 1 = 73
ESS_0 = 223815416.0 *with* df = K_0 −1 = 4 − 1 = 3
$\mathbf{RSS_0}$ = 411249980.0 ***with*** **$\mathbf{df_0 = N - K_0}$ = 74 − 4 = 70**

(2) The **$\mathbf{R^2}$ for the *restricted* OLS SRE** is

$$R_R^2 = \frac{\text{ESS}_0}{\text{TSS}} = 1 - \frac{\text{RSS}_0}{\text{TSS}} = \mathbf{0.3524}.$$

- Compute the ***sample value*** **of either of the general F-statistics**.

  ▪ $$F = \frac{(RSS_0 - RSS_1)/(df_0 - df_1)}{RSS_1/df_1} \sim F[df_0 - df_1, df_1] = F[K - K_0, N - K]$$

  RSS_0 = 411249980.0 *with* $df_0 = N - K_0 = 74 - 4 = 70$
  RSS_1 = 326680563.0 *with* $df_1 = N - K = 74 - 6 = 68$
  $RSS_0 - RSS_1$ = 411249980.0 – 326680563.0 = 84569417
  $df_0 - df_1 = 70 - 68 = 2$ *or* $K - K_0 = 6 - 4 = 2$.

  $$F_0 = \frac{(411249980 - 326680563)/(70 - 68)}{326680563/68} = \frac{84569417/2}{326680563/68} = \mathbf{8.80}$$

  ▪ $$F = \frac{(R_U^2 - R_R^2)/(df_0 - df_1)}{(1 - R_U^2)/df_1} \sim F[df_0 - df_1, df_1] = F[K - K_0, N - K]$$

  $$R_U^2 = \frac{ESS_1}{TSS} = 1 - \frac{RSS_1}{TSS} = \mathbf{0.4856}; \quad R_R^2 = \frac{ESS_0}{TSS} = 1 - \frac{RSS_0}{TSS} = \mathbf{0.\ 3524}.$$

  $df_0 = N - K_0 = 74 - 4 = 70$
  $df_1 = N - K = 74 - 6 = 68$
  $df_0 - df_1 = 70 - 68 = 2$ *or* $K - K_0 = 6 - 4 = 2$.

  $$F_0 = \frac{(0.4856 - 0.3524)/(70 - 68)}{(1 - 0.4856)/68} = \frac{0.1332/2}{0.5144/68} = \mathbf{8.80}$$

- ***Null distribution*** of the F-statistic: **F ~ F[2,68]** under H_0.

  The ***critical values*** **of the F[2,68]-distribution** are:

  for $\alpha = 0.05$: $F_\alpha[2,68] = F_{0.05}[2,68] = 3.13$
  for $\alpha = 0.01$: $F_\alpha[2,68] = F_{0.01}[2,68] = 4.93$

- **Inference at α = 0.05 (5% significance level)**:

  Since $F_0 = 8.80 > 3.13 = F_{0.05}[2,68]$, ***reject*** $\mathbf{H_0}$ at the ***5%*** significance level.

- **Inference at α = 0.01 (1% significance level)**:

  Since $F_0 = 8.80 > 4.93 = F_{0.01}[2,68]$, ***reject*** $\mathbf{H_0}$ at the ***1%*** significance level.

❑ ***<u>Conclusion -- TEST 3</u>***

Reject the ***restricted*** **model (Model 4)**

$$\text{price}_i = \beta_0 + \beta_1 \text{wgt}_i + \beta_2 \text{mpg}_i + \beta_4 \text{mpg}_i^2 + u_i \qquad \textbf{(4)}$$

in favour of the ***unrestricted*** **model (Model 2)**

$$\text{price}_i = \beta_0 + \beta_1 \text{wgt}_i + \beta_2 \text{mpg}_i + \beta_3 \text{wgt}_i^2 + \beta_4 \text{mpg}_i^2 + \beta_5 \text{wgt}_i \text{mpg}_i + u_i \qquad \textbf{(2)}$$

<u>TEST 4</u>: Test that the *marginal effect of wgt_i* in Model 2 does not depend on *mpg_i*, and that the *marginal effect of mpg_i* in Model 2 does not depend on *wgt_i*.

Model 2:

$$\text{price}_i = \beta_0 + \beta_1 \text{wgt}_i + \beta_2 \text{mpg}_i + \beta_3 \text{wgt}_i^2 + \beta_4 \text{mpg}_i^2 + \beta_5 \text{wgt}_i \text{mpg}_i + u_i \qquad \textbf{(2)}$$

- **Marginal effect of *wgt_i* in Model 2** =

$$\partial \text{price}_i / \partial \text{wgt}_i = \beta_1 + 2\beta_3 \text{wgt}_i + \beta_5 \text{mpg}_i$$

A sufficient condition for $\partial \text{price}_i / \partial \text{wgt}_i$ to be independent of mpg_i is: $\boldsymbol{\beta_5 = 0}$.

- **Marginal effect of *mpg_i* in Model 2** =

$$\partial \text{price}_i / \partial \text{mpg}_i = \beta_2 + 2\beta_4 \text{mpg}_i + \beta_5 \text{wgt}_i$$

A sufficient condition for $\partial \text{price}_i / \partial \text{mpg}_i$ to be independent of wgt_i is: $\boldsymbol{\beta_5 = 0}$.

- The ***null* hypothesis** is:

H_0: $\beta_5 = 0$

- The ***alternative* hypothesis** is:

H_1: $\beta_5 \neq 0$

Test statistics: For this test, **we can use the general F-statistic, the F-statistic for $\hat{\beta}_5$, or the t-statistic for $\hat{\beta}_5$.**

- The ***unrestricted* model** corresponding to the alternative hypothesis H_1 is simply PRE (2):

$$\text{price}_i = \beta_0 + \beta_1 \text{wgt}_i + \beta_2 \text{mpg}_i + \beta_3 \text{wgt}_i^2 + \beta_4 \text{mpg}_i^2 + \beta_5 \text{wgt}_i \text{mpg}_i + u_i \qquad \textbf{(2)}$$

Number of free (unrestricted) regression coefficients is $\mathbf{K = K_1 = 6}$.

- OLS estimation of the unrestricted model yields the ***unrestricted* OLS SRE**:

$$\text{price}_i = \hat\beta_0 + \hat\beta_1 \text{wgt}_i + \hat\beta_2 \text{mpg}_i + \hat\beta_3 \text{wgt}_i^2 + \hat\beta_4 \text{mpg}_i^2 + \hat\beta_5 \text{wgt}_i \text{mpg}_i + \hat u_i \qquad \textbf{(2*)}$$

(1) The **OLS decomposition equation for the unrestricted OLS-SRE** is

$$\underset{(N-1)}{\text{TSS}} = \underset{(K-1)}{\text{ESS}_1} + \underset{(N-K)}{\text{RSS}_1}\,.$$

TSS = 635065396.0 *with* df = N −1 = 74 − 1 = 73
ESS_1 = 308384833.0 *with* df = K −1 = 6 − 1 = 5
$\mathbf{RSS_1}$ **= 326680563.0** ***with*** $\mathbf{df_1}$ **= N − K = 74 − 6 = 68**

(2) The $\mathbf{R^2}$ **for the *unrestricted* OLS SRE** is

$$R_U^2 = \frac{\text{ESS}_1}{\text{TSS}} = 1 - \frac{\text{RSS}_1}{\text{TSS}} = \mathbf{0.4856}.$$

```
. regress price wgt mpg wgtsq mpgsq wgtmpg

  Source |       SS       df       MS                  Number of obs =      74
---------+------------------------------               F(  5,    68) =   12.84
   Model |   308384833     5  61676966.6               Prob > F      =  0.0000
Residual |   326680563    68  4804125.93               R-squared     =  0.4856
---------+------------------------------               Adj R-squared =  0.4478
   Total |   635065396    73  8699525.97               Root MSE      =  2191.8

------------------------------------------------------------------------------
   price |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
     wgt |  -31.88985   9.148215     -3.486   0.001      -50.14483   -13.63487
     mpg |  -3549.495   1126.464     -3.151   0.002      -5797.318   -1301.672
   wgtsq |   .0034574   .0008629      4.007   0.000       .0017355    .0051792
   mpgsq |   38.74472   12.62339      3.069   0.003       13.55514     63.9343
  wgtmpg |   .5421927   .1971854      2.750   0.008       .1487154    .9356701
   _cons |   92690.55   25520.53      3.632   0.001       41765.12      143616
------------------------------------------------------------------------------
```

- The ***restricted* model** corresponding to the null hypothesis H_0 is obtained by setting $\beta_5 = 0$ in the unrestricted model (2):

$$\text{price}_i = \beta_0 + \beta_1 \text{wgt}_i + \beta_2 \text{mpg}_i + \beta_3 \text{wgt}_i^2 + \beta_4 \text{mpg}_i^2 + u_i \qquad \textbf{(5)}$$

Number of free (unrestricted) regression coefficients is $\mathbf{K_0 = 5}$.

- OLS estimation of the restricted model (5) yields the ***restricted* OLS SRE**:

$$\text{price}_i = \tilde{\beta}_0 + \tilde{\beta}_1 \text{wgt}_i + \tilde{\beta}_2 \text{mpg}_i + \tilde{\beta}_3 \text{wgt}_i^2 + \tilde{\beta}_4 \text{mpg}_i^2 + \tilde{u}_i \qquad \textbf{(5*)}$$

(1) The **OLS decomposition equation for the restricted OLS-SRE** is

$$\underset{(N-1)}{TSS} = \underset{(K_0-1)}{ESS_0} + \underset{(N-K_0)}{RSS_0}.$$

TSS = 635065396.0 *with* df = N −1 = 74 – 1 = 73
ESS_0 = 272062621.0 *with* df = K_0 −1 = 5 – 1 = 4
RSS_0 = 363002775.0 ***with*** **$df_0 = N - K_0$ = 74 – 5 = 69**

(2) The **R^2 for the restricted OLS-SRE** is

$$R_R^2 = \frac{ESS_0}{TSS} = 1 - \frac{RSS_0}{TSS} = \mathbf{0.4284}.$$

```
. regress price wgt mpg wgtsq mpgsq

  Source |       SS       df       MS                  Number of obs =      74
---------+------------------------------               F(  4,    69) =   12.93
   Model |   272062621     4  68015655.2               Prob > F      =  0.0000
Residual |   363002775    69  5260909.79               R-squared     =  0.4284
---------+------------------------------               Adj R-squared =  0.3953
   Total |   635065396    73  8699525.97               Root MSE      =  2293.7

------------------------------------------------------------------------------
   price |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
     wgt |   -7.99723   2.994029     -2.671   0.009      -13.97015   -2.024306
     mpg |  -615.2419   377.5204     -1.630   0.108      -1368.375    137.8907
   wgtsq |   .0014407   .0004757      3.028   0.003       .0004916    .0023898
   mpgsq |   9.323582   7.009083      1.330   0.188      -4.659156    23.30632
   _cons |   24884.95   6878.057      3.618   0.001       11163.61     38606.3
------------------------------------------------------------------------------
```

- Compute the ***sample value*** **of either of the general F-statistics**.

  - $F = \dfrac{(RSS_0 - RSS_1)/(df_0 - df_1)}{RSS_1/df_1} \sim F[df_0 - df_1, df_1] = F[K - K_0, N - K]$

    RSS_0 = 363002775.0 *with* $df_0 = N - K_0 = 74 - 5 = 69$
    RSS_1 = 326680563.0 *with* $df_1 = N - K = 74 - 6 = 68$
    $RSS_0 - RSS_1$ = 363002775.0 – 326680563.0 = 36322212.0
    $df_0 - df_1 = 69 - 68 = 1$ *or* $K - K_0 = 6 - 5 = 1$.

    $$F_0 = \frac{(363002775 - 326680563)/(69 - 68)}{326680563/68} = \frac{36322212/1}{326680563/68} = \mathbf{7.561}$$

  - $F = \dfrac{(R_U^2 - R_R^2)/(df_0 - df_1)}{(1 - R_U^2)/df_1} \sim F[df_0 - df_1, df_1] = F[K - K_0, N - K]$

    $$R_U^2 = \frac{ESS_1}{TSS} = 1 - \frac{RSS_1}{TSS} = \mathbf{0.4856}; \quad R_R^2 = \frac{ESS_0}{TSS} = 1 - \frac{RSS_0}{TSS} = \mathbf{0.4284}.$$

    $df_0 = N - K_0 = 74 - 5 = 69$
    $df_1 = N - K = 74 - 6 = 68$
    $df_0 - df_1 = 69 - 68 = 1$ *or* $K - K_0 = 6 - 5 = 1$.

    $$F_0 = \frac{(0.4856 - 0.4284)/(69 - 68)}{(1 - 0.4856)/68} = \frac{0.0572/1}{0.5144/68} = \mathbf{7.561}$$

- ***Null distribution*** of the F-statistic: **F ~ F[1,68]** under H_0.

  The ***critical values*** **of the F[1,68]-distribution** are:

  for $\alpha = 0.05$: $F_\alpha[1,68] = F_{0.05}[1,68] = 3.98$
  for $\alpha = 0.01$: $F_\alpha[1,68] = F_{0.01}[1,68] = 7.02$

- **Inference at α = 0.05 (5% significance level)**:

  Since $F_0 = 7.561 > 3.98 = F_{0.05}[1,68]$, ***reject*** $\mathbf{H_0}$ at the ***5%*** significance level.

- **Inference at α = 0.01 (1% significance level)**:

  Since $F_0 = 7.561 > 7.02 = F_{0.01}[1,68]$, ***reject*** $\mathbf{H_0}$ at the ***1%*** significance level.

❑ ***Conclusion -- TEST 4***

Reject the ***restricted* model (Model 5)**

$$\text{price}_i = \beta_0 + \beta_1\text{wgt}_i + \beta_2\text{mpg}_i + \beta_3\text{wgt}_i^2 + \beta_4\text{mpg}_i^2 + u_i \qquad \textbf{(5)}$$

in favour of the ***unrestricted* model (Model 2)**

$$\text{price}_i = \beta_0 + \beta_1\text{wgt}_i + \beta_2\text{mpg}_i + \beta_3\text{wgt}_i^2 + \beta_4\text{mpg}_i^2 + \beta_5\text{wgt}_i\text{mpg}_i + u_i \qquad \textbf{(2)}$$

- **<u>Alternative formula for the F-statistic</u> for testing H_0: $\beta_5 = 0$ vs. H_1: $\beta_5 \neq 0$:**

$$F(\hat{\beta}_5) = \frac{\left(\hat{\beta}_5 - \beta_5\right)^2}{\hat{Var}(\hat{\beta}_5)} \sim F[1, N-K] = F[1, 74-6] = F[1, 68].$$

- The ***sample value*** **of $F(\hat{\beta}_5)$** under H_0: $\beta_5 = 0$ is calculated by substituting into this formula the following values:

$$\hat{\beta}_5 = 0.54219;$$
$$\beta_5 = 0;$$
$$\hat{Var}(\hat{\beta}_5) = \left[\hat{se}(\hat{\beta}_5)\right]^2 = 0.19718539^2 = 0.0388821.$$

Thus, the ***sample value*** **of $F(\hat{\beta}_5)$** under H_0: $\beta_5 = 0$ is:

$$F_0(\hat{\beta}_5) = \frac{\left(\hat{\beta}_5 - \beta_5\right)^2}{\hat{Var}(\hat{\beta}_5)} = \frac{(0.54219 - 0)^2}{0.0388821} = \frac{0.29397}{0.0388821} = \mathbf{7.561 = F_0}.$$

- ***<u>Result</u>:*** **$F_0(\hat{\beta}_5)$ = F_0 using formula (F1) or (F2) = <u>7.561</u>**

- **<u>A *two-tailed* t-test</u> of H_0: $\beta_5 = 0$ vs. H_1: $\beta_5 \neq 0$:**

  - The **t-statistic** is:

$$t(\hat{\beta}_5) = \frac{\hat{\beta}_5 - \beta_5}{\hat{se}(\hat{\beta}_5)} \sim t[N-K] = t[74-6] = t[68].$$

  - The ***sample value*** of this t-statistic under H_0: $\beta_5 = 0$ is calculated by substituting into this formula the following values:

$$\hat{\beta}_5 = 0.54219;$$
$$\beta_5 = 0;$$
$$\hat{se}(\hat{\beta}_5) = 0.197185.$$

  Thus, the ***sample value* of $t(\hat{\beta}_5)$** under H_0: $\beta_5 = 0$ is:

$$t_0(\hat{\beta}_5) = \frac{\hat{\beta}_5 - \beta_5}{\hat{se}(\hat{\beta}_5)} = \frac{0.54219 - 0}{0.197185} = \frac{0.54219}{0.197185} = \mathbf{2.750}.$$

  - ***Null distribution*** of t-statistic**:** $t(\hat{\beta}_5) \sim t[N–K] = t[70]$

  The ***two-tailed critical values* of the t[N–K] = t[70]-distribution** are:

  for $\alpha = 0.10$, $\alpha/2 = 0.05$: $t_{\alpha/2}[68] = t_{0.05}[68] = 1.667.$
  for $\alpha = 0.05$, $\alpha/2 = 0.025$: $t_{\alpha/2}[68] = t_{0.025}[68] = 1.995.$
  for $\alpha = 0.01$, $\alpha/2 = 0.005$: $t_{\alpha/2}[68] = t_{0.005}[68] = 2.650.$

  - **Inference at α = 0.01 (1% significance level)**:

  Since $\left|t_0(\hat{\beta}_5)\right| = 2.750 > 2.650 = t_{\alpha/2}[68] = t_{0.05}[68]$, ***reject* H_0** at the ***1%*** significance level.

(1) Relationship between values of the two test statistics $\mathbf{t_0(\hat{\beta}_5)}$ and $\mathbf{F_0(\hat{\beta}_5)}$:

$$F_0(\hat{\beta}_5) = \frac{(\hat{\beta}_5 - \beta_5)^2}{\hat{Var}(\hat{\beta}_5)} = \left[\frac{\hat{\beta}_5 - \beta_5}{\sqrt{\hat{Var}(\hat{\beta}_5)}}\right]^2 = \left[\frac{\hat{\beta}_5 - \beta_5}{\hat{se}(\hat{\beta}_5)}\right]^2 = \left(t_0(\hat{\beta}_5)\right)^2.$$

<u>Example -- Test 4</u>: For this particular test,

$$\left(t_0(\hat{\beta}_5)\right)^2 = (2.750)^2 = 7.56 = F_0(\hat{\beta}_5).$$

(2) The ***null distributions*** **of the two test statistics** are related according to a similar equality.

$$\left(t[N-K]\right)^2 \sim F[1, N-K].$$

The square of a t[N–K] distribution has the F[1, N–K] distribution.

***<u>Implication 1</u>:* The *square* of the α/2 critical value of the t[N–K] distribution *equals* the α-level critical value of the F[1, N–K] distribution:**

$$\left(t_{\alpha/2}[N-K]\right)^2 = F_\alpha[1, N-K].$$

<u>Example -- Test 4</u>: For α = 0.01 (the 1% significance level)

$F_\alpha[1,68] = F_{0.01}[1,68] = 7.023$; $t_{\alpha/2}[68] = t_{0.01}[68] = 2.65$

$\therefore \left(t_{0.025}[68]\right)^2 = 2.65^2 = 7.023 = F_{0.01}[1,68]$.

<u>Implication 2</u>:

the *two-tailed* p-value for $\mathbf{t_0(\hat{\beta}_5)}$ = the p-value for $\mathbf{F_0(\hat{\beta}_5)}$.

<u>Example -- Test 4</u>:

the ***two-tailed p-value*** **for $\mathbf{t_0(\hat{\beta}_5)}$ = 0.0076**
the ***p-value*** **for $\mathbf{F_0(\hat{\beta}_5)}$ = 0.0076**