ECON 351* -- NOTE 8

Hypothesis Testing in the Classical Normal Linear Regression Model

1. Components of Hypothesis Tests

1. A ***testable* hypothesis**, which consists of ***two parts***:

Part 1: a ***null*** **hypothesis, H_0**
Part 2: an ***alternative*** **hypothesis, H_1**

2. A ***feasible* test statistic**.

Definition: A **test statistic is a *random variable*** whose value for given sample data determines whether the null hypothesis H_0 is rejected or not rejected.

Definition: A **test statistic is *feasible*** if it satisfies two conditions:

(1) Its **probability distribution**, or **sampling distribution**, ***must be known*** completely when the null hypothesis H_0 is true, and it must have some other distribution when the null hypothesis is false.

(2) Its **value can be calculated from** the given ***sample data***.

3. A ***decision* rule** or **rejection rule**.

Definition: A **decision rule** *specifies* (1) the ***rejection* region** and (2) the ***non-rejection* region** of the test statistic.

(1) *Definition:* The ***rejection*** **region** is the ***set, or range, of values of the test statistic*** for which the null hypothesis ***H_0 is rejected*** **–** i.e., that have a low probability of occurring when the null hypothesis is true.

(2) *Definition*: The **nonrejection region** is the ***set***, or range, ***of values of the test statistic*** for which the null hypothesis ***H_0 is not rejected***, or ***retained***.

2. Procedure for Testing Hypotheses

Five Basic Steps

The procedure for testing hypotheses consists of ***five* basic steps**.

<u>**Step 1**</u>**:** **Formulate** the **null hypothesis H_0** and the **alternative hypothesis H_1**.

<u>**Step 2**</u>**:** **Specify** the **test statistic and its distribution** -- specifically its distribution when the null hypothesis H_0 is true.

The distribution of the test statistic when the null hypothesis H_0 is true is known as the ***null distribution of the test statistic***.

<u>**Step 3**</u>**:** **Calculate** the ***sample value*** **of the test statistic** under the **null hypothesis H_0** for the given sample data.

<u>**Step 4**</u>**:** **Select** the ***significance level*** α**,** and **determine** the corresponding ***rejection*** **region** and ***non-rejection*** **region** for the **test statistic**.

<u>**Step 5**</u>**:** **Apply** the ***decision rule*** of the test and **state the inference**, or conclusion, **implied by the sample value of the test statistic**.

We illustrate these five steps for an important class of hypothesis tests in applied econometrics -- namely tests of equality restrictions on individual regression coefficients.

Tests of Equality Restrictions on Individual Regression Coefficients

- These tests assess the probable empirical validity of statements or hypotheses of the following form:

  $\boldsymbol{\beta_j = b_j}$ where $\mathbf{b_j}$ is a ***specified constant***. (j = 0, 1)

- Such statements are conjectures about the population values of the regression coefficients β_j (j = 0, 1).

Examples

$\beta_1 = 0 \quad \Rightarrow \quad \partial E(Y_i | X_i)/\partial X_i = 0$, i.e., X_i is unrelated to $E(Y_i | X_i)$

$\beta_1 = 1.0 \quad \Rightarrow \quad \partial E(Y_i | X_i)/\partial X_i = 1$

$\beta_1 = 0.8 \quad \Rightarrow \quad \partial E(Y_i | X_i)/\partial X_i = 0.8$

$\beta_1 = -1.0 \quad \Rightarrow \quad \partial E(Y_i | X_i)/\partial X_i = -1.0$

Later we will consider more general hypotheses that take the form of **linear equality restrictions on *two or more* regression coefficients $\boldsymbol{\beta_j}$** (j = 0, 1).

<u>STEP 1</u>: Formulation of the Null and Alternative Hypotheses

<u>Step 1</u>: Formulate the *null* hypothesis H_0 and the *alternative* hypothesis H_1.

Components of a Statistical Test

A **statistical hypothesis test** consists of ***two opposing statements or propositions or conjectures*** about the model parameters:

1. The ***<u>null</u>* hypothesis**, denoted by $\mathbf{H_0}$.

- H_0 is the ***proposition being tested***.
- It specifies our conjecture about the true value(s) of the regression coefficient(s).

2. The ***<u>alternative</u>* hypothesis**, denoted by $\mathbf{H_1}$.

- H_1 is the ***counter-proposition to the null hypothesis H_0***.
- It specifies the set of alternative possibilities which is *presumed* to contain the truth if the null hypothesis is false.

Purpose of a Statistical Test

- A statistical test is designed and constructed so as **to provide sample evidence** respecting the ***probable empirical validity***, or ***truth***, ***of the null hypothesis H_0***.

- The test addresses the question: Are the sample estimates of the model parameters **--** ***consistent or inconsistent*** (***compatible or incompatible***) with the ***truth*** of the ***null hypothesis***?

  Consistency or ***compatibility*** means ***sufficiently close*** **to the value(s) specified by H_0 that we *retain* (do not reject) the *null hypothesis***.

- A statistical test does not test the empirical validity, or truth, of the alternative hypothesis H_1. Only the null hypothesis H_0 is being subjected to test.

Formulation of H_0 and H_1: Equality Restrictions on β_1

The Null Hypothesis H_0

H_0: $\beta_1 = b_1$ where **b_1 is a *specified constant*** (such as 0 or 0.9 or −1).

The Alternative Hypothesis H_1

For a null hypothesis of this general form, there are ***three*** possible alternative hypotheses.

(1) H_1: $\beta_1 \neq b_1$ a ***two-sided*** **alternative hypothesis**.

Rejection of the null hypothesis H_0: $\beta_1 = b_1$ implies that β_1 takes some other value, and that this other value is **either greater than or less than b_1**.

That is, H_1: $\beta_1 \neq b_1 \;\Rightarrow\;$ *either* $\beta_1 > b_1$ *or* $\beta_1 < b_1$.

(2) H_1: $\beta_1 > b_1$ a ***one-sided*** **(*right-sided*) alternative hypothesis**.

Rejection of the null hypothesis H_0: $\beta_1 = b_1$ in this case implies that β_1 takes some other value that is ***greater than*** **b_1**.

This alternative hypothesis completely discounts the possibility that $\beta_1 < b_1$. It implies that values of β_1 *less than* b_1 are considered to be logically unacceptable alternatives to the null hypothesis, an implication that presumably is based on economic theory.

(3) H_1: $\beta_1 < b_1$ a ***one-sided*** **(*left-sided*) alternative hypothesis**.

Rejection of the null hypothesis H_0: $\beta_1 = b_1$ in this case implies that β_1 takes some other value that is ***less than*** **b_1**.

This alternative hypothesis completely excludes the possibility that $\beta_1 > b_1$. It implies that the value of β_1 could not be *greater than* b_1 if in fact the null hypothesis H_0 is false.

<u>STEP 2</u>: Specify the Test Statistic and its Null Distribution

<u>**Step 2**</u>: **Specify the *test statistic* and its *null distribution*** when the null hypothesis H_0 is true.

Theoretical Prerequisite

Assumptions A1-A9 of the CNLRM -- especially the ***normality assumption A9***.

Important Results

1. Under **assumptions A1-A9** of the CNLRM, the following **t-statistics** are ***feasible test statistics*** for the OLS coefficient estimators $\hat{\beta}_0$ and $\hat{\beta}_1$:

$$t(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\hat{Var}(\hat{\beta}_1)}} = \frac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} = \frac{\hat{\beta}_1 - \beta_1}{\hat{\sigma} \Big/ \left(\sum_i x_i^2\right)^{1/2}} \sim t[N-2]$$

$$t(\hat{\beta}_0) = \frac{\hat{\beta}_0 - \beta_0}{\sqrt{\hat{Var}(\hat{\beta}_0)}} = \frac{\hat{\beta}_0 - \beta_0}{\hat{se}(\hat{\beta}_0)} = \frac{\hat{\beta}_0 - \beta_0}{\hat{\sigma}\left(\sum_i X_i^2\right)^{1/2} \Big/ N^{1/2}\left(\sum_i x_i^2\right)^{1/2}} \sim t[N-2]$$

2. For the **true (but unknown) values** of the **population regression coefficients** $\boldsymbol{\beta_0}$ **and** $\boldsymbol{\beta_1}$, each of the test statistics $t(\hat{\beta}_0)$ and $t(\hat{\beta}_1)$ has the **t-distribution with N – 2 degrees of freedom**, denoted as **t[N – 2]**.

STEP 3: Evaluate the Test Statistic Under H_0

Step 3: Calculate the *sample value* of the test statistic under the null hypothesis H_0 for the given sample data.

The **null hypothesis** is

H_0: $\beta_1 = b_1$ where **b_1** is ***a specified constant*** such as 0 or 0.9 or -1.

From Step 2, the **feasible test statistic** is the t-statistic for the OLS estimator $\hat{\beta}_1$:

$$t(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\hat{Var}(\hat{\beta}_1)}} = \frac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} \sim t[N-2].$$

To calculate the ***sample value*** **of $t(\hat{\beta}_1)$ under the null hypothesis H_0: $\beta_1 = b_1$**, simply **substitute** in the above formula for $t(\hat{\beta}_1)$

- **the value b_1 for β_1**, since b_1 is the value of β_1 specified by H_0;
- **the sample value of** $\hat{\beta}_1$, the point estimate of β_1 for the given sample data;
- **the sample value of** $\hat{se}(\hat{\beta}_1) = \sqrt{\hat{Var}(\hat{\beta}_1)}$, the estimated standard error of $\hat{\beta}_1$.

❑ The ***sample value*** **of $t(\hat{\beta}_1)$** evaluated **under the null hypothesis H_0** is therefore

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - b_1}{\sqrt{\hat{Var}(\hat{\beta}_1)}} = \frac{\hat{\beta}_1 - b_1}{\hat{se}(\hat{\beta}_1)}.$$

Note: The subscript "0" on $t_0(\hat{\beta}_1)$ indicates the value of $t(\hat{\beta}_1)$ under H_0.

❑ The ***null distribution*** **of $t_0(\hat{\beta}_1)$**, the **calculated sample value of $t(\hat{\beta}_1)$, is t[N–2]**, **the *t-distribution* with *N–2 degrees of freedom***:

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - b_1}{\sqrt{\hat{Var}(\hat{\beta}_1)}} = \frac{\hat{\beta}_1 - b_1}{\hat{se}(\hat{\beta}_1)} \sim t[N-2] \text{ under } H_0\text{: } \beta_1 = b_1.$$

STEP 4: Determine the Rejection and Non-Rejection Regions

Step 4: Select the *significance level* α, and **determine the corresponding *rejection* region** and ***non-rejection* region** for the calculated test statistic.

Background: Type I and Type II Errors

In performing any hypothesis test – i.e., in deciding to reject or retain a null hypothesis – there is always some chance of making mistakes. Such mistakes arise whenever the decision to retain or reject H_0 does not reflect the true but unknown state of the world.

Decision	*State of the World*	
	H_0 is true	**H_0 is false**
Retain (do not reject) **H_0**	Correct Decision Pr = $1 - \alpha$	Type II Error Pr = β
Reject H_0	Type I Error Pr = α	Correct Decision Pr = $1 - \beta$

1. A ***correct* decision** is made if:

- the null hypothesis H_0 is **false** and the decision is to **reject** it.
- the null hypothesis H_0 is **true** and the decision is to **retain (not to reject)** it.

2. An ***incorrect* decision** is made if:

- the null hypothesis H_0 is **true** and the decision is to **reject** it (a Type I error).
- the null hypothesis H_0 is **false** and the decision is to **retain (not to reject)** it (a Type II error).

❑ **<u>Definitions</u>:**

Type I error: ***rejecting*** $\mathbf{H_0}$ when $\mathbf{H_0}$ **is *true***.

Type II error: ***not rejecting*** $\mathbf{H_0}$ when $\mathbf{H_0}$ **is *false***.

❑ **<u>Probabilities of Type I and Type II Errors</u>**

$\alpha \equiv$ **Pr(Type I Error) = Pr($\mathbf{H_0}$ is rejected | $\mathbf{H_0}$ is true)**

$1 - \alpha$ = Pr(H_0 is not rejected | H_0 is true) = the **confidence level** of the test
= the ***probability*** of making a ***correct*** **decision** when the null hypothesis $\mathbf{H_0}$ **is *true***.

$\beta \equiv$ **Pr(Type II Error) = Pr($\mathbf{H_0}$ is not rejected | $\mathbf{H_0}$ is false)**

$1 - \beta$ = Pr(H_0 is rejected | H_0 is false) = the **power** of the test
= the ***probability*** of making a ***correct*** **decision** when the null hypothesis $\mathbf{H_0}$ **is *false***.

Analogy Between Statistical Hypothesis Tests and Criminal Court Trials

Presumption of Innocence:

The accused is presumed innocent until proven to be guilty beyond a reasonable doubt.

H_0: the accused is ***not guilty***

H_1: the accused is ***guilty*** as charged

The court must decide whether to retain or reject H_0 on the basis of admissible evidence.

	State of the World	
Court's Decision	**Accused is innocent**	**Accused is guilty**
Acquit (find not guilty)	Correct Decision	Type II Error
Convict (find guilty)	Type I Error	Correct Decision

1. The court makes a ***correct*** **decision** if:

- the accused is **innocent** and the court's decision is to **acquit**.
- the accused is **guilty** and the court's decision is to **convict**.

2. The court makes an ***incorrect*** **decision** if:

- the accused is **innocent** and the court's decision is to **convict** (the court has made a ***Type I error***).
- the accused is **guilty** and the court's decision is to **acquit** (the court has made a ***Type II error***).

The Significance Level of the Test

Definition: The ***significance level*** of the test is chosen to equal α, the **probability of making a Type I error**.

- <u>**Significance level**</u> **of the test =** $\alpha \equiv$ **Pr(Type I Error)**

  = Pr(H_0 is rejected | H_0 is true).

- <u>**Confidence level**</u> **of the test = 1 –** α **= Pr(H_0 is not rejected | H_0 is true).**

- <u>**Power of the Test**</u>

  Definition: The power of the test is defined to equal $1 - \beta$, the probability of making a correct decision when the null hypothesis **H_0 is *false***.

  Power of the test ≡ 1 – β **= Pr(H_0 is rejected | H_0 is false)**

  = 1 – Pr(Type II Error)

- <u>**Relationship Between Type I and Type II Errors**</u>

  For any given sample size N,

  α = Pr(Type I Error) is ***inversely related*** to Pr(Type II Error)

  $\Rightarrow$ α = Pr(Type I Error) is ***directly related*** to the **power** of the test.

  Result: For any given sample size N, there exists a ***trade-off*** between

  (1) α **= Pr(Type I Error)** = the **significance level** of the test

  and

  (2) β **= Pr(Type II Error)**

Implications of Trade-Off Between α and β

- By choosing a ***lower*** **significance level** α -- and thereby reducing the Pr(Type I Error) -- we necessarily

  (1) ***increase*** β = the Pr(Type II Error), and
  (2) ***decrease*** $1 - \beta$ = the power of the test.

- Conversely, by choosing a ***higher*** **significance level** α -- and thereby increasing the Pr(Type I Error) -- we necessarily

  (1) ***decrease*** β = the Pr(Type II Error), and
  (2) ***increase*** $1 - \beta$ = the power of the test.

❑ Comments on Choosing α, the Significance Level of the Test

- The value of the significance level α is usually chosen to be small. In practice, the values most frequently chosen are:

  $\alpha = 0.01$ (a 1% significance level);
  $\alpha = 0.05$ (a 5% significance level);
  $\alpha = 0.10$ (a 10% significance level).

- But the choice of value for α is more or less arbitrary. It presumably reflects the investigator's relative willingness to accept, or relative aversion to, Type I and Type II errors.

Rejection and Non-Rejection Regions for a Test Statistic

Definition of the Rejection and Non-Rejection Regions

- The ***rejection* region** is the ***set, or range, of values of the test statistic*** for which the null hypothesis **H_0 is *rejected***.

  Values of the test statistic in the rejection region have a low probability of occurring when the null hypothesis H_0 is true.

- The ***nonrejection* region** is the ***set, or range, of values of the test statistic*** for which the null hypothesis **H_0 is *not rejected,* or *retained***.

Distinguishing Between the Rejection and Nonrejection Regions

Question: How are the rejection and nonrejection regions delineated or separated?

Answer: By the ***critical values* of the test statistic** – or more correctly, by the ***critical values* of the *null distribution* of the test statistic**.

Critical Values of a Test Statistic

Definition: The ***critical values*** **of a test statistic** are defined as those values

that **separate the *rejection region* from the *non-rejection region***,

that **partition** the **sample values** of a test statistic into a ***rejection region*** and a ***non-rejection region***.

Determinants

The ***critical values*** of a test statistic are determined by the following factors:

1. the ***null distribution*** **of the test statistic** – the probability, or sampling, distribution of the test statistic when the null hypothesis H_0 is true;

2. the **chosen *significance level* for the test,** α;

3. the ***nature*** **of the hypothesis test**, specifically whether the test is

(1) a ***two-tail*, or *two-sided*, test**

or

(2) a ***one-tail*, or *one-sided*, test**, of which there are two types,

(2.1) a ***left-tail*** **test**

(2.2) a ***right-tail*** **test**.

Two-Tail and One-Tail Tests: Which is it?

Important Point: Whether a two-tail or one-tail test is appropriate depends on **the nature of the *alternative hypothesis* H_1**.

Definitions

- A ***two-tail* test** is one for which the alternative hypothesis **H_1 is a *two-sided* hypothesis** that incorporates the "not equal" condition "≠".

  Example: H_0: $\beta_1 = b_1$ where b_1 is a specified constant
  H_1: $\beta_1 \neq b_1$ is a ***two-sided* alternative hypothesis**
  ⇒ a ***two-tail* test**
  ⇒ a ***two-tail* rejection region**.

- A ***one-tail* test** is one for which the alternative hypothesis **H_1 is a *one-sided* hypothesis** that incorporates either the "less than" condition "<" or the "greater than" condition ">".

  - ***Case 1:*** A ***left-tail* test** is one in which H_1 incorporates the "less than" condition "<".

    Example: H_0: $\beta_1 = b_1$ where b_1 is a specified constant
    H_1: $\beta_1 < b_1$ is a ***left-sided* alternative hypothesis**
    ⇒ a ***one-tail* test**, specifically a ***left-tail* test**.
    ⇒ a ***left-tail* rejection region**.

  - ***Case 2:*** A ***right-tail* test** is one in which H_1 incorporates the "greater than" condition ">".

    Example: H_0: $\beta_1 = b_1$ where b_1 is a specified constant
    H_1: $\beta_1 > b_1$ is a ***right-sided* alternative hypothesis**
    ⇒ a ***one-tail* test**, specifically a ***right-tail* test**.
    ⇒ a ***right-tail* rejection region**.

Note: The **form of the alternative hypothesis H_1** – specifically the ***direction of the inequality*** in the alternative hypothesis H_1 – determines whether a one-tail test is a right-tail test or a left-tail test.

Determining Critical Values for Two-Tail Tests

Problem: Determine the critical values for the following **two-tail t-test**:

H_0: $\beta_1 = b_1$ *or* $\beta_1 - b_1 = 0$ where b_1 is a specified constant
H_1: $\beta_1 \neq b_1$ *or* $\beta_1 - b_1 \neq 0$ $\Leftarrow$ a ***two-sided*** **alternative hypothesis**.

1. The **appropriate *test statistic*** is the **t-statistic for $\hat{\beta}_1$**, the OLS estimate of the slope coefficient β_1:

$$t(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\hat{Var}(\hat{\beta}_1)}} = \frac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)}.$$

2. The ***sample value*** **of the test statistic $t(\hat{\beta}_1)$** is calculated by evaluating $t(\hat{\beta}_1)$ under the null hypothesis H_0. That is, in the expression for $t(\hat{\beta}_1)$, set β_1 equal to b_1, which is the value of β_1 specified by H_0. The resulting sample value of the test statistic $t(\hat{\beta}_1)$ under H_0 is

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - b_1}{\hat{se}(\hat{\beta}_1)}.$$

3. The ***null distribution*** **of $t_0(\hat{\beta}_1)$ is $t[N - 2]$**, the t-distribution with $N - 2$ degrees of freedom. In other words, if the null hypothesis H_0: $\beta_1 = b_1$ is true, then

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - b_1}{\hat{se}(\hat{\beta}_1)} \sim t[N-2] \quad \text{under } H_0\text{: } \beta_1 = b_1.$$

Question 1: What values of $\hat{\beta}_1$ and $t_0(\hat{\beta}_1)$ would lead us to reject H_0 against H_1?

Answer: Examine the numerator of the calculated t-statistic:

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - b_1}{\hat{se}(\hat{\beta}_1)}$$

Remember:

$\hat{\beta}_1$ = the ***estimated* value of** $\boldsymbol{\beta_1}$
b_1 = the ***hypothesized* value of** $\boldsymbol{\beta_1}$
$\hat{se}(\hat{\beta}_1) > 0$ ($\hat{se}(\hat{\beta}_1)$ is ***always* a *positive* number**)

- We would reject H_0 against H_1 if $\hat{\beta}_1$, the estimated value of β_1, is very different from b_1, the hypothesized value of β_1.

- More specifically, we would **reject H_0 against H_1** in either of the following two cases:

1. $\hat{\beta}_1 >> b_1 \quad \Rightarrow \quad \hat{\beta}_1 - b_1 >> 0 \quad \Rightarrow \quad t_0(\hat{\beta}_1) = \dfrac{\hat{\beta}_1 - b_1}{\hat{se}(\hat{\beta}_1)} >> 0$

   Values of $\hat{\beta}_1$ *much greater* than b_1 imply **large *positive* values** of $t_0(\hat{\beta}_1)$;

or

2. $\hat{\beta}_1 << b_1 \quad \Rightarrow \quad \hat{\beta}_1 - b_1 << 0 \quad \Rightarrow \quad t_0(\hat{\beta}_1) = \dfrac{\hat{\beta}_1 - b_1}{\hat{se}(\hat{\beta}_1)} << 0$

   Values of $\hat{\beta}_1$ *much less* than b_1 imply **large *negative* values** of $t_0(\hat{\beta}_1)$.

Question 2: How much larger or smaller than zero does the value of $t_0(\hat{\beta}_1)$ have to be for us to reject H_0 in favour of H_1?

Answer: It depends on **the *significance level* we choose** for the test and **the *null distribution* of our test statistic**.

Let α = the chosen significance level for the test (e.g., 0.01, 0.05, or 0.10)
= the probability of making a Type I error.

Allocate α equally between large positive values of t_0 and large negative values of t_0. We therefore have both an upper critical value and a lower critical value of the null distribution of our test statistic, which is the **t[N – 2]-distribution**.

$t_{\alpha/2}[N-2]$ = the ***upper*** **α/2 critical value** of the **t[N – 2]-distribution**;
$-t_{\alpha/2}[N-2]$ = the ***lower*** **α/2 critical value** of the **t[N – 2]-distribution**.

Implications: **If H_0: $\beta_1 = b_1$ is *true***, then the following two probability statements hold.

(1) $$\Pr\left(-t_{\alpha/2}[N-2] \le t_0(\hat{\beta}_1) \le t_{\alpha/2}[N-2]\right) = 1-\alpha \qquad \textbf{(1)}$$

(2) $$\Pr\left(t_0(\hat{\beta}_1) < -t_{\alpha/2}[N-2] \quad \text{or} \quad t_0(\hat{\beta}_1) > t_{\alpha/2}[N-2]\right) = \Pr\left(\left|t_0(\hat{\beta}_1)\right| > t_{\alpha/2}[N-2]\right) = \alpha \qquad \textbf{(2)}$$

where

$t_0(\hat{\beta}_1)$ = the **calculated *sample* value of the t-statistic** under H_0;
$\left|t_0(\hat{\beta}_1)\right|$ = the ***absolute value*** **of $t_0(\hat{\beta}_1)$**;
$t_{\alpha/2}[N-2]$ = the ***upper*** **α/2 critical value** of the **t[N – 2]-distribution**;
$-t_{\alpha/2}[N-2]$ = the ***lower*** **α/2 critical value** of the **t[N – 2]-distribution**;
α = the ***significance*** **level** for the test;
$1-\alpha$ = the ***confidence level*** for the test.

Determine the Rejection and Nonrejection Regions – Two-Tail Test

- The ***non-rejection*** **region for $t_0(\hat{\beta}_1)$** is defined by the double inequality in probability statement (1) above. It consists of all values of $t_0(\hat{\beta}_1)$ such that

$$-t_{\alpha/2}[N-2] \le t_0(\hat{\beta}_1) \le t_{\alpha/2}[N-2] \quad \Leftarrow$$ ***non-rejection*** **region for H_0: $\beta_1 = b_1$**.

- The ***rejection*** **region for $t_0(\hat{\beta}_1)$** is the two-sided **region** or ***two-tail*** **region** defined in probability statement (2) above. It consists of all values of $t_0(\hat{\beta}_1)$ such that

$$t_0(\hat{\beta}_1) < -t_{\alpha/2}[N-2] \quad \textit{or} \quad t_0(\hat{\beta}_1) > t_{\alpha/2}[N-2]$$

or

$$\left|t_0(\hat{\beta}_1)\right| > t_{\alpha/2}[N-2] \quad \Leftarrow$$ ***rejection*** **region for H_0: $\beta_1 = b_1$** is a ***two-tail*** **rejection region**.

<u>NOTE</u>*:* For a ***two-tail*** **test**, the **rejection region** is a ***two-tail*** **rejection region** consisting of **two parts**.

1) The ***lower*** **or** ***left-tail*** **rejection region** $t_0(\hat{\beta}_1) < -t_{\alpha/2}[N-2]$, which contains **unexpectedly** ***small*** **values of $t_0(\hat{\beta}_1)$ under H_0** – i.e., values that we would only expect to obtain with "small" probability $\alpha/2$ if the null hypothesis H_0: $\beta_1 = b_1$ is true.

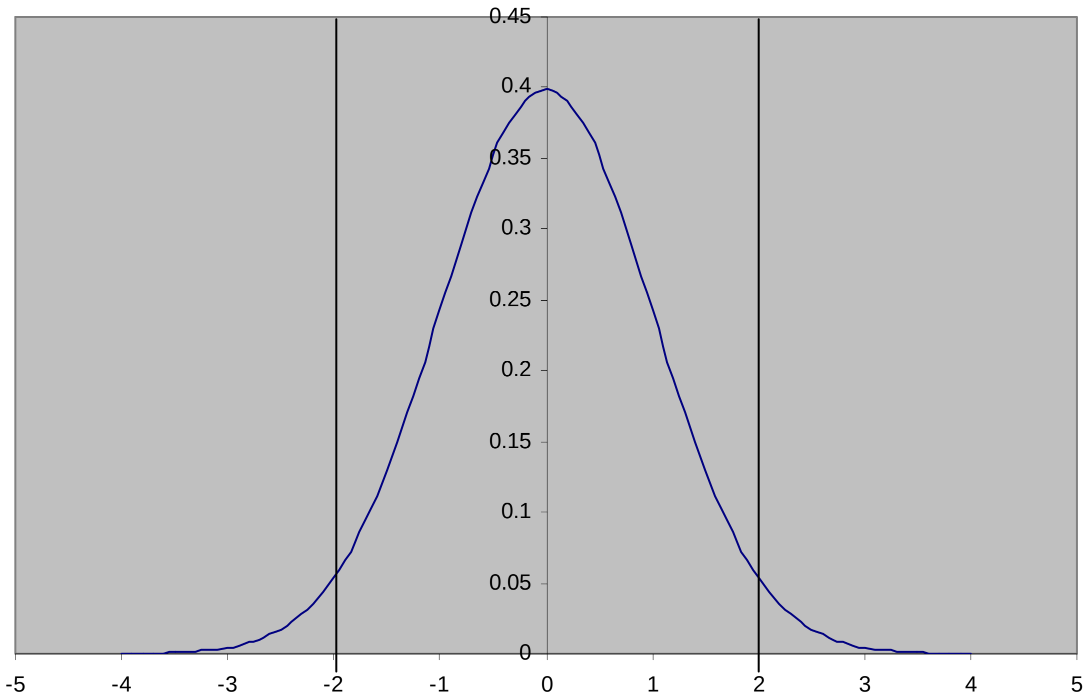
2) The ***upper*** **or** ***right-tail*** **rejection region** $t_0(\hat{\beta}_1) > t_{\alpha/2}[N-2]$, which contains **unexpectedly** ***large*** **values of $t_0(\hat{\beta}_1)$ under H_0** – i.e., values that we would only expect to obtain with "small" probability $\alpha/2$ if the null hypothesis H_0: $\beta_1 = b_1$ is true.

3) The **<u>rejection</u> region for a *two-tail* test** thus consists of ***both* the lower *left-hand* tail *and* the upper *right-hand* tail** of the t[N − 2]-distribution.

- The area in each tail under the t[N − 2]-distribution is $\alpha/2$.
- The sum of these two tail area probabilities equals the significance level α.
- Thus, for a two-tail test, the significance level α is allocated equally between the ***lower* $\alpha/2$ rejection region** and the ***upper* $\alpha/2$ rejection region**.

Rejection and Nonrejection Regions for a Two-Tail Test

left-tail rejection region	↓	nonrejection region	↓	right-tail rejection region
area = $\alpha/2$	↓	area = $1 - \alpha$	↓	area = $\alpha/2$



$-\,t_{\alpha/2}$ ↑ $t_{\alpha/2}$ ↑

$\Pr(t_0 < -\,t_{\alpha/2}) = \alpha/2$ ↑ $\Pr(-\,t_{\alpha/2} \le t_0 \le t_{\alpha/2}) = 1-\alpha$ ↑ $\Pr(t_0 > t_{\alpha/2}) = \alpha/2$

STEP 5: Apply the Decision Rule and State Inference

Step 5: Apply the *decision rule* of the test and **state the *inference***, or conclusion, implied by the sample value of the test statistic.

1. Formulation 1 of the Decision Rule for a Two-Tail Test

Formulation 1: Determine if the *sample value* t_0 of the test statistic **lies in the *rejection* or *nonrejection* region** at the chosen significance level α.

Decision Rule for a Two-Tail Test – Formulation 1

1. If the ***sample value* t_0 of the test statistic** lies in **the *rejection* region** at the chosen significance level, then ***reject* the null hypothesis H_0**.

   For a ***two-tail*** **test**: $|t_0| > t_{\alpha/2}[N-2] \quad \Rightarrow$ ***reject* H_0** at **significance level α.**

   ***Reject* H_0 in favour of H_1** at **significance level** α if

   (1) $t_0 > t_{\alpha/2}[N-2]$ meaning **t_0 lies in the *upper tail* rejection area**;

   or

   (2) $t_0 < -\, t_{\alpha/2}[N-2]$ meaning **t_0 lies in the *lower tail* rejection area**.

   ***Inference:* *Reject* H_0 in favour of H_1** at **significance level** α.

2. If the ***sample value* t_0 of the test statistic** lies in **the *nonrejection* region** at the chosen significance level, then ***retain* (do not reject) the null hypothesis H_0**.

   For a ***two-tail*** **test**: $|t_0| \le t_{\alpha/2}[N-2] \quad \Rightarrow$ ***retain* H_0** at **significance level α.**

   ***Retain* H_0 against H_1** at **significance level** α if

   $$-\, t_{\alpha/2}[N-2] \le t_0 \le t_{\alpha/2}[N-2]$$ meaning **t_0 lies in the *nonrejection* area**.

   ***Inference:* *Retain* H_0 against H_1** at **significance level** α.

Examples of Two-Tail Hypothesis Tests

The Model:

DATA: **auto1.dta** (a Stata-format data file)

MODEL: **$\text{price}_i = \beta_0 + \beta_1 \text{weight}_i + u_i$ (i = 1, ..., N)**

```
. regress price weight

  Source |       SS       df       MS                  Number of obs =      74
---------+------------------------------               F(  1,    72) =   29.42
   Model |  184233937     1   184233937                Prob > F      =  0.0000
Residual |  450831459    72  6261548.04                R-squared     =  0.2901
---------+------------------------------               Adj R-squared =  0.2802
   Total |  635065396    73  8699525.97                Root MSE      =  2502.3

------------------------------------------------------------------------------
   price |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
  weight |  2.044063   .3768341       5.424   0.000       1.292858    2.795268
   _cons | -6.707353    1174.43      -0.006   0.995       -2347.89    2334.475
------------------------------------------------------------------------------
```

$N = 74 \qquad N - 2 = 74 - 2 = \mathbf{72}$

$\hat{\beta}_1 = \mathbf{2.0441} \qquad \hat{se}(\hat{\beta}_1) = \mathbf{0.376834}$

$\alpha = 0.05 \quad \Rightarrow \quad \alpha/2 = 0.025$

$t_{\alpha/2}[N-2] = t_{0.025}[72] = \mathbf{1.9935}$

<u>Test 1</u>: Test the proposition that $weight_i$ is unrelated to $price_i$ at the 5 percent significance level ($\alpha = 0.05$).

- ***Null*** **and** ***Alternative*** **Hypotheses**

  $\mathbf{H_0: \beta_1 = 0}$

  $\mathbf{H_1: \beta_1 \neq 0}$ a ***<u>two-sided</u>*** **alternative hypothesis**.

- The ***feasible*** **test statistic** is the t-statistic for $\hat{\beta}_1$:

$$t(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\hat{Var}(\hat{\beta}_1)}} = \frac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} \sim t[N-2].$$

- **Compute the** ***sample value*** **of** $\mathbf{t(\hat{\beta}_2)}$ under the null hypothesis H_0:
  Set $\hat{\beta}_1 = 2.0441$, $\beta_1 = 0$ and $\hat{se}(\hat{\beta}_1) = 0.376834$ in the formula for $t(\hat{\beta}_1)$:

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} = \frac{2.0441 - 0}{0.376834} = \frac{2.0441}{0.376834} = \mathbf{5.424}$$

- The ***two-tail critical value*** **of the t[N – 2] distribution** at the **5 percent significance level** (at α **= 0.05**) is $t_{\alpha/2}[N-2] = t_{0.025}[72]$ = **1.9935**.

- ***Decision Rule:***

  If $|t_0| > t_{\alpha/2}[N-2]$ ***reject*** $\mathbf{H_0}$ at **significance level** α;
  If $|t_0| \le t_{\alpha/2}[N-2]$ ***retain*** $\mathbf{H_0}$ at **significance level** α.

- ***Inference:***

  $|t_0| = 5.424 > 1.9935 = t_{0.025}[72]$ $\Rightarrow$ ***reject*** $\mathbf{H_0}$ at **significance level** α **= 0.05**

  Reject $\mathbf{H_0}$**:** $\mathbf{\beta_1 = 0}$ in favour of $\mathbf{H_1}$**:** $\mathbf{\beta_1 \neq 0}$ at the ***5 percent*** **significance level**.

Test 2: Test the proposition that a **1-pound increase in weight$_i$** is associated with an **increase in average price$_i$ of 1 dollar**. Use the 5 percent significance level (α = .05).

- ***Null* and *Alternative* Hypotheses**

  H$_0$: $\beta_1 = 1$

  H$_1$: $\beta_1 \neq 1$ a ***two-sided*** **alternative hypothesis**.

- The ***feasible* test statistic** is the t-statistic for $\hat{\beta}_1$:

$$t(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\hat{Var}(\hat{\beta}_1)}} = \frac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} \sim t[N-2].$$

- **Compute the *sample value* of $t(\hat{\beta}_1)$** under the null hypothesis H$_0$:
  Set $\hat{\beta}_1 = 2.0441$, $\beta_1 = 1$ and $\hat{se}(\hat{\beta}_1) = 0.376834$ in the formula for $t(\hat{\beta}_1)$:

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} = \frac{2.0441 - 1}{0.376834} = \frac{1.0441}{0.376834} = \mathbf{2.771}$$

- The ***two-tail critical value* of the t[N – 2] distribution** at the **5 percent significance level** (at **α = 0.05**) is $t_{\alpha/2}[N-2] = t_{0.025}[72]$ = **1.9935**.

- ***Decision Rule:***

  If $|t_0| > t_{\alpha/2}[N-2]$ ***reject* H$_0$** at **significance level** α;
  If $|t_0| \leq t_{\alpha/2}[N-2]$ ***retain* H$_0$** at **significance level** α.

- ***Inference:***

  $|t_0| = 2.771 > 1.9935 = t_{0.025}[72]$ $\Rightarrow$ ***reject* H$_0$** at **significance level α = 0.05**

  ***Reject* H$_0$: $\beta_1 = 1$** in favour of **H$_1$: $\beta_1 \neq 1$** at the ***5 percent* significance level**.

<u>Test 3</u>: Test the proposition that a **1-pound increase in weight$_i$** is associated with an **increase in average price$_i$ of 2 dollars**. Perform the test at the 5 percent significance level (α = .05).

- ***Null*** **and** ***Alternative*** **Hypotheses**

  H$_0$: $\beta_1 = 2$

  H$_1$: $\beta_1 \neq 2$ a ***<u>two-sided</u>*** **alternative hypothesis**.

- The ***feasible*** **test statistic** is the t-statistic for $\hat{\beta}_1$:

$$t(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\hat{Var}(\hat{\beta}_1)}} = \frac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} \sim t[N-2].$$

- **Compute the** ***sample value*** **of $t(\hat{\beta}_1)$** under the null hypothesis H$_0$:
  Set $\hat{\beta}_1 = 2.0441$, $\beta_1 = 2$ and $\hat{se}(\hat{\beta}_1) = 0.376834$ in the formula for $t(\hat{\beta}_1)$:

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} = \frac{2.0441 - 2}{0.376834} = \frac{0.0441}{0.376834} = \mathbf{0.1170}$$

- The ***two-tail critical value*** **of the t[N − 2] distribution** at the **5 percent significance level** (at **α = 0.05**) is $t_{\alpha/2}[N-2] = t_{0.025}[72]$ = **1.9935**.

- ***Decision Rule:***

  If $|t_0| > t_{\alpha/2}[N-2]$ ***reject*** **H$_0$** at **significance level** α;
  If $|t_0| \leq t_{\alpha/2}[N-2]$ ***retain*** **H$_0$** at **significance level** α.

- ***Inference:***

  $|t_0| = 0.1170 < 1.9935 = t_{0.025}[72] \Rightarrow$ ***retain*** **H$_0$** at **significance level α = 0.05**

  Retain **H$_0$: $\beta_1 = 2$** against **H$_1$: $\beta_1 \neq 2$** at the ***5 percent*** **significance level**.

How to perform all three of these two-tail hypothesis tests at once

Test 1: H_0: $\beta_1 = 0$ versus H_1: $\beta_1 \neq 0$
Test 2: H_0: $\beta_1 = 1$ versus H_1: $\beta_1 \neq 1$
Test 3: H_0: $\beta_1 = 2$ versus H_1: $\beta_1 \neq 2$

Compute the *two-sided* 95 percent confidence interval for β_1.

$$\hat{\beta}_{1U} = \hat{\beta}_1 + t_{\alpha/2}[N-2]\hat{se}(\hat{\beta}_1) = \mathbf{2.0441 + 0.75121} = 2.79531 = \underline{\mathbf{2.795}}$$

$$\hat{\beta}_{1L} = \hat{\beta}_1 - t_{\alpha/2}[N-2]\hat{se}(\hat{\beta}_1) = \mathbf{2.0441 - 0.75121} = 1.29289 = \underline{\mathbf{1.293}}$$

Result: **The two-sided 95% confidence interval for β_1 is [1.293, 2.795].**

Decision Rule:

- If the ***hypothesized value* of β_1 lies *outside* the two-sided *95 percent confidence* interval** for β_1, ***reject* the null hypothesis H_0** at the ***5 percent* significance level**.

- If the ***hypothesized value* of β_1 lies *inside* the two-sided *95 percent confidence* interval** for β_1, ***retain* the null hypothesis H_0** at the ***5 percent* significance level**.

Test 1: Since **the value 0 lies *outside* the two-sided *95 percent confidence* interval** for β_1, ***reject* H_0: $\beta_1 = 0$ in favour of H_1: $\beta_1 \neq 0$** at the ***5 percent* significance level**.

Test 2: Since **the value 1 lies *outside* the two-sided *95 percent confidence* interval** for β_1, ***reject* H_0: $\beta_1 = 1$ in favour of H_1: $\beta_1 \neq 1$** at the ***5 percent* significance level**.

Test 3: Since **the value 2 lies *inside* the two-sided *95 percent confidence* interval** for β_1, ***retain* H_0: $\beta_1 = 2$ against H_1: $\beta_1 \neq 2$** at the ***5 percent* significance level**.

Determining Critical Values for One-Tail Tests

CASE 1 – A Left-Tail Test

Problem: Determine the critical values for the following **one-tail t-test**:

H_0: $\beta_1 = b_1$ *or* $\beta_1 - b_1 = 0$ where b_1 is a specified constant
H_1: $\beta_1 < b_1$ *or* $\beta_1 - b_1 < 0$ $\Leftarrow$ a ***left-sided*** **alternative hypothesis**.
$\Rightarrow$ a ***one-tail*** **test**, specifically a ***left-tail*** **test**.

1. The **appropriate** ***test statistic*** is the **t-statistic for $\hat{\beta}_1$**, the OLS estimate of the slope coefficient β_1:

$$t(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\hat{Var}(\hat{\beta}_1)}} = \frac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)}.$$

2. The ***sample value*** **of the test statistic $t(\hat{\beta}_1)$** is calculated by evaluating $t(\hat{\beta}_1)$ **under the null hypothesis H_0**. This involves evaluating $t(\hat{\beta}_1)$ using the ***equality form of the null hypothesis*** **H_0**, which is $\beta_1 = b_1$. Setting $\beta_1 = b_1$ in the expression for $t(\hat{\beta}_1)$ yields the sample value of the test statistic $t(\hat{\beta}_1)$ under H_0:

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - b_1}{\hat{se}(\hat{\beta}_1)}.$$

Note: The ***null*** **hypothesis for a** ***left-tail*** **test** is sometimes written as H_0: $\beta_1 \geq b_1$ rather than H_0: $\beta_1 = b_1$. But the computational procedure for testing

H_0: $\beta_1 \geq b_1$ against H_1: $\beta_1 < b_1$

is *exactly the same* as the procedure for testing

H_0: $\beta_1 = b_1$ against H_1: $\beta_1 < b_1$.

Question: Why do we use the equality form of the null hypothesis, $\beta_1 = b_1$, to calculate the sample value of the test statistic?

Answer: A two-part answer:

- A test that takes the null hypothesis as H_0: $\beta_1 = b_1$ is the most favorable to the null hypothesis, and hence is the least favorable to the alternative hypothesis H_1: $\beta_1 < b_1$. This means that, at any chosen significance level α, if we reject H_0: $\beta_1 = b_1$ in favor of the alternative hypothesis H_1: $\beta_1 < b_1$, then we would also reject H_0: $\beta_1 = b_1 + c$ in favor of H_1: $\beta_1 < b_1 + c$, where $c > 0$ is any *positive* constant.
- Calculating the value of $t(\hat{\beta}_1)$ for *all* values of $\beta_1 > b_1$ would be extremely tedious!! (How would you know when you're done?) Moreover, its unnecessary.

3. The ***null distribution* of $t_0(\hat{\beta}_1)$ is $t[N-2]$**, the t-distribution with N – 2 degrees of freedom. In other words, if the null hypothesis H_0 is true – i.e., if $\beta_1 = b_1$, then

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - b_1}{\hat{se}(\hat{\beta}_1)} \sim t[N-2] \text{ under } H_0\text{: } \beta_1 = b_1.$$

Question 1: What values of $\hat{\beta}_1$ and $t_0(\hat{\beta}_1)$ would lead us to reject H_0 against H_1?

H_0: $\beta_1 = b_1$ *or* $\beta_1 - b_1 = 0$
H_1: $\beta_1 < b_1$ *or* $\beta_1 - b_1 < 0$ $\Leftarrow$ a ***left-sided*** **alternative hypothesis**.

Answer: Examine the numerator of the calculated t-statistic:

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - b_1}{\hat{se}(\hat{\beta}_1)}$$

Remember:

$\hat{\beta}_1$ = the ***estimated*** **value of** $\boldsymbol{\beta_1}$
b_1 = the ***hypothesized*** **value of** $\boldsymbol{\beta_1}$
$\hat{se}(\hat{\beta}_1) > 0$ ($\hat{se}(\hat{\beta}_1)$ is ***always*** **a** ***positive*** **number**)

- We would reject H_0 against H_1 if $\hat{\beta}_1$ is much less than b_1, if **the *estimated* value of $\boldsymbol{\beta_1}$ is *much less than* $\mathbf{b_1}$, the *hypothesized* value of $\boldsymbol{\beta_1}$**.

- More specifically, we would **reject $\mathbf{H_0}$ against $\mathbf{H_1}$** in the following case:

$$\hat{\beta}_1 << b_1 \quad \Rightarrow \quad \hat{\beta}_1 - b_1 << 0 \quad \Rightarrow \quad t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - b_1}{\hat{se}(\hat{\beta}_1)} << 0$$

Values of $\boldsymbol{\hat{\beta}_1}$ *much less than* $\mathbf{b_1}$ imply **large *negative* values of $\mathbf{t_0(\hat{\beta}_1)}$**.

Question 2: **How much *less* than zero does the value of $t_0(\hat{\beta}_1)$ have to be** for us ***to reject* H_0 in favour of H_1?**

Answer: It depends on **the *significance level* we choose** for the test and **the *null distribution* of our test statistic**.

Let α = the chosen significance level for the test (e.g., 0.01, 0.05, or 0.10)
= the probability of making a Type I error.

Because only large negative values of t_0 favour the alternative hypothesis, there is only one critical value – a lower critical value of the t[N − 2]-distribution that delineates a lower or left-tail rejection area equal to α.

$-t_\alpha[N-2]$ = the ***lower* α critical value** of the **t[N − 2]-distribution**.

Implications: **If H_0: $\beta_1 = b_1$ is *true***, then the following two probability statements hold.

(1) $$\Pr\left(t_0(\hat{\beta}_1) \geq -t_\alpha[N-2]\right) = 1-\alpha \quad \textbf{(1)}$$

(2) $$\Pr\left(t_0(\hat{\beta}_1) < -t_\alpha[N-2]\right) = \alpha \quad \textbf{(2)}$$

where

$t_0(\hat{\beta}_1)$ = the **calculated *sample* value of the t-statistic** under H_0;
$-t_\alpha[N-2]$ = the ***lower* α-level critical value** of the t[N − 2]-distribution;
α = the ***significance* level** for the test;
$1-\alpha$ = the ***confidence* level** for the test.

Determine the Rejection and Nonrejection Regions -- Left-Tail Test

- The ***non-rejection*** **region for $t_0(\hat{\beta}_1)$** is defined by the inequality in probability statement (1) above.

(1) $$\Pr\left(t_0(\hat{\beta}_1) \ge -t_\alpha[N-2]\right) = 1-\alpha$$ **(1)**

It consists of all values of $t_0(\hat{\beta}_1)$ such that

$t_0(\hat{\beta}_1) \ge -t_\alpha[N-2]$ ⇐ ***non-rejection*** **region under H_0: $\beta_1 = b_1$**.

- The ***rejection*** **region for $t_0(\hat{\beta}_1)$** is the set of values defined by the inequality in probability statement (2) above.

(2) $$\Pr\left(t_0(\hat{\beta}_1) < -t_\alpha[N-2]\right) = \alpha$$ **(2)**

It consists of all values of $t_0(\hat{\beta}_1)$ such that

$t_0(\hat{\beta}_1) < -t_\alpha[N-2]$ ⇐ ***rejection*** **region under H_0: $\beta_1 = b_1$** is a ***one-tail left-tail*** **rejection region**.

NOTE:

1) For a ***left-tail*** **test**, the rejection region $t_0(\hat{\beta}_1) < -t_\alpha[N-2]$ consists only of the ***lower left-hand*** **tail** of the t-distribution with N − 2 degrees of freedom.

2) This **left-tail rejection region** contains **unexpectedly *small* values** of $t_0(\hat{\beta}_1)$ under H_0 – i.e., values that we would only expect to obtain with “small” probability α if the null hypothesis H_0: $\beta_1 = b_1$ is true.

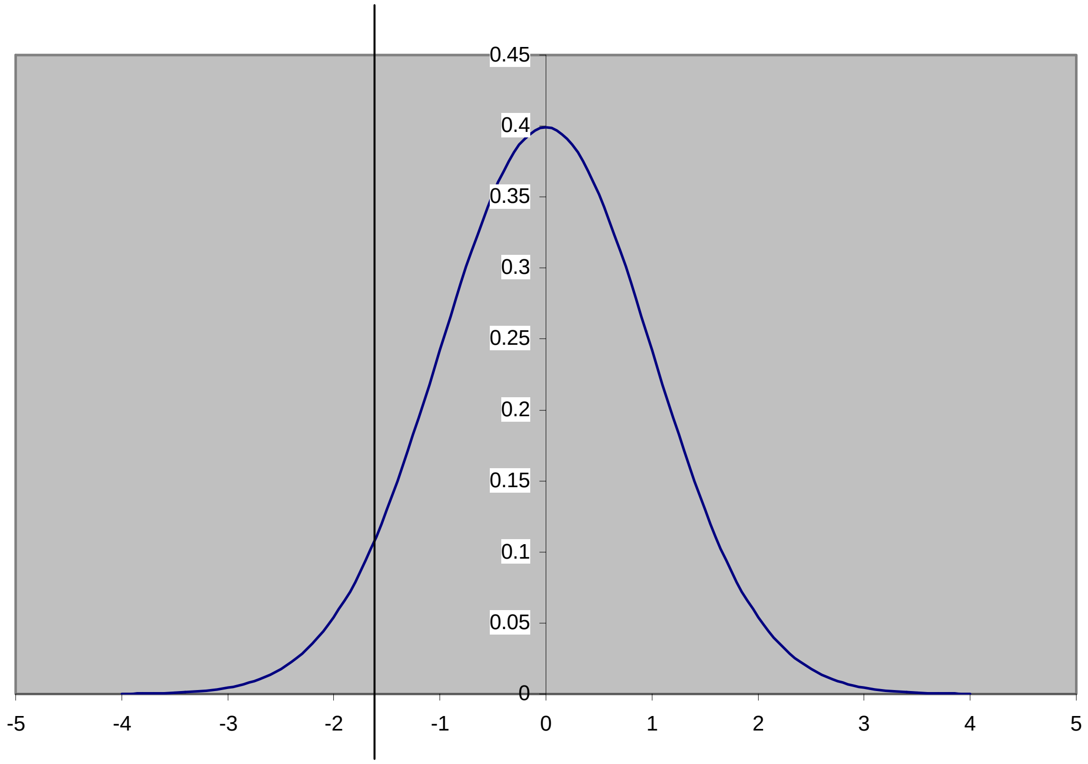
3) The ***left tail area*** **under the t[N − 2]-distribution** in this lower tail ***equals*** **the significance level α**. This area is called the ***lower*** **α-level (or lower 100α percent) tail area** of the t[N − 2]-distribution.

Rejection and Nonrejection Regions for a Left-Tail Test

left-tail **rejection region** ↓ ***nonrejection*** **region**

area = α ↓ area = 1 − α

← ↓ →



$-\,t_\alpha$

← ↑ →

↑

$\Pr(t_0 < -\,t_\alpha) = \alpha$ ↑ $\Pr(t_0 \geq -\,t_\alpha) = 1 - \alpha$

<u>STEP 5</u>: Apply the Decision Rule and State Inference

<u>Step 5</u>: Apply the *decision rule* of the test and **state the *inference***, or conclusion, implied by the sample value of the test statistic.

1. Formulation 1 of the Decision Rule for a Left-Tail Test

Formulation 1: Determine if the *sample value* t_0 of the test statistic **lies in the *rejection* or *nonrejection* region** at the chosen significance level α.

<u>Decision Rule for a Left-Tail Test -- Formulation 1</u>

1. If the ***sample value*** **t_0 of the test statistic** lies in **the *left-tail rejection* region** at the chosen significance level, then ***reject*** **the null hypothesis H_0**.

For a ***<u>left-tail</u>*** **test**: $t_0 < -t_\alpha[N-2]$ ⇒ ***reject*** **H_0** at **significance level α.**

Reject **H_0 in favour of H_1** at **significance level** α if

$$t_0 < -t_\alpha[N-2]$$ meaning **t_0 lies in the *lower left-tail* rejection area**.

<u>Inference</u>*: *Reject **H_0 in favour of H_1** at **significance level** α.

2. If the ***sample value*** **t_0 of the test statistic** lies in **the *nonrejection* region** at the chosen significance level, then ***retain*** **(do not reject) the null hypothesis H_0**.

For a ***<u>left-tail</u>*** **test**: $t_0 \geq -t_\alpha[N-2]$ ⇒ ***retain*** **H_0** at **significance level α.**

Retain **H_0 against H_1** at **significance level** α if

$$t_0 \geq -t_\alpha[N-2]$$ meaning **t_0 lies in the *nonrejection* area**.

<u>Inference</u>*: *Retain **H_0 against H_1** at **significance level** α.

CASE 2 – A Right-Tail Test

Problem: Determine the critical values for the following **one-tail t-test**:

H_0: $\beta_1 = b_1$ *or* $\beta_1 - b_1 = 0$ where b_1 is a specified constant
H_1: $\beta_1 > b_1$ *or* $\beta_1 - b_1 > 0$ ⇐ a ***right-sided*** **alternative hypothesis**.
⇒ a ***one-tail*** **test**, specifically a ***right-tail*** **test**.

NOTE: This one-tail test is a ***right-tail*** **test** because, as we will see, the ***rejection*** **region** for the test consists of the ***upper right-hand*** **tail** of the appropriate t-distribution.

1. Again, the **appropriate *test statistic*** is the **t-statistic for $\hat{\beta}_1$**, the OLS estimate of the slope coefficient β_1:

$$t(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\hat{\text{Var}}(\hat{\beta}_1)}} = \frac{\hat{\beta}_1 - \beta_1}{\hat{\text{se}}(\hat{\beta}_1)}.$$

2. The ***sample value* of the test statistic $t(\hat{\beta}_1)$** is calculated by evaluating $t(\hat{\beta}_1)$ **under the null hypothesis H_0**. This involves evaluating $t(\hat{\beta}_1)$ using the ***equality form of the null hypothesis H_0***, which is $\beta_1 = b_1$. Setting $\beta_1 = b_1$ in the expression for $t(\hat{\beta}_1)$ yields the sample value of the test statistic $t(\hat{\beta}_1)$ under H_0:

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - b_1}{\hat{\text{se}}(\hat{\beta}_1)}.$$

Note: The ***null* hypothesis for a *right-tail* test** can be written as H_0: $\beta_1 \le b_1$ rather than H_0: $\beta_1 = b_1$. But the computational procedure for testing

H_0: $\beta_1 \le b_1$ against H_1: $\beta_1 > b_1$

is *exactly the same* as the procedure for testing

H_0: $\beta_1 = b_1$ against H_1: $\beta_1 > b_1$.

Question: Why do we use the equality form of the null hypothesis, $\beta_1 = b_1$, to calculate the sample value of the test statistic?

Answer: A test that takes the null hypothesis as H_0: $\beta_1 = b_1$ is the most favorable to the null hypothesis, and hence is the least favorable to the alternative hypothesis H_1: $\beta_1 > b_1$. This means that, at any chosen significance level α, if we reject H_0: $\beta_1 = b_1$ in favor of the alternative hypothesis H_1: $\beta_1 > b_1$, then we would also reject H_0: $\beta_1 = b_1 - c$ in favor of H_1: $\beta_1 > b_1 - c$, where $c > 0$ is any *positive* constant.

3. The ***null distribution*** **of $t_0(\hat{\beta}_1)$ is $t[N - 2]$**, the t-distribution with N – 2 degrees of freedom. In other words, if the null hypothesis H_0 is true – i.e., if $\beta_1 = b_1$, then

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - b_1}{\hat{se}(\hat{\beta}_1)} \sim t[N-2] \quad \text{under } H_0\text{: } \beta_1 = b_1.$$

Question 1: What values of $\hat{\beta}_1$ and $t_0(\hat{\beta}_1)$ would lead us to reject H_0 against H_1?

H_0: $\beta_1 = b_1$ *or* $\beta_1 - b_1 = 0$
H_1: $\beta_1 > b_1$ *or* $\beta_1 - b_1 > 0$ $\Leftarrow$ a ***right-sided*** **alternative hypothesis**.

Answer: Examine the numerator of the calculated t-statistic:

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - b_1}{\hat{se}(\hat{\beta}_1)}.$$

Remember:

$\hat{\beta}_1$ = the ***estimated*** **value of** $\boldsymbol{\beta_1}$
b_1 = the ***hypothesized*** **value of** $\boldsymbol{\beta_1}$
$\hat{se}(\hat{\beta}_1) > 0$ ($\hat{se}(\hat{\beta}_1)$ is ***always*** **a** ***positive*** **number**)

- We would reject H_0 against H_1 if $\hat{\beta}_1$ is much greater than b_1, if **the *estimated* value of $\boldsymbol{\beta_1}$ is *much greater than* $\mathbf{b_1}$, the *hypothesized* value of $\boldsymbol{\beta_1}$**.

- More specifically, we would **reject $\mathbf{H_0}$ against $\mathbf{H_1}$** in the following case:

$$\hat{\beta}_1 >> b_1 \Rightarrow \quad \hat{\beta}_1 - b_1 >> 0 \quad \Rightarrow \quad t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - b_1}{\hat{se}(\hat{\beta}_1)} >> 0$$

Values of $\boldsymbol{\hat{\beta}_1}$ *much greater than* $\mathbf{b_1}$ imply **large *positive* values of $\mathbf{t_0(\hat{\beta}_1)}$**.

<u>Question 2</u>: **How much *greater* than zero does the value of $t_0(\hat{\beta}_1)$ have to be** for us ***to reject*** **H_0 in favour of H_1?**

Answer: It depends on **the *significance level* we choose** for the test and **the *null distribution* of our test statistic**.

Let α = the chosen significance level for the test (e.g., 0.01, 0.05, or 0.10)
= the probability of making a Type I error.

Because only large positive values of t_0 favour the alternative hypothesis, there is only one critical value – an upper critical value of the t[N – 2]-distribution that delineates an upper or right-tail rejection area equal to α.

$t_\alpha[N-2]$ = the ***upper*** **α critical value** of the **t[N – 2]-distribution**.

<u>Implications</u>: **If H_0: $\beta_1 = b_1$ is *true***, then the following two probability statements hold.

(1) $$\Pr\left(t_0(\hat{\beta}_1) \le t_\alpha[N-2]\right) = 1-\alpha \qquad \textbf{(1)}$$

(2) $$\Pr\left(t_0(\hat{\beta}_1) > t_\alpha[N-2]\right) = \alpha \qquad \textbf{(2)}$$

where

$t_0(\hat{\beta}_1)$ = the **calculated *sample* value of the t-statistic** under H_0;
$t_\alpha[N-2]$ = the ***<u>upper</u>*** **α-level critical value** of the t[N – 2]-distribution;
α = the ***significance*** **level** for the test;
$1-\alpha$ = the ***confidence*** **level** for the test.

Determine the Rejection and Nonrejection Regions – Right-Tail Test

- The ***non-rejection* region for $t_0(\hat{\beta}_1)$** is the set of values defined by the inequality in probability statement (1) above:

(1) $$\Pr\left(t_0(\hat{\beta}_1) \le t_\alpha[N-2]\right) = 1-\alpha$$ **(1)**

It consists of all values of $t_0(\hat{\beta}_1)$ such that

$t_0(\hat{\beta}_1) \le t_\alpha[N-2]$ ⇐ ***non-rejection* region under H_0: $\beta_1 = b_1$**.

- The ***rejection* region for $t_0(\hat{\beta}_1)$** is the set of values defined by the inequality in probability statement (2) above:

(2) $$\Pr\left(t_0(\hat{\beta}_1) > t_\alpha[N-2]\right) = \alpha$$ **(2)**

It consists of all values of $t_0(\hat{\beta}_1)$ such that

$t_0(\hat{\beta}_1) > t_\alpha[N-2]$ ⇐ ***rejection* region under H_0: $\beta_1 = b_1$**
is a ***one-tail right-tail* rejection region**.

NOTE:

1) For a ***right-tail* test**, the rejection region $t_0(\hat{\beta}_1) > t_\alpha[N-2]$ consists only of the ***upper right-hand* tail** of the t-distribution with N – 2 degrees of freedom.

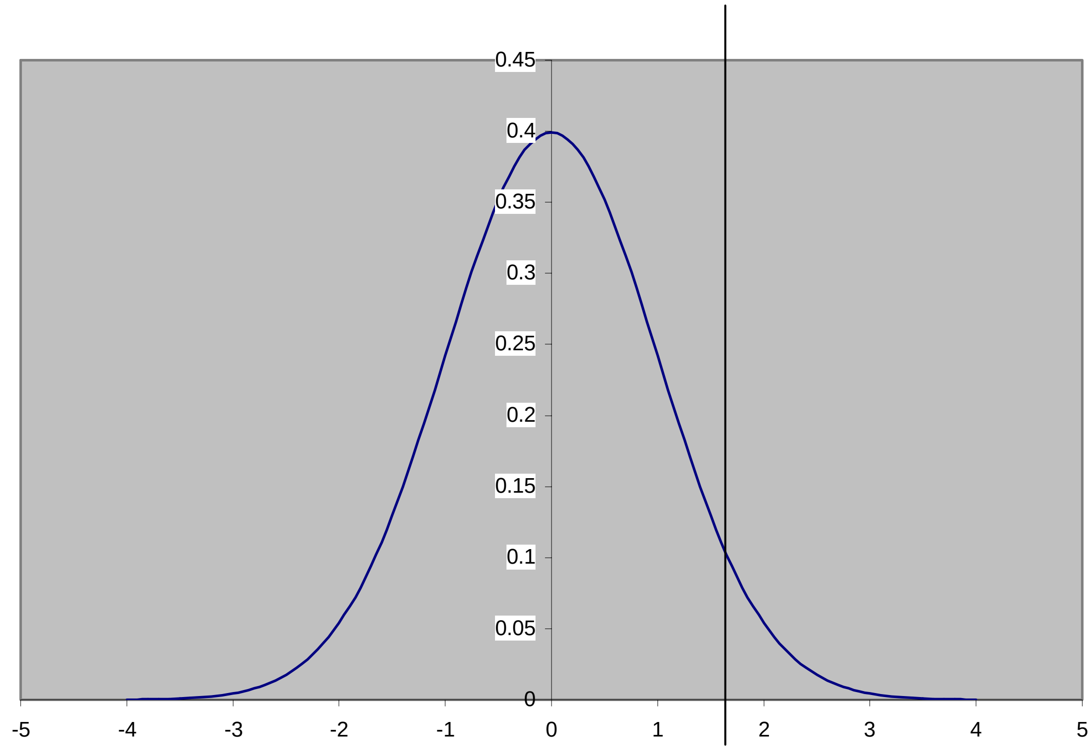
2) This **right-tail rejection region** contains **unexpectedly *large* values** of $t_0(\hat{\beta}_1)$ under H_0 – i.e., values that we would only expect to obtain with "small" probability α if the null hypothesis H_0: $\beta_1 = b_1$ is true.

3) The ***right tail area* under the t[N – 2]-distribution** in this upper tail **equals the significance level α**. This area is called the ***upper* α-level (or upper 100α percent) tail area** of the t[N – 2]-distribution.

Rejection and Nonrejection Regions for a Right-Tail Test

nonrejection **region** ↓ ***right-tail rejection*** **region**

area = $1 - \alpha$ ↓ area = α



t_α

↑

$\Pr(t_0 \leq t_\alpha) = 1 - \alpha$ ↑ $\Pr(t_0 > t_\alpha) = \alpha$

STEP 5: Apply the Decision Rule and State Inference

Step 5: **Apply the *decision rule*** of the test and **state the *inference***, or conclusion, implied by the sample value of the test statistic.

1. Formulation 1 of the Decision Rule for a Right-Tail Test

Formulation 1: Determine if the *sample value t_0* of the test statistic **lies in the *rejection* or *nonrejection* region** at the chosen significance level α.

Decision Rule for a Right-Tail Test -- Formulation 1

1. If the ***sample value* t_0 of the test statistic** lies in **the *right-tail rejection* region** at the chosen significance level, then ***reject* the null hypothesis H_0**.

For a ***right-tail*** **test**: $t_0 > t_\alpha[N-2] \;\Rightarrow$ ***reject*** **H_0** at **significance level α.**

Reject **H_0 in favour of H_1** at **significance level** α if

$t_0 > t_\alpha[N-2]$ meaning **t_0 lies in the *upper right-tail* rejection area**.

Inference: Reject **H_0 in favour of H_1** at **significance level** α.

2. If the ***sample value* t_0 of the test statistic** lies in **the *nonrejection* region** at the chosen significance level, then ***retain* (do not reject) the null hypothesis H_0**.

For a ***right-tail*** **test**: $t_0 \le t_\alpha[N-2] \;\Rightarrow$ ***retain*** **H_0** at **significance level α.**

Retain **H_0 against H_1** at **significance level** α if

$t_0 \le t_\alpha[N-2]$ meaning **t_0 lies in the *nonrejection* area**.

Inference: Retain **H_0 against H_1** at **significance level** α.

Examples of One-Tail Hypothesis Tests

The Model:

DATA: **auto1.dta** (a Stata-format data file)

MODEL: $\mathbf{price_i = \beta_0 + \beta_1 \, weight_i + u_i \quad (i = 1, ..., N)}$

```
. regress price weight

  Source |       SS       df       MS                  Number of obs =      74
---------+------------------------------               F(  1,    72) =   29.42
   Model |   184233937     1   184233937               Prob > F      =  0.0000
Residual |   450831459    72  6261548.04               R-squared     =  0.2901
---------+------------------------------               Adj R-squared =  0.2802
   Total |   635065396    73  8699525.97               Root MSE      =  2502.3

------------------------------------------------------------------------------
   price |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
  weight |   2.044063   .3768341      5.424   0.000       1.292858    2.795268
   _cons |  -6.707353    1174.43     -0.006   0.995       -2347.89    2334.475
------------------------------------------------------------------------------
```

$N = 74 \qquad N - 2 = 74 - 2 = \mathbf{72}$

$\hat{\beta}_1 = \mathbf{2.0441} \qquad \hat{se}(\hat{\beta}_1) = \mathbf{0.376834}$

$\alpha = 0.05 \quad \Rightarrow \quad \alpha/2 = 0.025$

$t_{\alpha/2}[N-2] = t_{0.025}[72] = \mathbf{1.9935}$

$t_{\alpha}[N-2] = t_{0.05}[72] = \mathbf{1.6663}$

<u>**Test 1 – A Left-Tail Test:**</u> Test the proposition that **$weight_i$ has a *negative* effect on $price_i$**. Perform the test at the 5 percent significance level ($\alpha = 0.05$).

- ***Null* and *Alternative* Hypotheses**

  H_0: $\beta_1 = 0$

  H_1: $\beta_1 < 0$ a ***one-sided <u>left-sided</u>* alternative hypothesis**.

- The ***feasible* test statistic** is the t-statistic for $\hat{\beta}_1$:

$$t(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\hat{Var}(\hat{\beta}_1)}} = \frac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} \sim t[N-2].$$

- **Compute the *sample value* of $t(\hat{\beta}_1)$** under the null hypothesis H_0.
  Set $\hat{\beta}_1 = 2.0441$, $\beta_1 = 0$ and $\hat{se}(\hat{\beta}_1) = 0.376834$ in the formula for $t(\hat{\beta}_1)$:

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} = \frac{2.0441 - 0}{0.376834} = \frac{2.0441}{0.376834} = \mathbf{5.424}$$

- The ***one-tail critical value* of the t[N – 2] distribution** at the **5 percent significance level** (at **α = 0.05**) is $t_\alpha[N-2] = t_{0.05}[72] = \mathbf{1.6663}$.

- ***Decision Rule -- Left-Tail Test:***

  If $t_0 < -t_\alpha[N-2]$ ***reject* H_0** at **significance level** α;
  If $t_0 \ge -t_\alpha[N-2]$ ***retain* H_0** at **significance level** α.

- ***Inference:***

  $t_0 = 5.424 > -1.6663 = -t_{0.05}[72]$ $\Rightarrow$ ***retain* H_0** at **significance level α = 0.05**

  ***Retain* H_0: $\beta_1 = 0$** against **H_1: $\beta_1 < 0$** at the ***5 percent* significance level**.

<u>Test 2 – A Right-Tail Test</u>: Test the proposition that **$weight_i$ has a *positive* effect on $price_i$**. Perform the test at the 5 percent significance level ($\alpha = 0.05$).

- ***Null* and *Alternative* Hypotheses**

H_0: $\beta_1 = 0$

H_1: $\beta_1 > 0$ a ***one-sided <u>right-sided</u>* alternative hypothesis**.

- The ***feasible* test statistic** is the t-statistic for $\hat{\beta}_1$:

$$t(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\hat{Var}(\hat{\beta}_1)}} = \frac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} \sim t[N-2].$$

- **Compute the *sample value* of $t(\hat{\beta}_1)$** under the null hypothesis H_0.
  Set $\hat{\beta}_1 = 2.0441$, $\beta_1 = 0$ and $\hat{se}(\hat{\beta}_1) = 0.376834$ in the formula for $t(\hat{\beta}_1)$:

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} = \frac{2.0441 - 0}{0.376834} = \frac{2.0441}{0.376834} = \mathbf{5.424}$$

- The ***one-tail critical value* of the t[N – 2] distribution** at the **5 percent significance level** (at α **= 0.05**) is $t_\alpha[N-2] = t_{0.05}[72] = \mathbf{1.6663}$.

- ***Decision Rule -- Right-Tail Test:***

If $t_0 > t_\alpha[N-2]$ ***reject*** H_0 at **significance level** α;
If $t_0 \le t_\alpha[N-2]$ ***retain*** H_0 at **significance level** α.

- ***Inference:***

$t_0 = 5.424 > \mathbf{1.6663} = t_{0.05}[72] \Rightarrow$ ***reject*** H_0 at **significance level** α **= 0.05**

Reject H_0: $\beta_1 = 0$ against H_1: $\beta_1 > 0$ at the ***5 percent*** **significance level**.

The **sample evidence *favours*** the alternative hypothesis H_1: $\beta_1 > 0$.

Interpretation of the Decision Rules

❑ An **hypothesis test** can lead to ***only two*** **possible decisions**:

either

(1) a decision *to reject* the null hypothesis H_0 against the alternative hypothesis H_1, in which case the sample evidence favours H_1 over H_0;

or

(2) a decision *to retain (not to reject)* the null hypothesis H_0 against the alternative hypothesis H_1, in which case the sample evidence favours H_0 over H_1.

❑ **Points to remember** in interpreting these alternative decisions.

1) An hypothesis test can never be interpreted as **either *proving* or *disproving* the truth of the null hypothesis**.

Reason: The decision to reject or retain H_0 on the basis of sample evidence is always subject -- explicitly or implicitly -- to some uncertainty, or margin of statistical error. That is, there is always some non-zero probability of committing a Type I or Type II error.

2) A decision *to retain (not to reject)* the null hypothesis H_0 should not be interpreted to mean that we "accept" H_0, or that H_0 is true.

Reason: Saying "we accept H_0" implies that we are concluding that the null hypothesis is true, but such a conclusion is incorrect. Nonrejection (or retention) of H_0 means only that the sample data provide insufficient evidence to reject H_0; it does not mean that H_0 is true beyond any doubt.

Explanation: see example in Gujarati (2003, p. 134).

We obtain an estimate of the slope coefficient of $\hat{\beta}_1 = 0.5091$ and a corresponding estimated standard error of $\hat{se}(\hat{\beta}_1) = 0.0357$.

- First, we perform a **two-tail test of H_0: β_1 = 0.50** against H_1: $\beta_1 \neq 0.50$.

  The sample value of the t-statistic under H_0 is calculated as:

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} = \frac{0.5091 - 0.50}{0.0357} = 0.25\,.$$

  But the sample value 0.25 is clearly insignificant at, say, the 5% significance level ($\alpha = 0.05$). Suppose we "accept" H_0 and conclude that the true value of β_1 is 0.50.

- Next, we perform a **two-tail test of H_0: β_1 = 0.48** against H_1: $\beta_1 \neq 0.48$.

  The sample value of the t-statistic under this H_0 is calculated as:

$$t_0(\hat{\beta}_1) = \frac{\hat{\beta}_1 - \beta_1}{\hat{se}(\hat{\beta}_1)} = \frac{0.5091 - 0.48}{0.0357} = 0.82\,.$$

  But the sample value 0.82 is also clearly insignificant at the 5% significance level ($\alpha = 0.05$). Do we now "accept" this H_0 and conclude that the true value of β_1 is 0.48?

- *Question:* Do we "accept" either or both of these hypothesized values of β_1 as the true value?

  The correct answer is NO. We do not know the exact true value of β_1.

♦ ***Conclusion:*** All we can legitimately conclude from these two hypothesis tests is that the sample evidence is consistent or compatible with both the null hypotheses we have tested. But the tests provide no reason to conclude that either hypothesized value, 0.50 or 0.48, is the true value of β_1.

3) **A decision *to reject* H_0** on the basis of given sample data **does not imply that we *must accept* the alternative hypothesis H_1**, or that H_1 *must be true*, or that H_1 does in fact *contain the truth*.

<u>Reason</u>: An hypothesis test is designed only to assess the probable empirical validity of the null hypothesis H_0; it is not designed to test the alternative hypothesis H_1.

- Regardless of whether a test outcome for some particular sample data indicates rejection or nonrejection of H_0, the set of alternative possibilities specified by the alternative hypothesis H_1 may or may not contain the truth.

- It is quite possible for a test of some null hypothesis H_0 against some alternative hypothesis H_1 to indicate rejection of H_0 *when H_1 is false* – that is, when the alternative possibilities specified by H_1 do not contain the truth.

Formulation 2 of the Decision Rule: the p-value Rule

Formulation 2: Determine if the **p-value for t_0,** the calculated sample value of the test statistic, **is *smaller* or *larger* than the chosen significance level** α.

Definition: The **p-value** (or **probability value**) associated with the calculated sample value of the test statistic is defined as the ***lowest*** **significance level at which the null hypothesis H_0 *can be rejected***, given the calculated sample value of the test statistic.

Interpretation

- The **p-value** is the **probability of obtaining a *sample value* of the test statistic *as extreme as* the one we computed *if the null hypothesis H_0 is true***.

- **P-values** serve as ***inverse*** **measures** of **the strength of evidence *against* the *null* hypothesis H_0**.

    - ***Small*** **p-values – p-values *close to zero* –** constitute ***strong*** **evidence against the null hypothesis H_0**.

    - ***Large*** **p-values – p-values *close to one* –** provide only ***weak*** **evidence against the null hypothesis H_0**.

Examples of p-values for common types of hypothesis tests

- For a ***two-tail* t-test**, let the calculated sample value of the t-statistic for a given null hypothesis be t_0. Then the p-value associated with the sample value t_0 is the probability that the null distribution of the test statistic takes a value greater than the absolute value of t_0, where the absolute value of t_0 is denoted as $|t_0|$. That is,

$$\textit{two-tail}\ \textbf{p-value for}\ t_0 = \Pr(|t| > |t_0|)$$
$$= \Pr(t > t_0) + \Pr(t < -t_0) = 2\Pr(t > t_0) \quad \text{if } t_0 > 0$$
$$= \Pr(t < t_0) + \Pr(t > -t_0) = 2\Pr(t < t_0) \quad \text{if } t_0 < 0$$

Remember: the t-distribution is symmetric about its mean of zero.

- For a ***one-tail* t-test**, let the calculated sample value of the t-statistic for a given null hypothesis be t_0. Then the p-value associated with the sample value t_0 is depends on whether the test is a ***right-tail*** or ***left-tail*** **test**.

(1) For a ***right-tail*** **t-test**, the p-value associated with the sample value t_0 is the probability that the null distribution of the test statistic takes a value ***greater than*** the calculated sample value t_0 – i.e.,

$$\textit{right-tail}\ \textbf{p-value for}\ t_0 = \Pr(t > t_0).$$

(2) For a ***left-tail*** **t-test**, the p-value associated with the sample value t_0 is the probability that the null distribution of the test statistic takes a value ***less than*** the calculated sample value t_0 – i.e.,

$$\textit{left-tail}\ \textbf{p-value for}\ t_0 = \Pr(t < t_0).$$

♦ For an **<u>F-test</u>**, let the calculated sample value of the F-statistic for a given null hypothesis be F_0. Then the p-value associated with the sample value F_0 is the probability that the null distribution of the test statistic takes a value greater than the calculated sample value F_0 – i.e.,

p-value for F_0 = $\Pr(F > F_0)$.

Note that the F-distribution is defined only over non-negative values that are greater than or equal to zero.

<u>Decision Rule -- Formulation 2</u>

1. If the **p-value** for the calculated sample value of the test statistic ***is less than*** the chosen **significance level** α, ***reject* the null hypothesis** at significance level α.

p-value $< \alpha \;\Rightarrow\;$ ***reject*** $\mathbf{H_0}$ at **significance level** α**.**

2. If the **p-value** for the calculated sample value of the test statistic ***is greater than or equal to*** the chosen **significance level** α***, retain (i.e., do not reject)* the null hypothesis** at significance level α.

p-value $\geq \alpha \;\Rightarrow\;$ ***retain*** $\mathbf{H_0}$ at **significance level** α**.**