Linear Regression Analysis: Terminology and Notation

Consider the generic version of the ***simple (two-variable)*** **linear regression model**.

It is represented by the following **population regression equation** (called the **PRE** for short):

$$Y_i = f(X_i) + u_i = \beta_0 + \beta_1 X_i + u_i$$

- **The PRF (population regression function):**

  $f(X_i) = \beta_0 + \beta_1 X_i$
  = the i-th value of the population regression function (PRF).

- **Observable Variables:**

  $Y_i \equiv$ the i-th value of the dependent variable Y
  $X_i \equiv$ the i-th value of the independent variable X

- **Unobservable Variable:**

  $u_i \equiv$ the random error term for the i-th member of the population

- **Unknown Parameters:** the regression coefficients

  $\boldsymbol{\beta_0}$ = the ***intercept*** **coefficient**
  $\boldsymbol{\beta_1}$ = the ***slope*** **coefficient on $\mathbf{X_i}$**

  The **true population values** of the regression coefficients β_0 and β_1 are ***unknown***.

Variables and Parameters

PRE: $Y_i = f(X_i) + u_i = \beta_0 + \beta_1 X_i + u_i$

- The **variables** of the regression model are $\mathbf{Y_i}$**,** $\mathbf{X_i}$**,** and $\mathbf{u_i}$.

  $\mathbf{Y_i}$ and $\mathbf{X_i}$ are the ***observable*** **variables**; their values can be observed or measured.

  Y_i is called any of the following: (1) the ***dependent*** **variable**
  (2) the ***regressand***
  (3) the ***explained*** **variable**.

  X_i is called any of the following: (1) the ***independent*** **variable**
  (2) the ***regressor***
  (3) the ***explanatory*** **variable**.

- $\mathbf{u_i}$ is an ***unobservable*** **random variable**; its value cannot be observed or measured. It is called a **random error term**.

- $\boldsymbol{\beta_0}$ **and** $\boldsymbol{\beta_1}$ are the **parameters** of the regression model, together with any unknown parameters of the probability distribution of the random error term u_i.

  $\boldsymbol{\beta_0}$ **and** $\boldsymbol{\beta_1}$ are called **regression *coefficients***; in particular,

  $\boldsymbol{\beta_0} \equiv$ the ***intercept*** **coefficient**,

  and

  $\boldsymbol{\beta_1} \equiv$ the ***slope*** **coefficient** of X.

  The **true population values** of the regression coefficients β_0 and β_1 are ***unknown***.

Simple Regression versus Multiple Regression

- A ***simple*** **regression model** has ***only two*** observable variables:

  (1) **one** ***dependent*** variable or ***regressand*** Y_i;
  (2) **one** ***independent*** variable or ***regressor*** X_i.

  Population regression equation – or PRE – for a simple linear regression model:

  $$Y_i = \beta_0 + \beta_1 X_i + u_i$$

- A ***multiple*** **regression model** has ***three or more*** observable variables:

  (1) **one** ***dependent*** variable or ***regressand*** $\mathbf{Y_i}$;
  (2) **two or more** ***independent*** variables or ***regressors*** $\mathbf{X_{1i}, X_{2i}, ..., X_{ki}}$, where

  $X_{ji} \equiv$ the i-th value of the j-th regressor X_j ($j = 1, 2, \ldots, k$).

  Population regression equation – or PRE – for a multiple linear regression model:

  $$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \cdots + \beta_k X_{ki} + u_i$$

The Simple Linear Regression Model

- The **PRE (population regression equation)** for the simple linear regression model:

$$Y_i = f(X_i) + u_i = \underbrace{\beta_0 + \beta_1 X_i}_{\text{PRF}} + \underbrace{u_i}_{\text{random error}} \tag{1a}$$

$f(X_i) = \beta_0 + \beta_1 X_i$
= the **PRF** (population regression function) for the i-th population member

$u_i = Y_i - f(X_i) = Y_i - \beta_0 - \beta_1 X_i$
= the **random error** for the i-th population member

β_0, β_1 = the unknown regression coefficients β_0 and β_1

Number of regression coefficients = K = 2.
Number of slope coefficients = K – 1 = 2 – 1 = 1.

- **Sample Data:** A ***random*** **sample of N members of the population** for which the observed values of Y and X are measured. Each sample observation is of the form

$$(\mathbf{Y_i, X_i}), \qquad \mathbf{i = 1, ..., N}$$

- The **SRE (sample regression equation)** for the simple linear regression model:

$$Y_i = \hat{f}(X_i) + \hat{u}_i = \hat{Y}_i + \hat{u}_i = \underbrace{\hat{\beta}_0 + \hat{\beta}_1 X_i}_{\text{SRF}} + \underbrace{\hat{u}_i}_{\text{residual}} \tag{1b}$$

$\hat{f}(X_i) = \hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$
= the **SRF (sample regression function)** for sample observation i

$\hat{u}_i = Y_i - \hat{f}(X_i) = Y_i - \hat{Y}_i = Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i$
= the **residual** for sample observation i

$\hat{\beta}_0, \hat{\beta}_1$ = ***estimators*** or ***estimates*** of the regression coefficients β_0 and β_1

The Multiple Linear Regression Model

- The **PRE (population regression equation)** for the ***multiple*** **linear regression model** is:

$$Y_i = f(X_{1i}, X_{2i}, \ldots, X_{ki}) + u_i = \underbrace{\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \cdots + \beta_k X_{ki}}_{\text{PRF}} + \underbrace{u_i}_{\text{random error}} \qquad (2a)$$

$f(X_{1i}, X_{2i}, \ldots, X_{ki}) = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \cdots + \beta_k X_{ki}$
= the **PRF** (population regression function) for the i-th population member

$u_i = Y_i - f(X_{1i}, X_{2i}, \ldots, X_{ki}) = Y_i - \beta_0 - \beta_1 X_{1i} - \beta_2 X_{2i} - \cdots - \beta_k X_{ki}$
= the **random error** for the i-th population member

$\beta_0, \beta_1, \beta_2, \ldots, \beta_k$ = the unknown regression coefficients β_0, β_1, β_2, …, β_k

Number of regression coefficients = K.
Number of slope coefficients = k = K – 1.

- **Sample Data:** A ***random*** **sample of N members of the population** for which the observed values of Y and X_1, X_2, …, X_k are measured. Each sample observation is of the form

$$(\mathbf{Y_i, X_{1i}, X_{2i}, \ldots, X_{ki}}), \qquad \mathbf{i = 1, ..., N}$$

- The **SRE (sample regression equation)** for the multiple linear regression model:

$$Y_i = \hat{f}(X_{1i}, X_{2i}, \ldots, X_{ki}) + \hat{u}_i = \hat{Y}_i + \hat{u}_i = \underbrace{\hat{\beta}_0 + \hat{\beta}_1 X_{1i} + \hat{\beta}_2 X_{2i} + \cdots + \hat{\beta}_k X_{ki}}_{\text{SRF}} + \underbrace{\hat{u}_i}_{\text{residual}} \qquad (2b)$$

$\hat{f}(X_{1i}, X_{2i}, \ldots, X_{ki}) = \hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_{1i} + \hat{\beta}_2 X_{2i} + \cdots + \hat{\beta}_k X_{ki}$
= the **SRF (sample regression function)** for sample observation i

$\hat{u}_i = Y_i - \hat{f}(X_{1i}, X_{2i}, \ldots, X_{ki}) = Y_i - \hat{Y}_i = Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i} - \cdots - \hat{\beta}_k X_{ki}$
= the **residual** for sample observation i

$\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_k$ = ***estimators*** or ***estimates*** of the regression coefficients

- Examples of **multiple regression models**

♦ A ***three-variable*** **linear regression model** has **two regressors**; its PRE is written as

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + u_i .$$

Total number of regression coefficients = K = 3
Number of *slope* coefficients = k = K – 1 = 3 – 1 = 2

♦ A ***four-variable*** **linear regression model** has **three regressors**; its PRE is written as

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i .$$

Total number of regression coefficients = K = 4
Number of *slope* coefficients = k = K – 1 = 4 – 1 = 3

♦ The ***general*** **multiple linear regression model** has **k = K – 1 regressors**; its PRE is written as

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \cdots + \beta_k X_{ki} + u_i .$$

Total number of regression coefficients = K.
Number of *slope* coefficients = k = K – 1.

Regression Analysis: A Hypothetical Numerical Example

Reference: D. Gujarati (1995), Chapter 2, pp. 32-36.

Purpose: To illustrate some of the **basic ideas of linear regression analysis**.

The Model: A simple consumption function representing the relationship between

$Y_i \equiv$ the weekly consumption expenditure of family i ($ per week);

$X_i \equiv$ the weekly disposable (after-tax) income of family i ($ per week);

The **PRE (population regression equation)** for this model can be written as

$$Y_i = \beta_0 + \beta_1 X_i + u_i \qquad \textbf{(1)}$$

The Population: consists entirely of **60 families**.

We assume that the weekly disposable incomes of these families take only **10 distinct values** -- i.e., X takes only the 10 distinct values

$\mathbf{X_i}$ **= 80, 100, 120, 140, 160, 180, 200, 220, 240, 260.**

We further assume that we can observe the entire population of 60 families.

The **data for the *complete* population** is given in **Table 2.1**.

<u>Table 2.1</u>: Population data points (Y_i, X_i) for the population of 60 families.

X_i values →	**80**	**100**	**120**	**140**	**160**	**180**	**200**	**220**	**240**	**260**
Y_i values ↓	55	65	79	80	102	110	120	135	137	150
	60	70	84	93	107	115	136	137	145	152
	65	74	90	95	110	120	140	140	155	175
	70	80	94	103	116	130	144	152	165	178
	75	85	98	108	118	135	145	157	175	180
	--	88	--	113	125	140	--	160	189	185
	--	--	--	115	--	--	--	162	--	191
Sum Y_i values	**325**	**462**	**445**	**707**	**678**	**750**	**685**	**1043**	**966**	**1211**
Number of Y_i	**5**	**6**	**5**	**7**	**6**	**6**	**5**	**7**	**6**	**7**

- **<u>Interpretation of Table 2.1</u>:**

  Each column of Table 2.1 represents the **population conditional distribution of Y** (families' weekly consumption expenditure) **for the corresponding value of X** (families' weekly disposable income).

  - The first column gives the conditional distribution of Y for $X_i = 80$; five families in the population have weekly disposable income equal to 80 dollars.

  - The fifth column gives the conditional distribution of Y for $X_i = 160$; six families in the population have weekly disposable income equal to 160 dollars.

  - The tenth (last) column gives the conditional distribution of Y for $X_i = 260$; seven families in the population have weekly disposable income equal to 260 dollars.

<u>Table 2.2</u>: Population conditional probabilities of Y for each population value of X.

- **<u>Notation</u>:**

  $p(Y|X_i) = p(Y_j|X_i) =$ the conditional probability of Y for $X = X_i$

  = the probability that the random variable Y takes the numerical value Y_j given that the variable X is equal to the numerical value X_i.

Conditional probabilities $p(Y|X_i)$ for the population data in Table 2.1

X_i values →	**80**	**100**	**120**	**140**	**160**	**180**	**200**	**220**	**240**	**260**
$p(Y|X_i)$ ↓	1/5	1/6	1/5	1/7	1/6	1/6	1/5	1/7	1/6	1/7
	1/5	1/6	1/5	1/7	1/6	1/6	1/5	1/7	1/6	1/7
	1/5	1/6	1/5	1/7	1/6	1/6	1/5	1/7	1/6	1/7
	1/5	1/6	1/5	1/7	1/6	1/6	1/5	1/7	1/6	1/7
	1/5	1/6	1/5	1/7	1/6	1/6	1/5	1/7	1/6	1/7
	--	1/6	--	1/7	1/6	1/6	--	1/7	1/6	1/7
	--	--	--	1/7	--	--	--	1/7	--	1/7
Sum Y_i values	**325**	**462**	**445**	**707**	**678**	**750**	**685**	**1043**	**966**	**1211**
Number of Y_i	**5**	**6**	**5**	**7**	**6**	**6**	**5**	**7**	**6**	**7**
$E(Y|X_i)$	**65**	**77**	**89**	**101**	**113**	**125**	**137**	**149**	**161**	**173**

- **<u>Interpretation of Table 2.2</u>:**

  Each column of Table 2.2 contains the **population conditional probabilities of Y** (families' weekly consumption expenditure) **for the corresponding value of X** (families' weekly disposable income).

***<u>Examples</u>*: Computing the Conditional Probabilities of Indivdual Y Values**

1. Consider the column corresponding to $X_i = 80$.

   There are ***five*** different values of Y for $X_i = 80$:

   $Y \mid X_i = 80$: 55, 60, 65, 70, 75.

   The probability of observing any one family whose weekly disposable income is $X_i = 80$ equals 1/5: e.g.,

   $$p(Y = 55 \mid X_i = 80) = \frac{1}{5}.$$
   $$p(Y = 60 \mid X_i = 80) = \frac{1}{5}.$$
   $$p(Y = 65 \mid X_i = 80) = \frac{1}{5}.$$
   $$p(Y = 70 \mid X_i = 80) = \frac{1}{5}.$$
   $$p(Y = 75 \mid X_i = 80) = \frac{1}{5}.$$

2. Consider the column corresponding to $X_i = 160$.

   There are ***six*** different values of Y for $X_i = 160$:

   $Y \mid X_i = 160$: 102, 107, 110, 116, 118, 125.

   The probability of observing any one family whose weekly disposable income is $X_i = 160$ equals 1/6: e.g.,

   $$p(Y = 102 \mid X_i = 160) = \frac{1}{6}.$$
   $$p(Y = 110 \mid X_i = 160) = \frac{1}{6}.$$

Population Conditional Means of Y

For each of the 10 population values of X_i, we can compute from Tables 2.1 and 2.2 the corresponding ***conditional* mean value** of the **population values of Y**.

For each of the values X_i of X, the population mean value of Y is called

(1) the **population conditional *mean* of Y**

or

(2) the **population conditional *expectation* of Y**.

- **Notation:**

$$E(Y \mid X_i) = E(Y \mid X = X_i)$$

= the **population conditional mean of Y for X = X_i**

= the "expected value of Y given that X takes the specific value X_i"

- **Definition:**

$$E(Y \mid X_i) = E(Y \mid X = X_i) = \sum_{X=X_i} p(Y \mid X_i) Y$$

where

$p(Y \mid X_i)$ = the conditional probability of Y when X = X_i;

$p(Y \mid X_i)Y$ = the product of each population value of Y and its corresponding conditional probability for X = X_i.

In words, the above formula for $E(Y \mid X_i) = E(Y \mid X = X_i)$ says that for the value X_i of X,

(1) **multiply** each population value of Y by its associated conditional probability $p(Y \mid X_i)$ to get the product $p(Y \mid X_i)Y$

(2) then **sum these products** $p(Y \mid X_i)Y$ over all the population values of Y corresponding to X = X_i.

- **Illustrative Calculations of** $\mathrm{E}(Y \mid X_i)$**:**

1. For X_i = 80, $p(Y \mid X_i)$ = 1/5**:**

$$\begin{aligned} E(Y \mid X_i = 80) &= \frac{1}{5}55 + \frac{1}{5}60 + \frac{1}{5}65 + \frac{1}{5}70 + \frac{1}{5}75 \\ &= \frac{55+60+65+70+75}{5} \\ &= \frac{325}{5} \\ &= 65 \end{aligned}$$

2. For X_i = 160, $p(Y \mid X_i)$ = 1/6**:**

$$\begin{aligned} E(Y \mid X_i = 160) &= \frac{1}{6}102 + \frac{1}{6}107 + \frac{1}{6}110 + \frac{1}{6}116 + \frac{1}{6}118 + \frac{1}{6}125 \\ &= \frac{102+107+110+116+118+125}{6} \\ &= \frac{678}{6} \\ &= 113 \end{aligned}$$

3. For X_i = 260, $p(Y \mid X_i)$ = 1/7**:**

$$\begin{aligned} E(Y \mid X_i = 260) &= \frac{1}{7}150 + \frac{1}{7}152 + \frac{1}{7}175 + \frac{1}{7}178 + \frac{1}{7}180 + \frac{1}{7}185 + \frac{1}{7}191 \\ &= \frac{150+152+175+178+180+185+191}{7} \\ &= \frac{1211}{7} \\ &= 173 \end{aligned}$$

Table 2.3: **Population Conditional Means of Y**

Table 2.3

X_i	$E(Y \mid X_i)$
80	65
100	77
120	89
140	101
160	113
180	125
200	137
220	149
240	161
260	173

- **Interpretation of Table 2.3:**

Table 2.3 tabulates the relationship between $E(Y \mid X_i)$ and X_i for this particular population of 60 families.

This population relationship between $E(Y \mid X_i)$ and X_i is called either

(1) the **population regression function**, or **PRF**.

or

(2) the **population conditional mean function**, or **population CMF**

So **Table 2.3 is a *tabular representation* of the PRF** for the population of 60 families.

Properties of the Population Regression Function, or PRF:

Table 2.3

X_i	$E(Y \mid X_i)$
80	65
100	77
120	89
140	101
160	113
180	125
200	137
220	149
240	161
260	173

1. $E(Y \mid X_i)$ is a ***function* of X_i**: i.e., $E(Y \mid X_i) = f(X_i)$.

2. $E(Y \mid X_i)$ is an ***increasing* function of X_i**: i.e.,

$$\Delta X_i > 0 \;\Rightarrow\; \Delta E(Y \mid X_i) > 0 \quad \text{and} \quad \Delta X_i < 0 \;\Rightarrow\; \Delta E(Y \mid X_i) < 0.$$

3. $E(Y \mid X_i)$ is a ***linear* function of X_i**: i.e.,

- A plot of the 10 points in Table 2.3 lie on a straight line.
- Each 20-dollar increase in X induces a constant 12-dollar increase in $E(Y \mid X_i)$: i.e.,

$$\Delta X_i = 20 \;\Rightarrow\; \Delta E(Y \mid X_i) = 12 \;\Rightarrow\; \frac{\Delta E(Y \mid X_i)}{\Delta X_i} = \frac{12}{20} = \mathbf{0.60}.$$

4. The **population regression function (PRF)** -- also called the **population conditional mean function** -- takes the general linear form

$$E(Y \mid X_i) = \beta_0 + \beta_1 X_i .$$

5. The **population values** of the **regression coefficients β_1 and β_2** for this hypothetical population of 60 families are:

$$\beta_0 = 17 \qquad \text{and} \qquad \beta_1 = 0.60 .$$

6. The **population regression function, or PRF**, for this particular population of 60 families is therefore

$$E(Y \mid X_i) = \beta_0 + \beta_1 X_i = 17 + 0.60\, X_i .$$

❑ ***Summary* -- The Population Regression Function (PRF)**

The **PRF**, or **population regression function**, for this hypothetical population of 60 families is a ***linear* function of the population values X_i** of the regressor X; it takes the form

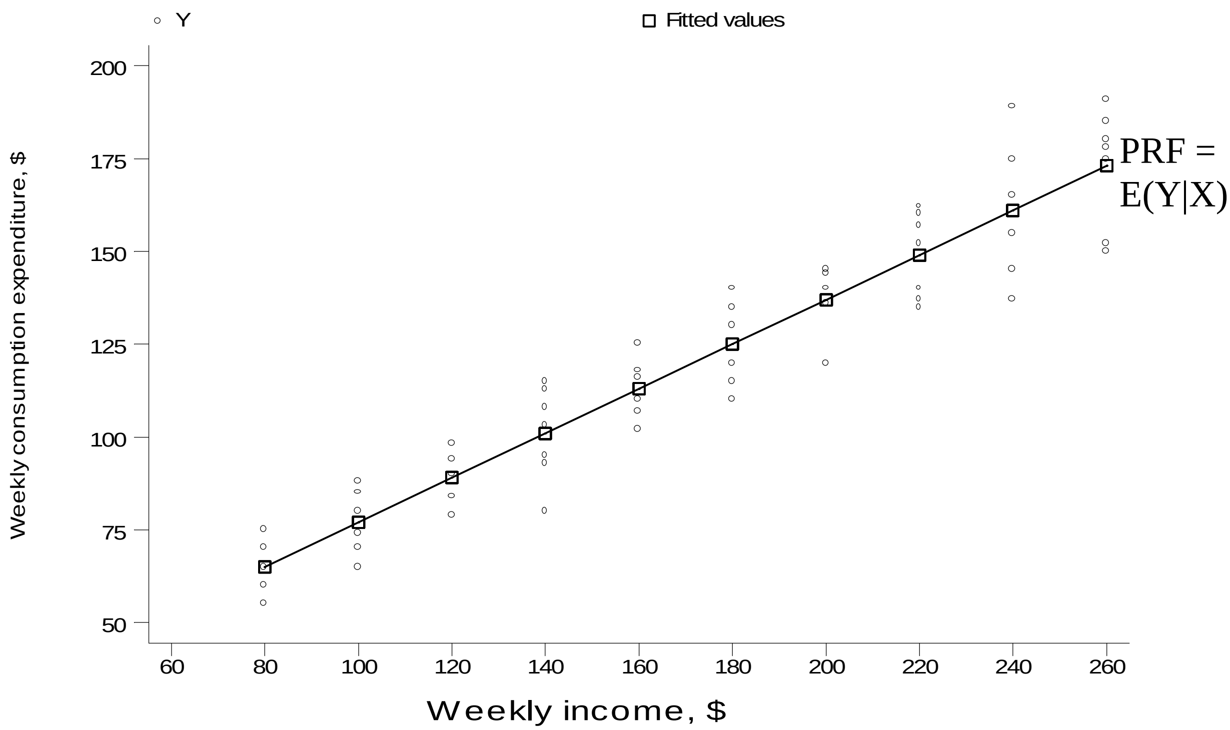
$$f(X_i) = E(Y \mid X_i) = \beta_0 + \beta_1 X_i = 17 + 0.60\, X_i .$$

where

$\beta_0 = 17$ is the population value of the *intercept* coefficient

$\beta_1 = 0.60$ is the population value of the *slope* coefficient of X_i.

- **Figure 2.1** **Plot of Population Data Points, Conditional Means E(Y|X), and the Population Regression Function PRF**



1. The ***small*** **dots** in Figure 2.1 constitute a **scatterplot** of the **population values of Y and X** for the population of 60 families:

   Each small dot corresponds to **a single population data point** of the form **(Y_i, X_i) i = 1, 2, ..., 60**.

2. The ***solid*** **line** in Figure 2.1 is the **population regression line** for the population of 60 families.

   Each pair of population values of $\left(E(Y|X_i), X_i\right)$, is represented by a ***large square*** **dot** in Figure 2.1.

   This population regression line is the locus of the 10 points in Table 2.3 -- i.e., it connects the 10 points of the form $\left(E(Y|X_i), X_i\right)$, i = 1, ..., 10.

The Random Error Terms

- ***Definition:*** The ***unobservable*** **random error term** for the i-th population member is denoted as u_i and defined as

  $$u_i = Y_i - E(Y|X_i) \qquad \forall \; i.$$

  For each population member -- for each of the 60 families in our hypothetical population -- the **random error term u_i** equals the deviation of that population member's individual Y_i value from the population conditional mean value of Y for the corresponding value X_i of X.

  Terminology: The random error term u_i is also known as the stochastic error term, the random disturbance term, or the stochastic disturbance term

- ***Implication 1:*** By simple re-arrangement of the above definition of u_i, it is obvious that each individual population value Y_i of Y can be written as

  $$\begin{aligned} Y_i &= E(Y|X_i) + u_i \\ &= \beta_0 + \beta_1 X_i + u_i \qquad \text{since } E(Y|X_i) = \beta_0 + \beta_1 X_i\,. \end{aligned}$$

  This equation is called the **population regression equation**, or **PRE**.

  Interpretation: **The PRE** indicates that **each population value Y_i** of Y can be expressed as the **sum of two components**:

  (1) $E(Y|X_i) = \beta_0 + \beta_1 X_i$
  = **the population conditional mean of Y for X = X_i**
  = the mean weekly consumption expenditure for all families in the population who have weekly disposable income X = X_i.

  (2) u_i = **the random error term for the i-th population member**
  = $Y_i - E(Y|X_i)$
  = the deviation of family i's weekly consumption expenditure Y_i from the population mean value $E(Y | X_i)$ of all families in the population that have the same weekly disposable income X = X_i.

Implication 2: The **population *conditional* mean value of the random error terms** for each population value X_i of X equals 0 -- i.e.,

$$E(u_i \mid X_i) = 0 \qquad \forall \; i.$$

Proof:

1. Take the conditional expectation for X = X_i of both sides of the PRE:

$$\begin{aligned} E(Y_i|X_i) &= E[E(Y \mid X_i)] + E(u_i|X_i) \\ &= E(Y \mid X_i) + E(u_i|X_i) \qquad \text{since } E(Y \mid X_i) \text{ is a constant.} \end{aligned}$$

2. Since $E(Y_i \mid X_i) = E(Y \mid X_i)$, the above equation implies that $E(u_i \mid X_i) = 0$.

- ***What do the Random Error Terms u_i Represent?***

The random error terms represent all the *unknown* and *unobservable* variables *other than* X that determine the *individual population values* Y_i of the dependent variable Y.

They arise from the following factors:

1. ***Omitted variables*** **that determine the population Y_i values**
2. ***Intrinsic randomness*** **in individual behaviour**

- ***Random Errors for Hypothetical Population of 60 Families***

Random Error Terms for X_i = 100

Y_i	$E(Y \mid X_i = 100)$	$u_i = Y_i - E(Y \mid X_i = 100)$
65	77	–12
70	77	–7
74	77	–3
80	77	3
85	77	8
88	77	11
Sum = 462		Sum = 0
Mean = 462/6 = 77		Mean = 0

Random Error Terms for X_i = 180

Y_i	$E(Y \mid X_i = 180)$	$u_i = Y_i - E(Y \mid X_i = 180)$
110	125	–15
115	125	–10
120	125	–5
130	125	5
135	125	10
140	125	15
Sum = 750		Sum = 0
Mean = 750/6 = 125		Mean = 0

Random Error Terms for X_i = 240

Y_i	$E(Y \mid X_i = 240)$	$u_i = Y_i - E(Y \mid X_i = 240)$
137	161	–24
145	161	–16
155	161	–6
165	161	4
175	161	14
189	161	28
Sum = 966		Sum = 0
Mean = 966/6 = 161		Mean = 0

The Sample Regression Function

- ***Important Point 1:*** Since in practice we do not observe the entire relevant population, and never know the true PRF, **we must *estimate* the PRF from *sample* data**.

- ***Objective of Regression Analysis*:** To estimate the PRF (population regression function) from sample data consisting of N randomly selected observations (Y_i, X_i), i = 1, ..., N taken from the population.

- ***Form of the Sample Regression Function (SRF)*:** The sample regression function, or SRF, takes the general form

  $$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i \qquad (i = 1, ..., N)$$

  where

  $\hat{Y}_i$ = an ***estimate* of the PRF**, $f(X_i) = E(Y_i \mid X_i) = \beta_0 + \beta_1 X_i$;
  $\hat{\beta}_0$ = an ***estimate* of the *intercept* coefficient β_0**;
  $\hat{\beta}_1$ = an ***estimate* of the *slope* coefficient β_1**.

- ***Nature of the Sample Data:*** A ***sample* is a *randomly-selected subset* of population members.**

  1. The sample observations {(Y_i, X_i): i = 1, ..., N} are typically a small subset of the parent population of all population data points (Y_i, X_i).

     Sample size N is much smaller than the number of population data points.

  2. ***Each* random sample** from a given population **yields *one estimate* of the PRF --** i.e., one estimate of the numerical value of β_0, and one estimate of the numerical value of β_1.

- ***<u>Important Point 2</u>*:** **Each random sample** from the same population **yields a *different* SRF --** i.e., a different numerical value of $\hat{\beta}_0$, and a different numerical value of $\hat{\beta}_1$.

  Example: Consider **two random samples of 10 observations** from the population of 60 families. Each sample consists of one family for each of the 10 different population values of X.

Tables 2.4 and 2.5

Sample 1		**Sample 2**	
X_i	Y_i	X_i	Y_i
80	70	80	55
100	65	100	88
120	90	120	90
140	95	140	80
160	110	160	118
180	115	180	120
200	120	200	145
220	140	220	135
240	155	240	145
260	150	260	175

Because the two samples contain **different Y_i values** for the 10 X_i values, they will yield **different SRFs --** a different numerical value of $\hat{\beta}_0$, and a different numerical value of $\hat{\beta}_1$.

- ***Sample 1 SRF* (SRF$_1$):** $\hat{Y}_i = 24.46 + 0.5091X_i$,

  where the Sample 1 coefficient estimates are $\hat{\beta}_0(1) = 24.46$ and $\hat{\beta}_1(1) = 0.5091$

- ***Sample 2 SRF* (SRF$_2$):** $\hat{Y}_i = 17.17 + 0.5761X_i$,

  where the Sample 2 coefficient estimates are $\hat{\beta}_0(2) = 17.17$ and $\hat{\beta}_1(2) = 0.5761$

- **Figure 2.2** **Plot of Sample Data Points and Sample Regression Functions for Random Samples 1 and 2**

SRF_1 is the SRF based on Sample 1: $\hat{Y}_i = 24.46 + 0.5091X_i$

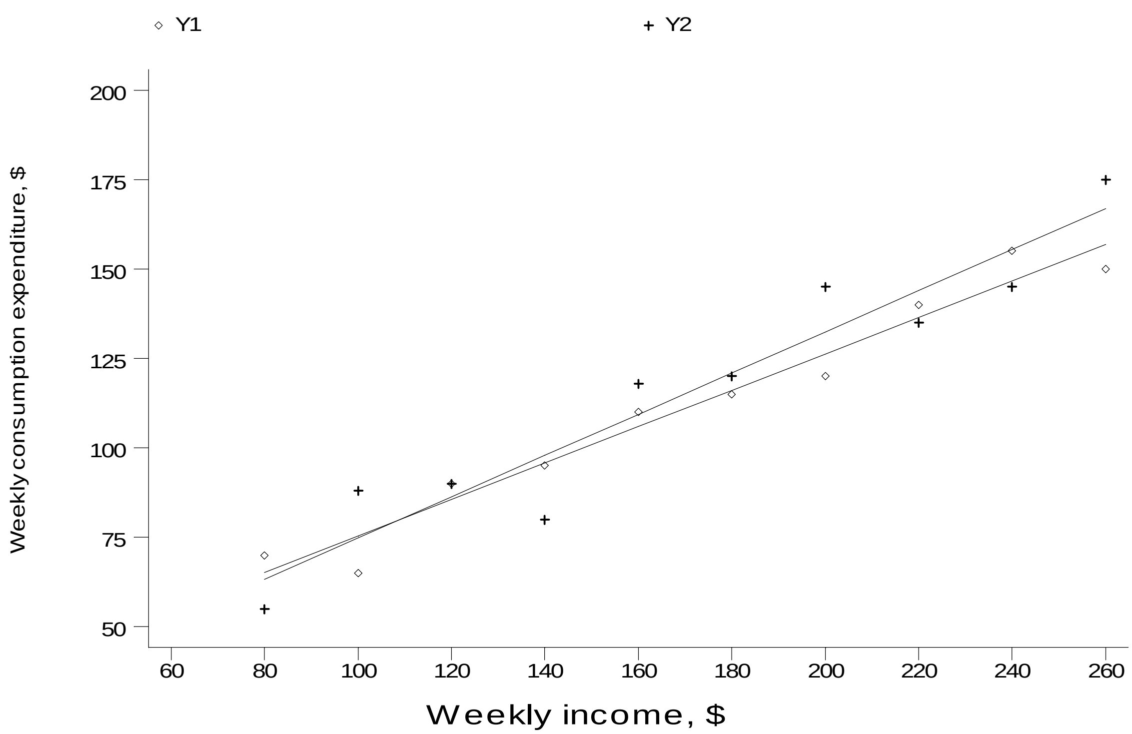
SRF_2 is the SRF based on Sample 2: $\hat{Y}_i = 17.17 + 0.5761X_i$

SRF_1 is the ***flatter*** regression line, **SRF_2** is the ***steeper*** regression line.

Important Points:

(1) Neither of these SRFs is identical to the true PRF. Each is merely an approximation to the true PRF.

(2) How good an approximation any SRF provides to the true PRF depends on how the SRF is constructed from sample data -- i.e., on the properties of the coefficient estimators $\hat{\beta}_0$ and $\hat{\beta}_1$.



The Sample Regression Equation (SRE)

- The **sample regression equation (SRE)** is the sample counterpart of the population regression equation (PRE)

$$Y_i = f(X_i) + u_i = E(Y_i \mid X_i) + u_i = \beta_0 + \beta_1 X_i + u_i \qquad \Leftarrow \text{ the PRE}$$

- ***Form of the Sample Regression Equation (SRE):*** The sample regression equation, or SRE, takes the general form

$$Y_i = \hat{Y}_i + \hat{u}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i + \hat{u}_i \qquad (i = 1, ..., N) \qquad \Leftarrow \text{ the SRE}$$

where

$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$ = an ***estimate* of the *PRF***, $f(X_i) = E(Y_i \mid X_i) = \beta_0 + \beta_1 X_i$;
$\hat{\beta}_0$ = an ***estimate*** of the ***intercept* coefficient β_0**;
$\hat{\beta}_1$ = an ***estimate*** of the ***slope* coefficient β_1**.
$\hat{u}_i$ = the ***residual*** for sample observation i.

- ***Interpretation of the SRE:*** The SRE represents each sample value of Y -- each Y_i value -- as the **sum of *two* components**:

(1) the ***estimated* (or *predicted*) mean value of Y** for each sample value X_i of X, i.e.,

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i \qquad (i = 1, ..., N);$$

(2) the ***residual*** corresponding to the i-th sample observation, i.e.,

$$\hat{u}_i = Y_i - \hat{Y}_i = Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i \qquad (i = 1, ..., N).$$

$\hat{u}_i$ = the residual for the i-th sample observation
= the ***observed*** Y-value (Y_i) – the ***estimated*** Y-value ($\hat{Y}_i$)

Compare the Population and Sample Regression Equations: the PRE and SRE

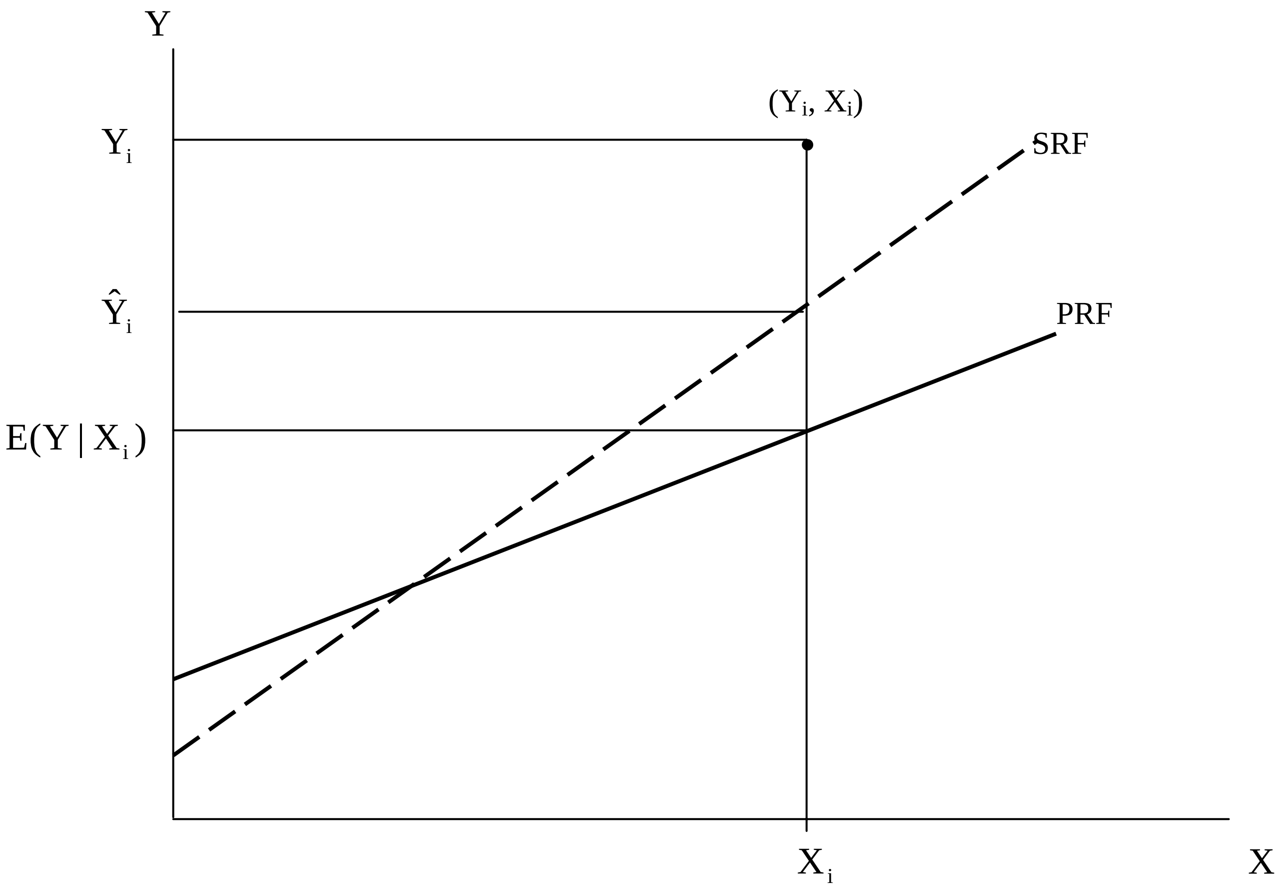
- **The PRE for Y_i** is:

$$Y_i = f(X_i) + u_i = E(Y_i \mid X_i) + u_i = \beta_0 + \beta_1 X_i + u_i$$

- **The SRE for Y_i** is:

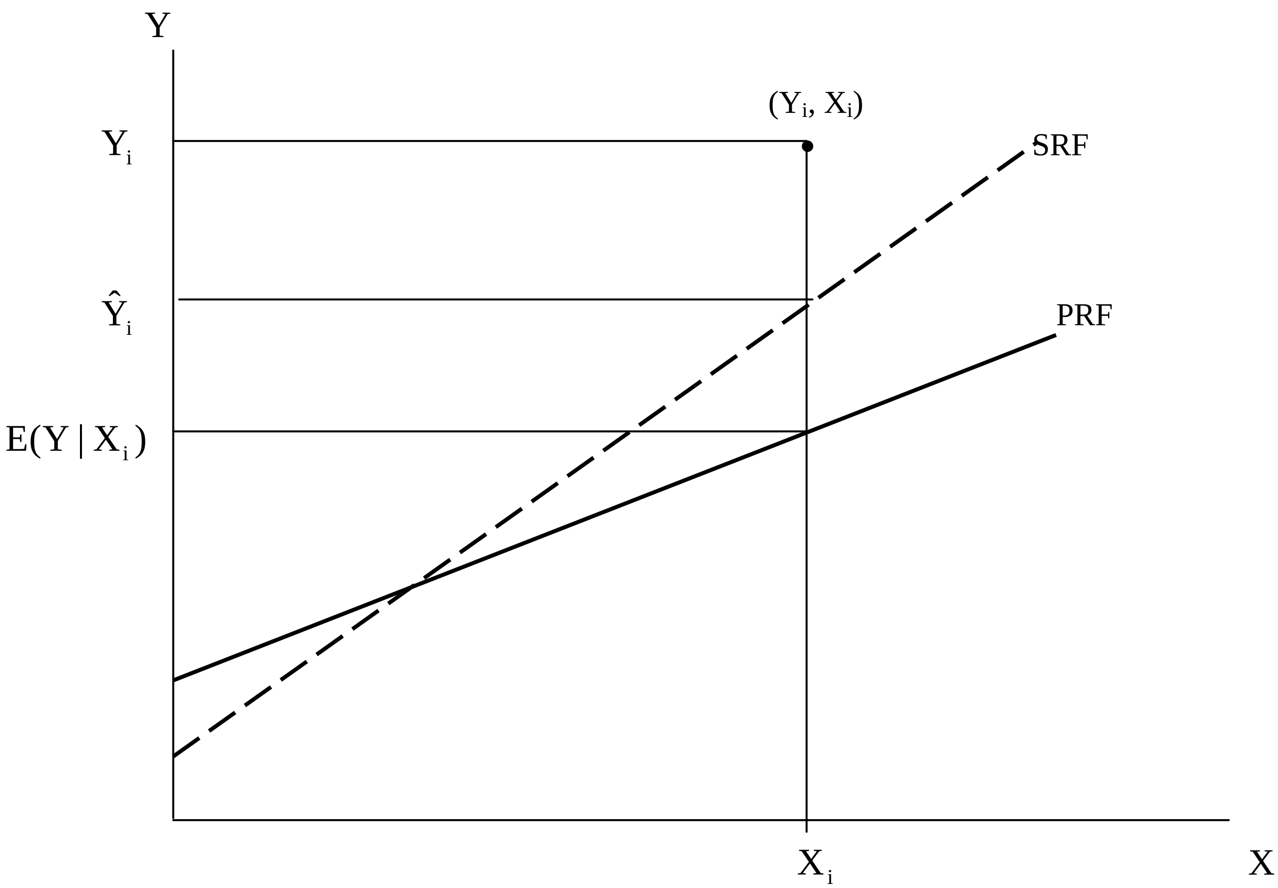
$$Y_i = \hat{Y}_i + \hat{u}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i + \hat{u}_i$$

- **Figure 2.3: Comparison of Population and Sample Regression Lines**



- The ***population*** **regression line** is a plot of the PRF: $E(Y_i \mid X_i) = \beta_0 + \beta_1 X_i$.
- The ***sample*** **regression line** is a plot of the SRF: $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$.

- **<u>Figure 2.3</u>: Comparison of Population and Sample Regression Lines**



At $X = X_i$:

♦ The population regression equation (PRE) represents the population value Y_i of Y as the sum of two parts:

$$Y_i = E(Y_i \mid X_i) + u_i = \beta_0 + \beta_1 X_i + u_i, \text{ where } E(Y_i \mid X_i) = \beta_0 + \beta_1 X_i$$

$$u_i = Y_i - E(Y_i \mid X_i) = Y_i - \beta_0 - \beta_1 X_i = \text{distance between } Y_i \text{ and } E(Y_i \mid X_i)$$

♦ The sample regression equation (SRE) represents the population value Y_i of Y as the sum of two parts:

$$Y_i = \hat{Y}_i + \hat{u}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i + \hat{u}_i, \text{ where } \hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$$

$$\hat{u}_i = Y_i - \hat{Y}_i = Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i = \text{distance between } Y_i \text{ and } \hat{Y}_i.$$