Thus we write for observation t

$$\ell_t = \ell_t(\boldsymbol{y}^t, \boldsymbol{\theta}_0) + n^{-1/2} a_t(\boldsymbol{y}^t). \tag{13.27}$$

We can see from this that the log of the density of the t^{th} observation is taken to be as given by a parametrized model for a parameter vector $\boldsymbol{\theta}_0$ satisfying the restrictions of the null hypothesis, plus a term that vanishes with $n^{-1/2}$ as $n \to \infty$. The fact that any density function is normalized so as to integrate to unity means that the functions a_t in (13.27) must be chosen so as to obey the normalization condition

$$\int \exp\bigl(\ell_t + n^{-1/2} a_t\bigr) dy_t = 1.$$

It can readily be shown that this implies that

$$E_0\bigl(a_t(\boldsymbol{y}^t)\bigr) = O(n^{-1/2}), \tag{13.28}$$

where E_0 denotes an expectation calculated using $\ell_t(\boldsymbol{y}^t, \boldsymbol{\theta}_0)$ as log density. To leading order asymptotically, then, the random variables a_t have mean zero.

The fact that ℓ_t is written in (13.27) as the sum of two terms does not restrict the applicability of the analysis at all, because one can think of (13.27) as arising from a first-order Taylor-series approximation to any drifting DGP. An example would be the sequence of local alternatives

$$\ell_t\bigl(\boldsymbol{y}^t, \boldsymbol{\theta}_0 + n^{-1/2}\boldsymbol{\delta}\bigr).$$

By arguments similar to those of Section 12.3, one can show that a Taylor-series approximation to this can be written in the form of (13.27).

We will now state without proof the results that correspond to equations (12.11), (12.12), and (12.13) in the NLS context. They are discussed and proved in Davidson and MacKinnon (1987), a paper that many readers may, however, find somewhat difficult because of the nature of the mathematics employed. These results provide asymptotically valid expressions for the various ingredients of the classical test statistics under the drifting DGP specified by (13.27). The first result is that the estimators $\hat{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{\theta}}$ are still root-n consistent for $\boldsymbol{\theta}_0$:

$$\ddot{\boldsymbol{\theta}} = \boldsymbol{\theta}_0 + O(n^{-1/2}),$$

from which we may conclude that $\ddot{\mathfrak{I}}$ and $\ddot{\boldsymbol{R}}$ are consistent for $\mathfrak{I}_0$ and $\boldsymbol{R}_0$, just as they are under the null hypothesis:

$$\ddot{\mathfrak{I}} = \mathfrak{I}_0 + O(n^{-1/2}); \quad \text{and} \quad \ddot{\boldsymbol{R}} = \boldsymbol{R}_0 + O(n^{-1/2}).$$

We may also conclude from the consistency of $\ddot{\boldsymbol{\theta}}$ that all the Taylor expansions used in developing equations (13.23), (13.25), and (13.26) are still valid, as are these equations themselves.