determinant to be minimized can be expressed as a function of $\boldsymbol{\eta}$ and $\boldsymbol{\alpha}$ alone, as follows:

$$\left|\big(\Delta\boldsymbol{Y} - \boldsymbol{Y}_{-p}\boldsymbol{\eta}\boldsymbol{\alpha}^\top\big)^\top \boldsymbol{M}_\Delta\big(\Delta\boldsymbol{Y} - \boldsymbol{Y}_{-p}\boldsymbol{\eta}\boldsymbol{\alpha}^\top\big)\right|. \tag{20.38}$$

Let us write $\boldsymbol{Y}^*_{-p}$ for $\boldsymbol{M}_\Delta\boldsymbol{Y}_{-p}$ and $\Delta\boldsymbol{Y}^*$ for $\boldsymbol{M}_\Delta\Delta\boldsymbol{Y}$. Then (20.38) can be expressed as

$$\left|\big(\Delta\boldsymbol{Y}^* - \boldsymbol{Y}^*_{-p}\boldsymbol{\eta}\boldsymbol{\alpha}^\top\big)^\top\big(\Delta\boldsymbol{Y}^* - \boldsymbol{Y}^*_{-p}\boldsymbol{\eta}\boldsymbol{\alpha}^\top\big)\right|. \tag{20.39}$$

It is now easy to concentrate this expression with respect to $\boldsymbol{\alpha}$, for, if we hold $\boldsymbol{\eta}$ fixed, the residuals in (20.39) depend linearly on $\boldsymbol{\alpha}$. If $\boldsymbol{V} \equiv \boldsymbol{Y}^*_{-p}\boldsymbol{\eta}$, we obtain the determinant

$$\big|(\Delta\boldsymbol{Y}^*)^\top\boldsymbol{M}_V\Delta\boldsymbol{Y}^*\big|. \tag{20.40}$$

By use of the same trick we had recourse to in Section 18.5, we can treat (20.40) as one factor in the decomposition of the determinant of a larger matrix. Consider

$$\begin{vmatrix} (\Delta\boldsymbol{Y}^*)^\top\Delta\boldsymbol{Y}^* & (\Delta\boldsymbol{Y}^*)^\top\boldsymbol{V} \\ \boldsymbol{V}^\top\Delta\boldsymbol{Y}^* & \boldsymbol{V}^\top\boldsymbol{V} \end{vmatrix}.$$

By the result (A.26) of Appendix A, this matrix can be factorized either as

$$\big|\boldsymbol{V}^\top\boldsymbol{V}\big|\big|(\Delta\boldsymbol{Y}^*)^\top\boldsymbol{M}_V\Delta\boldsymbol{Y}^*\big|$$

or as

$$\big|(\Delta\boldsymbol{Y}^*)^\top\Delta\boldsymbol{Y}^*\big|\big|\boldsymbol{V}^\top\boldsymbol{M}^*\boldsymbol{V}\big|,$$

where $\boldsymbol{M}^*$ projects orthogonally onto $\mathcal{S}^\perp(\Delta\boldsymbol{Y}^*)$. Since $|(\Delta\boldsymbol{Y}^*)^\top\Delta\boldsymbol{Y}^*|$ does not depend on $\boldsymbol{\eta}$, we see that minimizing (20.40) is equivalent to minimizing the ratio

$$\frac{|\boldsymbol{V}^\top\boldsymbol{M}^*\boldsymbol{V}|}{|\boldsymbol{V}^\top\boldsymbol{V}|} = \frac{\big|\boldsymbol{\eta}^\top(\boldsymbol{Y}^*_{-p})^\top\boldsymbol{M}^*\boldsymbol{Y}^*_{-p}\boldsymbol{\eta}\big|}{\big|\boldsymbol{\eta}^\top(\boldsymbol{Y}^*_{-p})^\top\boldsymbol{Y}^*_{-p}\boldsymbol{\eta}\big|} \tag{20.41}$$

with respect to $\boldsymbol{\eta}$. The minimum of (20.40) is then the minimum of (20.41) times $|(\Delta\boldsymbol{Y}^*)^\top\Delta\boldsymbol{Y}^*|$.

The least variance ratio problem that had to be solved in the LIML context (see (18.49)) involved a ratio of quadratic forms rather than the determinants that appear in (20.41). Even so, the present problem can be solved by the same technique as (18.49), namely, by converting the problem into an eigenvalue-eigenvector problem. Before we go into details, notice that (20.41) is invariant if $\boldsymbol{\eta}$ is replaced by $\boldsymbol{\eta}\boldsymbol{B}$, for any nonsingular $r \times r$ matrix $\boldsymbol{B}$. This is precisely what we noted earlier in speaking of the nonuniqueness of (20.36). We therefore cannot expect to obtain a unique minimizing $\boldsymbol{\eta}$ but only an r–dimensional subspace.