and evaluate it at the ML estimates $\hat{\boldsymbol{\theta}}$ to obtain $\hat{\boldsymbol{Z}}$. Then one performs an OPG regression, with regressors $\hat{\boldsymbol{G}}$ and $\hat{\boldsymbol{Z}}$, and uses n minus the SSR as the test statistic. Provided the matrix $[\hat{\boldsymbol{G}} \quad \hat{\boldsymbol{Z}}]^\top[\hat{\boldsymbol{G}} \quad \hat{\boldsymbol{Z}}]$ has full rank asymptotically, the test statistic will be asymptotically distributed as $\chi^2\big(\frac{1}{2}k(k+1)\big)$. When some of the columns of $\hat{\boldsymbol{G}}$ and $\hat{\boldsymbol{Z}}$ are perfectly collinear, as quite often happens, the number of degrees of freedom for the test must of course be reduced accordingly.

It is illuminating to consider as an example the univariate nonlinear regression model

$$y_t = x_t(\boldsymbol{\beta}) + u_t, \quad u_t \sim \mathrm{NID}(0, \sigma^2),$$

where $x_t(\boldsymbol{\beta})$ is a twice continuously differentiable function that depends on $\boldsymbol{\beta}$, a p–vector of parameters, and also on exogenous and predetermined variables which vary across observations. Thus the total number of parameters is $k = p + 1$. For this model, the contribution to the loglikelihood function from the t^{th} observation is

$$\ell_t(\boldsymbol{\beta}, \sigma) = -\frac{1}{2}\log(2\pi) - \log(\sigma) - \frac{1}{2\sigma^2}\big(y_t - x_t(\boldsymbol{\beta})\big)^2.$$

Thus the contribution from the t^{th} observation to the regressor corresponding to the i^{th} element of $\boldsymbol{\beta}$ is

$$G_{ti}(\boldsymbol{\beta}, \sigma) = \frac{1}{\sigma^2}\big(y_t - x_t(\boldsymbol{\beta})\big)X_{ti}(\boldsymbol{\beta}), \tag{16.66}$$

where, as usual, $X_{ti}(\boldsymbol{\beta})$ denotes the derivative of $x_t(\boldsymbol{\beta})$ with respect to β_i. Similarly, the contribution from the t^{th} observation to the regressor corresponding to σ is

$$G_{t,k}(\boldsymbol{\beta}, \sigma) = -\frac{1}{\sigma} + \frac{1}{\sigma^3}\big(y_t - x_t(\boldsymbol{\beta})\big)^2. \tag{16.67}$$

Using (16.66) and (16.67), it is easy to work out the regressors for the OPG version of the IM test. We make the definitions

$$\hat{e}_t \equiv \frac{1}{\hat{\sigma}}\big(y_t - x_t(\hat{\boldsymbol{\beta}})\big), \;\; \hat{X}_{ti} \equiv X_{ti}(\hat{\boldsymbol{\beta}}), \;\; \text{and} \;\; X^*_{tij}(\boldsymbol{\beta}) \equiv \frac{\partial X_{ti}(\boldsymbol{\beta})}{\partial \beta_j}.$$

Then, up to multiplicative factors that can have no effect on the fit of the regression, and hence no effect on the value of the IM test statistic, the regressors for the test regression are

$$\text{for } \beta_i: \quad \hat{e}_t\hat{X}_{ti}; \tag{16.68}$$

$$\text{for } \sigma: \quad \hat{e}_t^2 - 1; \tag{16.69}$$

$$\text{for } \beta_i \times \beta_j: \quad (\hat{e}_t^2 - 1)\hat{X}_{ti}\hat{X}_{tj} + \hat{\sigma}\hat{e}_t\hat{X}^*_{tij}; \tag{16.70}$$

$$\text{for } \sigma \times \beta_i: \quad (\hat{e}_t^3 - 3\hat{e}_t)\hat{X}_{ti}; \tag{16.71}$$

$$\text{for } \sigma \times \sigma: \quad \hat{e}_t^4 - 5\hat{e}_t^2 + 2. \tag{16.72}$$