$h \equiv 1/\sigma$, and the loglikelihood function can be shown to be globally concave in the latter parametrization. This implies that it must have a unique maximum no matter how it is parametrized. The $(k+1) \times (k+1)$ covariance matrix of the ML estimates may as usual be estimated in several ways. Unfortunately, as with the truncated regression model discussed in the previous section, the only artificial regression that is presently known to be applicable to this model is the OPG regression.

There is an interesting relationship among the tobit, truncated regression, and probit models. Suppose, for simplicity, that $x_t(\boldsymbol{\beta}) = \boldsymbol{X}_t\boldsymbol{\beta}$. Then the tobit loglikelihood function can be rewritten as

$$\sum_{y_t>0} \log\Bigl(\frac{1}{\sigma}\phi\Bigl(\frac{1}{\sigma}\bigl(y_t - \boldsymbol{X}_t\boldsymbol{\beta}\bigr)\Bigr)\Bigr) + \sum_{y_t=0} \log\Bigl(\Phi\Bigl(-\frac{1}{\sigma}\boldsymbol{X}_t\boldsymbol{\beta}\Bigr)\Bigr). \tag{15.49}$$

Now let us both add and subtract the term $\sum_{y_t>0} \log\bigl(\Phi(\boldsymbol{X}_t\boldsymbol{\beta}/\sigma)\bigr)$ in (15.49), which then becomes

$$\begin{aligned} &\sum_{y_t>0} \log\Bigl(\frac{1}{\sigma}\phi\Bigl(\frac{1}{\sigma}\bigl(y_t - \boldsymbol{X}_t\boldsymbol{\beta}\bigr)\Bigr)\Bigr) - \sum_{y_t>0} \log\Bigl(\Phi\Bigl(\frac{1}{\sigma}\boldsymbol{X}_t\boldsymbol{\beta}\Bigr)\Bigr) \\ &+ \sum_{y_t=0} \log\Bigl(\Phi\Bigl(-\frac{1}{\sigma}\boldsymbol{X}_t\boldsymbol{\beta}\Bigr)\Bigr) + \sum_{y_t>0} \log\Bigl(\Phi\Bigl(\frac{1}{\sigma}\boldsymbol{X}_t\boldsymbol{\beta}\Bigr)\Bigr). \end{aligned} \tag{15.50}$$

The first line here is the loglikelihood function for a truncated regression model; it is just (15.43) with $y^l = 0$ and $x_t(\boldsymbol{\beta}) = \boldsymbol{X}_t\boldsymbol{\beta}$ and with the set of observations to which the summations apply adjusted appropriately. The second line is the loglikelihood function for a probit model with index function $\boldsymbol{X}_t\boldsymbol{\beta}/\sigma$. Of course, if all we had was the second line here, we could not identify $\boldsymbol{\beta}$ and σ separately, but since we also have the first line, that is not a problem.

Expression (15.50) makes it clear that the tobit model is like a truncated regression model combined with a probit model, with the coefficient vectors in the latter two models restricted to be proportional to each other. Cragg (1971) argued that this restriction may sometimes be unreasonable and proposed several more general models as plausible alternatives to the tobit model. It may sometimes be desirable to test the tobit model against one or more of these more general models; see Lin and Schmidt (1984) and Greene (1990a, Chapter 21).

As we mentioned earlier, it is easy to modify the tobit model to handle different types of censoring. For example, one possibility is a model with **double censoring**. Suppose that

$$\begin{aligned} &y_t^* = x_t(\boldsymbol{\beta}) + u_t, \quad u_t \sim \text{NID}(0, \sigma^2), \\ &y_t = y_t^* \text{ if } y_t^l \le y_t^* \le y_t^u;\ \ y_t = y_t^l \text{ if } y_t^* < y_t^l;\ \ y_t = y_t^u \text{ if } y_t^* > y_t^u. \end{aligned}$$