By the FWL Theorem,

$$\hat{\boldsymbol{\beta}}_2 = \big(\boldsymbol{X}_2^\top \boldsymbol{M}_1 \boldsymbol{X}_2\big)^{-1} \boldsymbol{X}_2^\top \boldsymbol{M}_1 \boldsymbol{y} \quad \text{and}$$
$$\big((\boldsymbol{X}^\top \boldsymbol{X})^{-1}\big)_{22} = \big(\boldsymbol{X}_2^\top \boldsymbol{M}_1 \boldsymbol{X}_2\big)^{-1}.$$

Thus (13.41) becomes

$$W = n\left(\frac{\boldsymbol{y}^\top \boldsymbol{M}_1 \boldsymbol{X}_2 (\boldsymbol{X}_2^\top \boldsymbol{M}_1 \boldsymbol{X}_2)^{-1} \boldsymbol{X}_2^\top \boldsymbol{M}_1 \boldsymbol{y}}{\boldsymbol{y}^\top \boldsymbol{M}_X \boldsymbol{y}}\right) = n\left(\frac{\boldsymbol{y}^\top \boldsymbol{P}_{M_1 X_2} \boldsymbol{y}}{\boldsymbol{y}^\top \boldsymbol{M}_X \boldsymbol{y}}\right).$$

From (13.37) and (13.39), we obtain

$$W = \left(\frac{rn}{n-k}\right)F; \quad LR = n \log\left(1 + \frac{W}{n}\right). \tag{13.42}$$

Since W is equal to $n/(n-k)$ times rF, it is evident that

$$W = rF + O(n^{-1}).$$

Finally, we turn to the LM statistic. We first observe from (8.83) that the gradient with respect to the regression parameters $\boldsymbol{\beta}$ of the loglikelihood function for a linear regression model with normal errors is

$$\boldsymbol{g}(\boldsymbol{y}, \boldsymbol{\beta}, \sigma) = \frac{1}{\sigma^2} \sum_{t=1}^{n} \boldsymbol{X}_t^\top (y_t - \boldsymbol{X}_t \boldsymbol{\beta}) = \sigma^{-2} \boldsymbol{X}^\top (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}).$$

Thus, from (13.03), the LM statistic is

$$\begin{aligned} LM &= \tilde{\boldsymbol{g}}_2^\top \big(\tilde{\boldsymbol{I}}^{-1}\big)_{22} \tilde{\boldsymbol{g}}_2 \\ &= \tilde{\sigma}^{-4} (\boldsymbol{y} - \boldsymbol{X}\tilde{\boldsymbol{\beta}})^\top \boldsymbol{X}_2 \big(\tilde{\sigma}^2 (\boldsymbol{X}_2^\top \boldsymbol{M}_1 \boldsymbol{X}_2)^{-1}\big) \boldsymbol{X}_2^\top (\boldsymbol{y} - \boldsymbol{X}\tilde{\boldsymbol{\beta}}). \end{aligned} \tag{13.43}$$

Since the LM test is based on the restricted model (13.35), we use the ML estimate of σ from that model:

$$\tilde{\sigma}^2 = \tfrac{1}{n} \boldsymbol{y}^\top \boldsymbol{M}_1 \boldsymbol{y}.$$

Substituting this into (13.43), we see that

$$\begin{aligned} LM &= n\left(\frac{\boldsymbol{y}^\top \boldsymbol{M}_1 \boldsymbol{X}_2 (\boldsymbol{X}_2^\top \boldsymbol{M}_1 \boldsymbol{X}_2)^{-1} \boldsymbol{X}_2^\top \boldsymbol{M}_1 \boldsymbol{y}}{\boldsymbol{y}^\top \boldsymbol{M}_1 \boldsymbol{y}}\right) \\ &= n\left(\frac{\boldsymbol{y}^\top \boldsymbol{P}_{M_1 X_2} \boldsymbol{y} / \boldsymbol{y}^\top \boldsymbol{M}_X \boldsymbol{y}}{1 + \boldsymbol{y}^\top \boldsymbol{P}_{M_1 X_2} \boldsymbol{y} / \boldsymbol{y}^\top \boldsymbol{M}_X \boldsymbol{y}}\right) \\ &= n\left(\frac{rF}{n - k + rF}\right). \end{aligned} \tag{13.44}$$