The use of regressions (21.14) and (21.15) has been advocated for some time in the operations research literature; see Lavenberg and Welch (1981) and Ripley (1987). These procedures were exposited and developed further in Davidson and MacKinnon (1992b), in which it is shown how to use them for estimating quantiles as well as moments and tail areas and how to construct τ's that are approximately optimal for several cases of interest. In particular, for the estimation of test sizes and powers, a way is suggested to construct better, but more complicated, control variates than the two-value ones discussed above.

To illustrate the use of control variates, we will consider a simple example that was discussed by Hendry (1984). It is the stationary AR(1) model with normal errors:

$$y_t = \beta y_{t-1} + u_t, \quad u_t \sim N(0, \sigma^2), \;\; t = 1, \ldots, n. \tag{21.16}$$

We assume that $|\beta| < 1$, which is just the stationarity condition, and that $y_0 = 0$. Stationarity implies that $y_t \sim N\big(0, \sigma^2/(1-\beta^2)\big)$. Suppose that we are interested in the mean of $\hat{\beta}$, the OLS estimate of β. It is easy to see that both the value of $\hat{\beta}$ and its probability distribution are invariant to the value of σ in the DGP, say σ_0, but that its properties may well depend on both β_0 and the sample size n. A serious investigation would therefore involve seeing how the mean of $\hat{\beta}$ depends on β_0 and n; see Section 21.7 below. Since we are here merely interested in illustrating the use of control variates, we will consider only a few particular cases.[3]

The OLS estimate $\hat{\beta}$, assuming that y_0 is known, is

$$\hat{\beta} = \frac{\sum_{t=1}^{n} y_t y_{t-1}}{\sum_{t=1}^{n} y_{t-1}^2}.$$

Under the DGP characterized by β_0, this becomes

$$\frac{\sum_{t=1}^{n} (\beta_0 y_{t-1} + u_t) y_{t-1}}{\sum_{t=1}^{n} y_{t-1}^2} = \beta_0 + \frac{\sum_{t=1}^{n} u_t y_{t-1}}{\sum_{t=1}^{n} y_{t-1}^2}. \tag{21.17}$$

Although the numerator of the second term on the right-hand side of (21.17) has mean zero, it is not independent of the denominator, and so $E(\hat{\beta}) \neq \beta_0$. However, asymptotic theory tells us that $\hat{\beta}$ is consistent and asymptotically normal, since $n^{1/2}(\hat{\beta} - \beta_0) \overset{a}{\sim} N(0, 1 - \beta_0^2)$.

Now consider the control variate

$$\tau = n^{-1/2} \sum_{t=1}^{n} u_t y_{t-1}, \tag{21.18}$$

[3] Note that, although (21.16) looks like a regression model, antithetic variates are not useful here. If one generates two sets of data using disturbance vectors $\boldsymbol{u}$ and $-\boldsymbol{u}$, the estimates of β that one obtains are identical.