Because the determinant of the sum of two positive definite matrices is always greater than the determinants of either of those matrices (see Appendix A), it follows from (18.35) that (18.34) will exceed $|\boldsymbol{Y}^\top \boldsymbol{M}_X \boldsymbol{Y}|$ for all $\boldsymbol{A} \neq \mathbf{0}$. This implies that $\hat{\boldsymbol{\Pi}}$ minimizes (18.34), and so we have proved that equation-by-equation OLS estimates of the URF are also ML estimates for the entire system.

If one does not have access to a regression package that calculates (18.33) easily, there is another way to do so. Consider the **recursive system**

$$
\begin{aligned}
\boldsymbol{y}_1 &= \boldsymbol{X}\boldsymbol{\eta}_1 + \boldsymbol{e}_1 \\
\boldsymbol{y}_2 &= \boldsymbol{X}\boldsymbol{\eta}_2 + \boldsymbol{y}_1\alpha_1 + \boldsymbol{e}_2 \\
\boldsymbol{y}_3 &= \boldsymbol{X}\boldsymbol{\eta}_3 + [\boldsymbol{y}_1 \quad \boldsymbol{y}_2]\boldsymbol{\alpha}_2 + \boldsymbol{e}_3 \\
\boldsymbol{y}_4 &= \boldsymbol{X}\boldsymbol{\eta}_4 + [\boldsymbol{y}_1 \quad \boldsymbol{y}_2 \quad \boldsymbol{y}_3]\boldsymbol{\alpha}_3 + \boldsymbol{e}_4,
\end{aligned}
\tag{18.36}
$$

and so on, where $\boldsymbol{y}_i$ denotes the i^{th} column of $\boldsymbol{Y}$. This system of equations can be interpreted as simply a reparametrization of the URF (18.03). It is easy to see that if one estimates these equations by OLS, all the residual vectors will be mutually orthogonal: $\hat{\boldsymbol{e}}_2$ will be orthogonal to $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_3$ will be orthogonal to $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_1$, and so on. According to the URF, all the $\boldsymbol{y}_i$'s are linear combinations of the columns of $\boldsymbol{X}$ plus random errors. Therefore, the equations of (18.36) are correct for any arbitrary choice of the α parameters: The $\boldsymbol{\eta}_i$'s simply adjust to whatever choice is made. If, however, we *require* that the error terms $\boldsymbol{e}_i$ should be orthogonal, then this serves to identify a particular unique choice of the α's. In fact, the recursive system (18.36) has exactly the same number of parameters as the URF (18.03): g vectors $\boldsymbol{\eta}_i$, each with k elements, $g-1$ vectors $\boldsymbol{\alpha}_i$, with a total of $g(g-1)/2$, and g variance parameters, for a total of $gk + (g^2+g)/2$. The URF has gk parameters in $\boldsymbol{\Pi}$ and $(g^2+g)/2$ in the covariance matrix $\boldsymbol{\Omega}$, for the same total. What has happened is that the α parameters in (18.36) have replaced the off-diagonal elements of the covariance matrix of $\boldsymbol{V}$ in the URF.

Since the recursive system (18.36) is simply a reparametrization of the URF (18.03), it should come as no surprise that the loglikelihood function for the former is equal to (18.33). Because the residuals of the various equations in (18.36) are orthogonal, the value of the loglikelihood function for (18.36) is simply the sum of the values of the loglikelihood functions from OLS estimation of the individual equations. This result, which readers can easily verify numerically, sometimes provides a convenient way to compute the loglikelihood function for the URF. Except for this purpose, recursive systems are not generally of much interest. They do not convey any information that is not already provided by the URF, and the parametrization depends on an arbitrary ordering of the equations.