Similarly, when we test H_0 against H_2, the NCP is

$$\begin{aligned}\Lambda_{21} = \frac{\rho_0^2}{\sigma_0^2} &\operatorname*{plim}_{n\to\infty}\Big(\frac{1}{n}\boldsymbol{u}_{-1}{}^{\top}\boldsymbol{M}_X(\boldsymbol{X}_{-1}\boldsymbol{\beta}_0+\boldsymbol{u}_{-1})\Big)\\ &\times \operatorname*{plim}_{n\to\infty}\Big(\frac{1}{n}(\boldsymbol{X}_{-1}\boldsymbol{\beta}_0+\boldsymbol{u}_{-1})^{\top}\boldsymbol{M}_X(\boldsymbol{X}_{-1}\boldsymbol{\beta}_0+\boldsymbol{u}_{-1})\Big)^{-1}\\ &\times \operatorname*{plim}_{n\to\infty}\Big(\frac{1}{n}(\boldsymbol{X}_{-1}\boldsymbol{\beta}_0+\boldsymbol{u}_{-1})^{\top}\boldsymbol{M}_X\boldsymbol{u}_{-1}\Big).\end{aligned}$$

This simplifies to

$$\begin{aligned}&\frac{\rho_0^2}{\sigma_0^2}\sigma_0^2\Big(\sigma_0^2+\operatorname{plim}\frac{1}{n}\big\|\boldsymbol{M}_X\boldsymbol{X}_{-1}\boldsymbol{\beta}_0\big\|^2\Big)^{-1}\sigma_0^2\\ &=\rho_0^2\Big(1+\sigma_0^{-2}\operatorname{plim}\frac{1}{n}\big\|\boldsymbol{M}_X\boldsymbol{X}_{-1}\boldsymbol{\beta}_0\big\|^2\Big)^{-1}.\end{aligned}$$

Evidently, $\cos^2\phi$ for the test of H_0 against H_2 is the right-hand expression here divided by ρ_0^2, which is

$$\left(1+\frac{\operatorname{plim} n^{-1}\|\boldsymbol{M}_X\boldsymbol{X}_{-1}\boldsymbol{\beta}_0\|^2}{\sigma_0^2}\right)^{-1}. \tag{12.34}$$

This last result is worth comment. We have found that $\cos^2\phi$ for the test against H_2 when the data were generated by H_1, expression (12.34), is identical to $\cos^2\phi$ for the test against H_1 when the data were generated by H_2, expression (12.33). This result is true not just for this example, but for every case in which both alternatives involve one-degree-of-freedom tests. Geometrically, this equivalence simply reflects the fact that when $\boldsymbol{z}$ is a vector, the angle between $\alpha n^{-1/2}\boldsymbol{M}_X\boldsymbol{a}$ and the projection of $\alpha n^{-1/2}\boldsymbol{M}_X\boldsymbol{a}$ onto $\mathcal{S}(\boldsymbol{X},\boldsymbol{z})$, which is

$$\alpha n^{-1/2}\boldsymbol{M}_X\boldsymbol{z}\big(\boldsymbol{z}^{\top}\boldsymbol{M}_X\boldsymbol{z}\big)^{-1}\boldsymbol{z}^{\top}\boldsymbol{M}_X\boldsymbol{a},$$

is the same as the angle between $\alpha n^{-1/2}\boldsymbol{M}_X\boldsymbol{a}$ and $\alpha n^{-1/2}\boldsymbol{M}_X\boldsymbol{z}$. The reason for this is that $(\boldsymbol{z}^{\top}\boldsymbol{M}_X\boldsymbol{z})^{-1}\boldsymbol{z}^{\top}\boldsymbol{M}_X\boldsymbol{a}$ is a scalar when $\boldsymbol{z}$ is a vector. Hence, if we reverse the roles of $\boldsymbol{a}$ and $\boldsymbol{z}$, the angle is unchanged. This geometrical fact also results in two numerical facts. First, in the regressions

$$\begin{aligned}\boldsymbol{y} &= \boldsymbol{X}\boldsymbol{\alpha}+\gamma\boldsymbol{z} + \text{residuals}\quad\text{and}\\ \boldsymbol{z} &= \boldsymbol{X}\boldsymbol{\beta}+\delta\boldsymbol{y} + \text{residuals},\end{aligned}$$

the t statistic on $\boldsymbol{z}$ in the first is equal to that on $\boldsymbol{y}$ in the second. Second, in the regressions

$$\begin{aligned}\boldsymbol{M}_X\boldsymbol{y} &= \gamma\boldsymbol{M}_X\boldsymbol{z} + \text{residuals}\quad\text{and}\\ \boldsymbol{M}_X\boldsymbol{z} &= \delta\boldsymbol{M}_X\boldsymbol{y} + \text{residuals},\end{aligned}$$

the t statistics on γ and δ are numerically identical and so are the uncentered R^2's.